ELEMENTS OF QUATERNIONS.

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OF

QUATERNIONS.

BY THE LATE

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SECOND EDITION.

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ADVERTISEMENT TO THE FIRST EDITION.

In my late father's Will no instructions were left as to the publication of his Writings, nor specially as to that of the "Elements of Quaternions," which, but for his late fatal illness, would have been before now, in all their completeness, in the hands of the Public.

My brother, the Rev. A. H. Hamilton, who was named Executor, being too much engaged in his clerical duties to undertake the publication, deputed this task to me.

It was then for me to consider how I could best fulfil my triple duty in this matter—First, and chiefly, to the dead; secondly, to the present public; and thirdly, to succeeding generations. I came to the conclusion that my duty was to publish the work as I found it, adding merely proof-sheets, partially corrected by my late father, and from which I removed a few typographical errors, and editing only in the literal sense of giving forth.

Shortly before my father's death, I had several conversations with him on the subject of the "Elements." In these he spoke of anticipated applications of Quaternions to Electricity, and to all questions in which the idea of Polarity is involved—applications which he never in his own lifetime expected to be able fully to develop, bows to be reserved for the hands of another Ulysses. He also discussed a good deal the nature of his own forthcoming Preface; and I may intimate that, after dealing with its more important topics, he intended to advert to the great labour which the writing of the "Elements" had cost him—labour both mental and mechanical; as, besides a mass of subsidiary and unprinted calculations, he wrote out all the manuscript, and corrected the proof-sheets, without assistance.

And here I must gratefully acknowledge the generous act of the Board of Trinity College, Dublin, in relieving us of the remaining pecuniary liability, and thus incurring the main expense, of the publication of this volume. The announcement of their intention to do so, gratifying as it was, surprised me the less, when I remembered that they had, after the publication of my father's former book, "Lectures on Quaternions," defrayed its entire cost; an extension of their liberality beyond what

was recorded by him at the end of his Preface to the "Lectures," which doubtless he would have acknowledged, had he lived to complete the Preface of the "ELEMENTS."

He intended also, I know, to express his sense of the care bestowed upon the typographical correctness of this volume by Mr. M. H. Gill of the University Press, and upon the delineation of the figures by the Engraver, Mr. Oldham.

I annex the commencement of a Preface, left in manuscript by my father, and which he might possibly have modified or rewritten. Believing that I have thus best fulfilled my part as trustee of the unpublished "ELEMENTS," I now place them in the hands of the scientific public.

WILLIAM EDWIN HAMILTON.

January 1st, 1866.

PREFACE TO THE FIRST EDITION.

[1.] THE volume now submitted to the public is founded on the same principles as the "LECTURES,"(1) which were published on the same subject about ten years ago: but the plan adopted is entirely new, and the present work can in no sense be considered as a second edition of that former one. The *Table of Contents*, by collecting into one view the headings of the various Chapters and Sections, may suffice to give, to readers already acquainted with the subject, a notion of the course pursued: but it seems proper to offer here a few introductory remarks, especially as regards the method of exposition, which it has been thought convenient on this occasion to adopt.

[2.] The present treatise is divided into Three Books, each designed to develop one guiding conception or view, and to illustrate it by a sufficient but not excessive number of examples or applications. The First Book relates to the *Conception of a Vector*, considered as a *directed right line*, in space of three dimensions. The Second Book introduces a *First Conception of a Quaternion*, considered as *the Quotient of two such Vectors*. And the Third Book treats of *Products and Powers of Vectors*, regarded as constituting a *Second Principal Form* of the *Conception of Quaternions in Geometry*.

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* This fragment, by the Author, was found in one of his manuscript books by the Editor. [W. E. Hamilton.]

PREFACE TO THE SECOND EDITION.

Sir William Rowan Hamilton died on the 2nd of September, 1865, leaving his great work on Quaternions unfinished. He intended to have added some account of the operator* ∇, an Index, and an Appendix containing notes on *Anharmonic Coordinates*, on the *Barycentric Calculus*, and on proofs of his geometrical theorems stated in Nichol's Cyclopædia. At the time of his death, with the exception of a fragment of the preface, and a small portion of the table of contents, all the manuscript he had prepared was in type. As he rarely commenced writing before his thoughts were fully matured, he has left no outline of the additions contemplated.

In this edition, printed by direction of the Board of Trinity College, Dublin, the original text has been faithfully preserved, except in a few places where trifling errors have been corrected. I have added notes, distinguished in every case by square brackets, wherever I thought they were wanted. I have rendered the work more convenient by increasing the number of cross-references, by including in the page-headings the numbers of the articles (for the original references are generally given to articles and not to pages), by dividing the work into two volumes, and by the addition of an index. The table of contents has been amplified by a brief analysis of each article, designed as far as possible to assist the reader in following and in recapitulating the arguments in the text. Hamilton indicated "a *minimum* course of study, amounting to rather less than 200 pages (or parts of pages)," suitable for a first perusal, and he intended to have prepared a table containing references to this course. Such a table will be found at the end of the table of contents, but for the convenience of students of Physics, and of those desirous of obtaining a working knowledge of Hamilton's powerful engine of research, I have amplified it somewhat, duly noting, however, the minimum course.

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* In the second volume I hope to devote an appendix to this important subject.

I infer from the fragment of the author's preface that he proposed to sketch an outline of the method of exposition, of an elementary character and adapted to those readers to whom the subject is new. To those readers chiefly I address the following remarks :—

According to the plan of this work, whenever a new conception or notation is introduced, a series of illustrative examples immediately follows. Most of these involve no real difficulty, but occasionally a long and difficult investigation occurs even in the early parts of the book. Intricate investigations, which are merely illustrative, are everywhere omitted from the selected course.

The First Book deals with Vectors, considered without reference to angles or to rotations. In a word, it is concerned with the application of the signs +, −, and = to the algebra of vectors. The sign − is first introduced, and the sign + follows from the formula of relation $(b - a) + a = b$. Sections 3 and 4 (pp. 7–11) are occupied with a series of propositions concerning the commutative and associative laws of the addition of vectors, and the multiplication of vectors by *scalars*, or algebraical coefficients. Propositions such as these often appear to a student to be mere truisms, and unfortunately it is not easy to find elementary examples to convince him of the contrary. The addition of vector-arcs, he will find on p. 156, is not commutative, though it is associative.† With the exception of a few passages noted in the table of a selected course, there is nothing in chaps. II. and III. essential to a good knowledge of the subject. They contain, however, an account of an extremely elegant theory of anharmonic coordinates, independent of any non-projective property, and intricate and powerful investigations of geometric nets and of systems of barycentres.

The Second Book treats of Quaternions considered as quotients of vectors, and as involving angular relations. It opens with a first conception of a quaternion as a quotient of two vectors, and thus the division of vectors is introduced before that of multiplication, just as in the First Book subtraction precedes addition. If $q = \beta : \alpha$ is the quotient of two vectors, β and α, it is natural to define the product $q.\alpha$ by the relation $q.\alpha = \beta$. It is soon found, if any vector γ is selected in the plane of α and β, that the product $q.\gamma$ is a *vector* in the same plane whose length bears to that of γ the same ratio as the length of β to that of α, and which makes the same angle with γ that β

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* In fact the commutative law of addition depends on a property of a parallelogram, and therefore ultimately on the validity of Euclid's fifth postulate. It does not hold except for Euclidean space.

makes with a. Thus, from the first conception of a quaternion as a quantity expressing the relative length and direction of two given vectors, we have come to consider a quaternion as an operator on a special set of vectors, viz. those in its own plane. Observe that, so far, we have not arrived at the conception of the product of two vectors, nor of the product of a quaternion and an arbitrary vector. We have only reached the limited conception of the product $q \cdot \gamma$ of a quaternion q and a vector γ in its plane, and while an interpretation is assigned to $q \cdot \gamma$, as yet the product $\gamma \cdot q$ is unknown.

After reviewing a class of quaternions derived by fixed laws from a given quaternion, a special class of quaternions, called versors or radial quotients, is considered in detail. The product of a pair of versors is found (p. 147) to depend on the order in which they are multiplied, that is qq' is not generally equal to $q'q$, or the commutative law of algebraic multiplication is not true for versors, nor *à fortiori* for quaternions.

The multiplication of a special set of versors of a restricted kind occupies section 10, chap. I.; and on p. 160 the famous formula

$$i^2 = j^2 = k^2 = ijk = -1 \qquad \text{(A)}$$

is deduced, in which i, j, and k are right versors* in three mutually perpendicular planes. This section contains the first example of a product of more than two versors, and it is shown that the multiplication of these specially related right versors is associative. Warned by the failure of the commutative law, it is necessary to determine if the remaining laws of algebra are valid in quaternions. In algebra, if we first form the product bc and then multiply by a, we have the same result as if we multiplied c by the product ab, and this associative law is expressed in symbols by the equation $a \, . \, bc = ab \, . \, c$. This is also true for quaternions, and it may be regarded as the chief feature which distinguishes quaternions from other systems of vector analysis. For example, Grassmann's multiplication is sometimes associative, but sometimes it is not. It is necessary to prove, moreover, that quaternion multiplication is distributive, or that $a(b + c) = ab + ac$. This is not true if b and c are vector arcs, even when a is a number as shown on p. 156. Some of Hamilton's early investigations led him to a non-distributive system of multiplication in 1830.†

Next a quaternion is decomposed in two ways:—(1) in section 11, into the *product* of its tensor and its versor; (2) in section 12, into the *sum* of its

* A right versor turns a vector in its plane through a right angle.

† Preface to Lectures on Quaternions, paragraph [41]. Scheffler has reproduced this system.

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scalar and its right or vector part. This right or vector part, it is ultimately shown, may be identified with a vector; at present it is regarded as a right quaternion, or a quotient of two perpendicular vectors. By the first of these decompositions, "the multiplication of any two quaternions is reduced to the arithmetical operation of multiplying their tensors, and the geometrical operation of multiplying their versors"; and by the second the addition of quaternions is reduced to the algebraical addition of their scalar parts, and the geometrical addition of their vector parts. Thus it is proved (Arts. 206, 207) that the addition of the vector parts is reducible to the addition of vectors, and, as the addition both of scalars and of vectors is commutative and associative, so likewise is the addition of quaternions.

The multiplication of right quaternions, or of the vector parts of quaternions, is proved in Art. 211 to be distributive; and, as any quaternion is the sum of a scalar and a vector part, it is also proved that the general multiplication of quaternions is distributive. A long series of examples follows, some of which are not easy, including Hamilton's well-known construction of the ellipsoid.

Section 14 is entitled "On the reduction of the general Quaternion to the Standard Quadrinomial Form ($q = w + ix + jy + kz$); with a First Proof of the Associative Principle of the Multiplication of Quaternions." This proof depends on the general Distributive Property lately proved, and on the Associative Property of the particular set of versors i, j, k (Art. 161); but in chap. III. various proofs are given which are independent of these properties. The first proof is sufficient for all practical purposes.

The laws of combination of quaternions are now established. Addition (and subtraction) is associative and commutative; multiplication (and division) is associative and distributive, but not commutative.

Passing over the second and third chapters in this Second Book, which are chiefly complementary to the development of the theory, we find in chap. I., Book III., three lines of argument traced out in justification of the identification of the vector part of a quaternion with a vector. In fact a restriction is imposed, or a simplification is introduced, and this restriction or simplification is shown to be consistent with the results already obtained.* In much the same way as a couple or an angular

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* Compare the note to p. 175, in which Hamilton remarks: "We have thus a new point of *agreement*, or of *connexion*, between *right quaternions* and their *index-vectors*, tending to justify the ultimate assumption (not yet made), of *equality* between the former and the latter."

velocity is sometimes represented by a right line, a right quaternion and a vector of appropriate length, perpendicular to the plane of the quaternion, are now represented by the same symbol.*

The scope of the remainder of this volume is, I think, sufficiently indicated in the table of contents. The foregoing sketch of the development of the calculus of Quaternions necessarily presents but a meagre view of the nature of this work; however, my object has been to carry out, as far as I could, the intention of its illustrious author expressed in the fragment of his preface.

CHARLES JASPER JOLY.

THE OBSERVATORY, DUNSINK,
December, 1898.

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* With but slight change, much of Books I. and II. might have been extended to space of n-dimensions. In Book III. advantage is taken of the peculiar simplicity of space of those dimensions in which but one direction is perpendicular to a given plane, and a legitimate reduction of the number of symbols is consequently made.

TABLE OF CONTENTS.

BOOK I.

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* This *Chapter* may be referred to, as I. I.; the next as I. II.; the first Chapter of the Second Book, as II. I.; and similarly for the rest.

† This *Section* may be referred to, as I. I. 1; the next, as I. I. 2; the sixth Section of the second Chapter of the Third Book, as III. II. 6; and so on. [Article 180 is referred to as (180), and the third sub-article of (180) as (180 (3.)).]

[‡ This is, in words, $b - a$ is added to a and their sum is b, but *not* a is added to $b - a$ and their sum is b. See (6) and (7).]

[§ In (180 (3.)) it is shown that the addition of vector arcs is not commutative.]

This short First Chapter should be read with care by a beginner; any misconception of the meaning of the word "Vector" being fatal to progress in the Quaternions. The Chapter contains explanations also of the connected, but not all equally important, words or phrases, "revector," "provector," "transvector," "actual and null vectors," "opposite and successive vectors," "origin and term of a vector," "equal and unequal vectors," "addition and subtraction of vectors," "multiples and fractions of vectors," &c.; with the notation B − A, for the Vector (or directed right line) AB: and a deduction of the result, essential but *not peculiar*† to quaternions, that (what is here called) the *vector-sum*, of the two co-initial sides of a parallelogram, is the intermediate and co-initial *diagonal*. The term "Scalar" is also introduced, in connexion with *coefficients of vectors*.

CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

After reading these two first Sections of the second Chapter, and perhaps the three first Articles (31–33, pages 20–22) of the following Section, a student to whom the subject is new may find it convenient to pass at once, in his first perusal, to the third

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[* $m\ (\beta \pm \alpha) = m\beta \pm m\alpha$ is only true if $\alpha + \beta = \beta + \alpha$. See (180 (3.)).]

† Compare the second Note to page 206.

[‡ OA · BC denotes the point of intersection of the lines OA and BC, DE · ABC the point of intersection of the line DE with the plane ABC.]

CHAPTER III.

Applications of Vectors to Space.

It has already been recommended to the student to read the first two Articles of this Section, even in his first perusal of the Volume ; and then to pass to the Second Book.

An application of *finite differences*, to a question connected with *barycentres*, occurs in p. 84. The anharmonic generation of a ruled *hyperboloid* (or paraboloid) is employed to illustrate anharmonic equations; and (among other examples) certain *cones*, of the second and third orders, have their vector equations assigned. In the last Section, a *definition of differentials* (of *vectors* and *scalars*) is proposed, which is afterwards extended to *differentials of quaternions*, and which is independent of developments and of infinitesimals, but involves the conception of *limits*. *Vectors* of *Velocity* and *Acceleration* are mentioned; and a hint of *Hodographs* is given.

BOOK II.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

Very little, if any, of this Chapter II. I., should be omitted, even in a first perusal, since it contains the most essential conceptions and notations of the Calculus of Quaternions, at least so far as *quotients* of vectors are concerned, with numerous geometrical illustrations. Still there are a few investigations respecting circumscribed cones, imaginary intersections, and ellipsoids, in the thirteenth Section, which a student *may* pass over, and which will be indicated in the proper place in this Table.

In the five foregoing Sections it is shown, among other things, that the *plane* of a quaternion is generally an *essential element* of its constitution, so that *diplanar quaternions* are *unequal;* but that the *square* of *every right radial* (or *right versor*) is equal to *negative unity, whatever* its *plane* may be. The *Symbol* $\sqrt{-1}$ admits then of a *real interpretation*, in this as in several other systems; but *when* thus treated as *real*, it is in the present Calculus *too vague* to be useful: on which account it is found convenient to *retain the old signification* of that symbol, as denoting the (uninterpreted) *Imaginary of Algebra*, or what may here be called the *scalar imaginary*, in investigations respecting *non-real intersections*, or *non-real contacts*, in geometry.

In this last Section (II. ii. 7) the short first Article 258, and the following Art. 259, as far as the formula VIII. in p. 294, should be read, as a preparation for the Third Book, to which the Student may next proceed.

CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS.

This Chapter may be omitted, in a first perusal.

BOOK III.

CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS OR POWER OF A VECTOR, AS A QUATERNION.

The first six Sections of this Chapter ought to be read, even in a first perusal of the work.

This *first interpretation* treats the *product* $\beta . \alpha$, as equal to the *quotient* $\beta : \alpha^{-1}$; where α^{-1} (or $\mathrm{R}\alpha$) is the previously defined *Reciprocal* (II. ii. 7) of the vector α, namely a *second vector*, which has an *inverse length*, and an *opposite direction*. *Multiplication of Vectors* is thus proved to be (like that of Quaternions) a *Distributive*, but *not* generally a *Commutative Operation*. The *Square of a Vector* is shown to be always a *Negative Scalar*, namely the *negative* of the *square* of the *tensor* of that vector, or of the *number* which expresses its *length*; and some geometrical applications of this fertile principle, to *spheres*, &c., are given. The *Index* of the *Right Part* of a *Product of Two Coinitial Vectors*, OA, OB, is proved to be a right line, *perpendicular to the Plane of the Triangle* OAB, and representing by its *length* the *Double Area* of that triangle; while the *Rotation round this Index, from the Multiplier to the Multiplicand*, is *positive*. This *right part*, or *vector part*, $\mathrm{V}\alpha\beta$, of the product *vanishes*, when the *factors* are *parallel* (to one common line); and the *scalar* part, $\mathrm{S}\alpha\beta$, when they are *rectangular*.

In this *second interpretation*, which is found to agree in all its results with the first, but is better adapted to an extension of the theory, as in the following Sections, to *ternary products* of vectors, a *product of two vectors* is treated as the product of the two *right quaternions*, of which those vectors are the *indices* (II. i. 5). It is shown that, on the same plan, the *Sum of a Scalar and a Vector is a Quaternion*.

These three Sections may be passed over, in a first reading. They contain, however, theorems respecting *composition of successive rotations* (pp. 360, 361, see also p. 368); expressions for the *semi-area of a spherical polygon*, or for *half the opening of an arbitrary pyramid*, as the *angle of a quaternion product*, with an extension, by limits, to the semi-area of a spherical figure bounded by a *closed curve*, or to half the opening of an arbitrary *cone* (pp. 368, 369); a construction (pp. 390–392), for a *series of spherical parallelograms*, so called from a *partial analogy* to parallelograms in a *plane;* a theorem (p. 393), connecting a certain system of *such* (spherical) parallelograms with the *foci* of a *spherical conic*, inscribed in a certain quadrilateral; and the *conception* (pp. 384, 394) of a *Fourth Unit in Space* (u, or $+1$), which is of a *scalar* rather than a *vector* character, as admitting merely of *change of sign*, through reversal of an *order of rotation*, although it presents itself in this theory as the *Fourth Proportional* ($ij^{-1}k$) *to Three Rectangular Vector Units.*

It may be well to read this section (III. i. 10), especially for the *Exponential Connexions* which it establishes, between *Quaternions* and *Spherical Trigonometry*, or rather *Polygonometry*, by a species of *extension of Moivre's theorem*, from the *plane* to *space*, or to the *sphere*. For example, there is given (in p. 417) an *equation of six terms*, which holds good for *every spherical pentagon*, and is deduced in this way from an *extended exponential formula.* The calculations in the sub-articles to Art. 312 (pp. 409–414) may however be passed over; and perhaps Art. 315, with its sub-articles (p. 420). But Art. 314, and its sub-articles, pp. 417–419, should be read, on account of the *exponential forms* which they contain, of equations of the *circle, ellipse, logarithmic spirals* (circular and elliptic), *helix*, and *screw surface.*

It may suffice to read Art. 316, and its first eleven sub-articles, pp. 421–423. In this

Pages

Section, the adopted *Logarithm*, lq, of a quaternion q, is the *simplest root*, q', of the transcendental equation,

$$1 + q' + \frac{q'^2}{2} + \frac{q'^3}{2 \cdot 3} + \&c. = q;$$

and its expression is found to be,

$$lq = l\mathrm{T}q + \angle q \,.\, \mathrm{UV}q \qquad \text{(H)}$$

in which T and U are the signs of *tensor* and *versor*, while $\angle q$ is the *angle* of q, supposed usually to be between 0 and π. Such *logarithms* are found to be often *useful* in this Calculus, although they do not *generally* possess the elementary property, that the *sum* of the logarithms of two quaternions is equal to the logarithm of their *product*: this apparent paradox, or at least *deviation from ordinary algebraic rules*, arising necessarily from the corresponding property of *quaternion multiplication*, which has been already seen to be *not generally* a *commutative operation* ($q'q''$ *not* $= q''q'$, unless q' and q'' be *complanar*). And *here*, perhaps, a student might consider his *first perusal* of this work as *closed.**

CHAPTER II.

On Differentials and Developments of Functions of Quaternions; and on some Applications of Quaternions to Geometrical and Physical Questions.

It has been already said, that this Chapter may be omitted in a first perusal of the work.

In the view here adopted (comp. I. iii. 7), *differentials* are *not necessarily*, nor even *generally*, *small*. But it is shown at a later stage (Art. 401), that the principles of this Calculus *allow* us, whenever any advantage may be thereby gained, to treat differentials as *infinitesimals;* and so to *abridge calculation*, at least in many applications.

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* If he should choose to proceed to the *Differential Calculus of Quaternions* in the next Chapter (III. ii.), and to the *Geometrical* and other *Applications* in the third Chapter (III. iii.) of the present Book, it might be useful to read at this stage the last Section (I. iii. 7) of the First Book, which treats of *Differentials of Vectors* (pp. 96–102); and perhaps the omitted parts of the Section II. i. 13, namely Articles 213–220, with their sub-articles (pp. 220–242), which relate, among other things, to a *Construction of the Ellipsoid*, suggested by the present Calculus. But the writer will now abstain from making any further suggestions of this kind, after having indicated as above what appeared to him a *minimum* course of study, amounting to rather less than 200 pages (or parts of pages) of this Volume, which will be recapitulated for the convenience of the student at the end of the present Table.

Partial differentials and *derivatives* are introduced; and differentials of *functions of functions*.

One of the most important *rules* is, to differentiate the *factors* of a quaternion *product*, *in situ*; thus (by p. 446),

$$\mathrm{d}\,.\,qq' = \mathrm{d}q\,.\,q' + q\,.\,\mathrm{d}q'. \tag{I}$$

The formula (p. 439),

$$\mathrm{d}\,.\,q^{-1} = -\,q^{-1}\mathrm{d}q\,.\,q^{-1}, \tag{J}$$

for the differential of the *reciprocal* of a quaternion (or vector), is also very often useful; and so are the equations (p. 456),

$$\frac{\mathrm{d\,T}q}{\mathrm{T}q} = \mathrm{S}\frac{\mathrm{d}q}{q}; \qquad \frac{\mathrm{d\,U}q}{\mathrm{U}q} = \mathrm{V}\frac{\mathrm{d}q}{q}; \tag{K}$$

and (p. 454),

$$\mathrm{d}\,.\,\alpha^t = \frac{\pi}{2}\alpha^{t+1}\,\mathrm{d}t; \tag{L}$$

q being any quaternion, and α any constant vector-unit, while t is a variable scalar. It is important to *remember* (comp. III. i. 11), that we have *not* in *quaternions* the *usual* equation.

$$\mathrm{dl}q = \frac{\mathrm{d}q}{q};$$

unless q and dq be *complanar;* and therefore that we have *not generally*,

$$\mathrm{dl}\rho = \frac{\mathrm{d}\rho}{\rho},$$

if ρ be a *variable vector;* although we *have*, in this Calculus, the scarcely less simple equation, which is useful in questions respecting *orbital motion*,

$$\mathrm{dl}\frac{\rho}{\alpha} = \frac{\mathrm{d}\rho}{\rho}, \tag{M}$$

if α be a constant vector, and if the *plane* of α and ρ be *given* (or constant).

In this Section principles are established (pp. 469–473), respecting quaternion *functions* which *vanish together*; and a form of development (pp. 473–475) is assigned, *analogous** *to Taylor's Series*, and like it capable of being concisely expressed by the *symbolical equation*, $1 + \Delta = \epsilon^{d}$ (p. 480). As an example of partial and successive differentiation, the expression (pp. 480–481),

$$\rho = rk^t j^s k j^{-s} k^{-t},$$

which may represent *any vector*, is operated on; and an application is made, by means of *definite integration* (pp. 482, 483), to deduce the known area and volume of a sphere, or of portions thereof; together with the theorem, that the *vector sum* of the *directed elements* of a *spheric segment* is *zero*: each *element of surface* being represented by an *inward normal*, proportional to the elementary area, and corresponding in hydrostatics to the *pressure of a fluid* on that element.

In this Section it is shown, among other things, that a *Linear* and *Vector Symbol*, ϕ, of *Operation on a Vector*, ρ, satisfies (p. 494) a *Symbolic and Cubic Equation*, of the form,

$$0 = m - m'\phi + m''\phi^2 - \phi^3; \tag{N}$$

whence

$$m\phi^{-1} = m' - m''\phi + \phi^2 = \psi, \tag{N'}$$

= *another symbol of linear operation*, which it is shown how to deduce otherwise

* At a later stage (Art. 375), a *new Enunciation of Taylor's Theorem* is given, with a *new proof*, but still in a *form* adapted to quaternions.

from ϕ, as well as the three scalar constants, m, m', m''. The connected *algebraical cubic* (pp. 517, 518),

$$M = m + m'c + m''c^2 + c^3 = 0, \qquad \text{(O)}$$

is found to have important applications; and it is proved* (pp. 519, 520) that if $S\lambda\phi\rho = S\rho\phi\lambda$, independently of λ and ρ, in which case the function ϕ is said to be *self-conjugate*, then this last cubic has *three real roots*, c_1, c_2, c_3; while, in the same case, the *vector equation*,

$$V\rho\phi\rho = 0, \qquad \text{(P)}$$

is satisfied by a system of *Three Real and Rectangular Directions:* namely (compare pp. 527, 528, and the Section III. iii. 7), those of the *axes* of a (biconcyclic) system of *surfaces* of the *second order*, represented by the *scalar equation*,

$$S\rho\phi\rho = C\rho^2 + C', \text{ in which } C \text{ and } C' \text{ are constants.} \qquad \text{(Q)}$$

Cases are discussed; and *general forms* (called cyclic, rectangular, focal, bifocal, &c., from their chief geometrical uses) are assigned, for the vector and scalar functions $\phi\rho$ and $S\rho\phi\rho$: one useful pair of such (*cyclic*) forms being, with real and constant values of g, λ, μ,

$$\phi\rho = g\rho + V\lambda\rho\mu, \qquad S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho. \qquad \text{(R)}$$

And finally it is shown (pp. 560, 561) that if fq be a *linear and quaternion function of a quaternion*, q, then the *Symbol of Operation*, f, satisfies a certain *Symbolic and Biquadratic Equation*, analogous to the *cubic* equation in ϕ, and capable of similar applications.

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* A simplified proof, of some of the chief results for this important *case of self-conjugation*, is given at a later stage, in the few first sub-articles to Art. 415.

Table of Pages for the Figures in Volume I.

Table of a Selected Course.

This Course is recommended to those desirous of obtaining a good working knowledge of Quaternions. The passages which are not distinguished by a dagger are suitable for a first perusal.

BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES OR TO ROTATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

SECTION 1.

On the Conception of a Vector; and on Equality of Vectors.

Art. 1.—A right line AB, considered as having not only *length*, but also *direction*, is said to be a Vector. Its initial point A is said to be its *origin;* and its final point B is said to be its *term*. A vector AB is conceived to be (or to construct) the *difference* of its two extreme points; or, more fully, to be the result of the *subtraction* of its own origin from its own term; and, in conformity with this *conception*, it is also denoted by the *symbol* B − A: a notation which will be found to be extensively useful, on account of the analogies which it serves to express between geometrical and algebraical operations. When the extreme points A and B are *distinct*, the vector AB or B − A is said to be an *actual* (or an *effective*) vector; but when (as a limit) those two points are conceived to *coincide*, the vector AA or A − A, which then results, is said to be *null*. *Opposite* vectors, such as AB and BA, or B − A and A − B, are sometimes called vector and *revector*. *Successive* vectors, such as AB and BC, or B − A and C − B, are occasionally said to be vector and *provector:* the line AC, or C − A, which is drawn from the origin A of the first to the term C of the second, being then said to be the *transvector*. At a later stage, we shall have to consider *vector-arcs* and *vector-angles;* but at present, our only *vectors* are (as above) *right lines*.

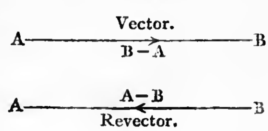

Fig. 1.

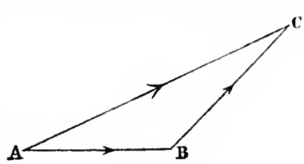

Fig. 2.

2. Two vectors are said to be EQUAL to each other, or the *equation* AB = CD, or B − A = D − C, is said to hold good, when (and only when) the origin

and term of the one can be brought to *coincide* respectively with the corresponding points of the other, by *transports* (or by *translations*) *without rotation*. It follows that *all null vectors are equal*, and may therefore be denoted by a *common symbol*, such as that used for *zero*; so that we may write,

$$A - A = B - B = \&c. = 0;$$

but that two *actual* vectors, AB and CD, are *not* (in the present *full sense*) *equal* to each other, unless they have not merely *equal lengths*, but also *similar directions*. If then they do not happen to be *parts* of one *common line*, they must be *opposite sides* of a *parallelogram*, ABDC; the two lines AD, BC becoming thus the two *diagonals* of such a figure, and consequently *bisecting* each other, in some point E.

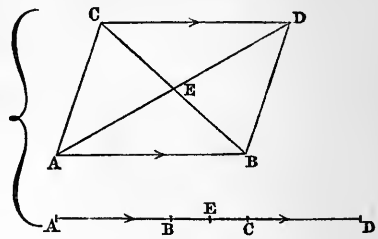

Fig. 3.

Conversely, if the two equations,

$$D - E = E - A, \quad \text{and} \quad C - E = E - B,$$

are satisfied, so that the two lines AD and BC are *commedial*, or have a *common middle point* E, then *even* if they be parts of *one* right line, the equation $D - C = B - A$ is satisfied. *Two radii*, AB, AC, of any *one circle* (or *sphere*), can *never* be *equal vectors*; because their *directions differ*.

Fig. 4.

3. An *equation* between *vectors*, considered as an *equidifference* of *points*, admits of *inversion* and *alternation*; or in symbols, if

$$D - C = B - A,$$

then $$C - D = A - B, \quad \text{and} \quad D - B = C - A.$$

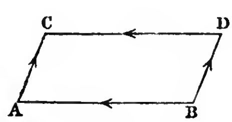

Fig. 5.

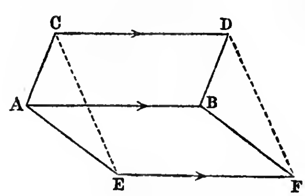

Fig. 6.

Two vectors, CD and EF, which are equal to the *same third* vector, AB, are also equal to *each other*; and these *three* equal vectors are, in general, the three parallel *edges* of a *prism*.

SECTION 2.

On Differences and Sums of Vectors taken two by two.

4. In order to be able to write, as in algebra,

$$(C' - A') - (B - A) = C - B, \quad \text{if} \quad C' - A' = C - A,$$

we next define, that when a first vector AB is *subtracted* from a second vector AC which is *co-initial* with it, or from a third vector A'C' which is *equal* to that second vector, the *remainder* is that fourth vector BC, which is drawn from the term B of the first to the term C of the second vector: so that if a vector be subtracted from a *transvector* (Art. 1), the remainder is the *provector* corresponding. It is evident that this *geometrical subtraction* of *vectors* answers to a *decomposition* of *vections* (or of *motions*); and that, by such a decomposition of a *null* vection into two *opposite* vections, we have the formula,

$$0 - (B - A) = (A - A) - (B - A) = A - B;$$

so that, if an *actual* vector AB be subtracted from a *null* vector AA, the remainder is the *revector* BA. If then we agree to *abridge*, generally, an expression of the form $0 - a$ to the shorter form, $-a$, we may write briefly, $-AB = BA$; a and $-a$ being thus symbols of *opposite* vectors, while a and $-(-a)$ are, for the same reason, symbols of one *common* vector: so that we may write, as in algebra, the *identity*,

$$-(-a) = a.$$

5. Aiming still at agreement with algebra, and adopting on that account the *formula of relation* between the *two signs*, + and −,

$$(b - a) + a = b,$$

in which we shall say as usual that $b - a$ is *added* to a, and that their *sum* is b, while relatively to it they may be jointly called *summands*, we shall have the two following consequences:—

I. If a *vector*, AB or $B - A$, be *added* to its own *origin* A, the *sum* is its *term* B (Art. 1); and

II. If a *provector* BC be added to a vector AB, the sum is the *transvector* AC; or in symbols,

$$\text{I.}\ .\ (B - A) + A = B; \quad \text{and} \quad \text{II.}\ .\ (C - B) + (B - A) = C - A.$$

In fact, the first equation is an *immediate* consequence of the general formula which, as above, *connects* the *signs* + and −, when combined with the *conception* (Art. 1) of a *vector* as a *difference* of *two points;* and the second is a result of the same formula, combined with the *definition* of the geometrical *subtraction* of *one* such vector from *another*, which was assigned in Art. 4, and according to which we have (as in algebra) for *any three points* A, B, C, the *identity*,

$$(C - A) - (B - A) = C - B.$$

It is clear that this *geometrical addition* of *successive vectors* corresponds (comp. Art. 4) to a *composition* of *successive vections*, or *motions;* and that the *sum* of two *opposite* vectors (or of vector and *revector*) is a *null line;* so that

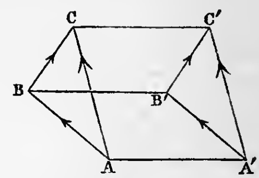

Fig. 7.

$$BA + AB = 0, \text{ or } (A - B) + (B - A) = 0.$$

It follows also that the *sums* of *equal pairs* of successive vectors are *equal;* or more fully that

$$\text{if } B' - A' = B - A, \text{ and } C' - B' = C - B, \text{ then } C' - A' = C - A;$$

the *two triangles*, ABC and A′B′C′, being in general the two *opposite faces* of a *prism* (comp. Art. 3).

6. Again, in order to have, as in algebra,

$$(C' - B') + (B - A) = C - A, \text{ if } C' - B' = C - B,$$

we shall define that if there be two successive vectors, AB, BC, and if a third vector B′C′ be *equal* to the second, but *not successive* to the first, the *sum* obtained by adding the third to the first is that fourth vector, AC, which is drawn from the origin A of the first to the term C of the second. It follows that the sum of any two *co-initial* sides, AB, AC, of any *parallelogram* ABDC, is the intermediate and co-initial *diagonal* AD; or, in symbols,

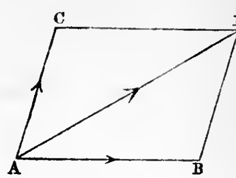

Fig. 8.

$$(C - A) + (B - A) = D - A, \text{ if } D - C = B - A;$$

because we have then (by 3) $C - A = D - B$.

7. The *sum* of any *two* given vectors has thus a *value* which is independent of their *order;* or, in symbols, $\alpha + \beta = \beta + \alpha$. If *equal* vectors be added to *equal* vectors, the *sums* are equal vectors, even if the *summands* be not given as *successive* (comp. 5); and if a *null* vector be added to an *actual* vector, the sum is that actual vector; or, in symbols, $0 + \alpha = \alpha$. If then we agree to *abridge* generally (comp. 4) the expression $0 + a$ to $+ a$, and if a still denote a

vector, then $+\alpha$, and $+(+\alpha)$, &c., are *other* symbols for the *same* vector; and we have, as in algebra, the identities,

$$-(-\alpha)=+\alpha, \quad +(-\alpha)=-(+\alpha)=-\alpha, \quad (+\alpha)+(-\alpha)=0, \ \&c.$$

SECTION 3.

On Sums of three or more Vectors.

8. The *sum* of *three* given vectors, α, β, γ, is next defined to be that fourth vector,

$$\delta=\gamma+(\beta+\alpha), \quad \text{or briefly,} \quad \delta=\gamma+\beta+\alpha,$$

which is obtained by adding the third to the sum of the first and second; and in like manner the sum of *any number* of vectors is formed by adding the *last* to the sum of all that precede it: also, for any four vectors, $\alpha, \beta, \gamma, \delta$, the sum $\delta+(\gamma+\beta+\alpha)$ is denoted simply by $\delta+\gamma+\beta+\alpha$, without parentheses, and so on for any number of summands.

9. The sum of any number of *successive* vectors, AB, BC, CD, is thus the line AD, which is drawn from the origin A of the first, to the term D of the last; and because, when there are *three* such vectors, we can draw (as in fig. 9) the two *diagonals* AC, BD of the (plane or gauche) quadrilateral ABCD, and may then at pleasure regard AD, either as the sum of AB, BD, or as the sum of AC, CD, we are allowed to establish the following general *formula of association*, for the case of *any three summand lines*, α, β, γ:

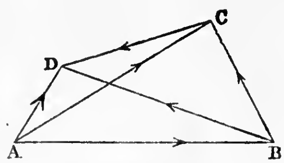

Fig. 9.

$$(\gamma+\beta)+\alpha=\gamma+(\beta+\alpha)=\gamma+\beta+\alpha;$$

by combining which with the *formula of commutation* (Art. 7), namely, with the equation,

$$\alpha+\beta=\beta+\alpha,$$

which had been previously established for the case of any *two* such summands, it is easy to conclude that the *Addition of Vectors* is always both an *Associative* and a *Commutative Operation*. In other words, the *sum* of *any number* of *given vectors* has a *value* which is independent of their *order*, and of the mode of *grouping* them; so that if the *lengths* and *directions* of the summands be *preserved*, the length and direction of the *sum* will *also* remain unchanged: except that this last *direction* may be regarded as *indeterminate*, when the *length* of the *sum-line* happens to *vanish*, as in the case which we are about to consider.

10. When any n summand-lines, AB, BC, CA, or AB, BC, CD, DA, &c., arranged in any one order, are the *n successive sides* of a *triangle* ABC, or of a *quadrilateral* ABCD, or of *any other closed polygon*, their *sum* is a *null line*, AA; and conversely, when the sum of any given system of n vectors is thus equal to *zero*, they may be made (*in any order, by transports without rotation*) the n successive *sides* of a *closed* polygon (plane or gauche). Hence, if there be *given* any such polygon (P), suppose a *pentagon* ABCDE, it is possible to construct another closed polygon (P′), such as A′B′C′D′E′, with an *arbitrary initial point* A′, but with the *same number* of sides, A′B′, . . E′A′, which *new sides* shall be *equal* (as vectors) to the *old* sides AB, . . EA, taken in any *arbitrary order*. For example, if we draw *four* successive vectors, as follows,

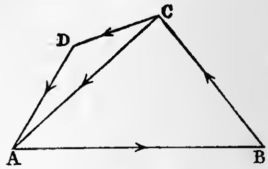

Fig. 10.

$$\text{A}'\text{B}' = \text{CD}, \quad \text{B}'\text{C}' = \text{AB}, \quad \text{C}'\text{D}' = \text{EA}, \quad \text{D}'\text{E}' = \text{BC},$$

and then *complete* the *new pentagon* by drawing the line E′A′, this *closing side* of the *second* figure (P′) will be *equal* to the *remaining side* DE of the *first* figure (P).

11. Since a *closed* figure ABC . . is *still* a closed one, when all its points are *projected* on any assumed *plane*, by any system of parallel ordinates (although the *area* of the projected figure A′B′C′ . . . may happen to *vanish*), it follows that if the *sum* of any number of given vectors $\alpha, \beta, \gamma, ..$ be *zero*, and if we project them all *on any one plane* by *parallel lines* drawn from their extremities, the *sum* of the *projected vectors* $\alpha', \beta', \gamma', ..$ will *likewise* be *null*; so that these *latter* vectors, like the *former*, *can* be so *placed* as to *become* the *successive sides* of a *closed polygon*, even if they be not *already* such. (In fig. 11, A″B″C″ is considered as *such* a polygon, namely, as a *triangle with evanescent area*; and we have the equation,

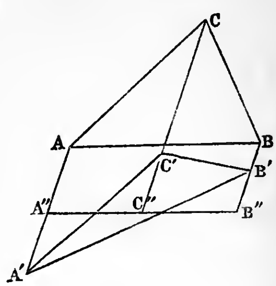

Fig. 11.

$$\text{A}''\text{B}'' + \text{B}''\text{C}'' + \text{C}''\text{A}'' = 0,$$

as well as $\quad \text{A}'\text{B}' + \text{B}'\text{C}' + \text{C}'\text{A}' = 0,$ and $\text{AB} + \text{BC} + \text{CA} = 0.$)

SECTION 4.

On Coefficients of Vectors.

12. The *simple* or *single* vector, α, is also denoted by 1α, or by $1 . \alpha$, or by $(+ 1)\,\alpha$; and in like manner, the *double* vector, $\alpha + \alpha$, is denoted by 2α, or $2 . \alpha$, or $(+ 2)\,\alpha$, &c.; the *rule* being, that for any algebraical integer, m, regarded as a *coefficient* by which the vector α is *multiplied*, we have always,

$$1\alpha + m\alpha = (1 + m)\,\alpha;$$

the symbol $1 + m$ being here interpreted as in algebra. Thus, $0\alpha = 0$, the zero on the one side denoting a *null coefficient*, and the zero on the other side denoting a *null vector*; because by the rule,

$$1\alpha + 0\alpha = (1 + 0)\,\alpha = 1\alpha = \alpha, \text{ and } \therefore\ 0\alpha = \alpha - \alpha = 0.$$

Again, because $(1)\,\alpha + (-1)\,\alpha = (1 - 1)\,\alpha = 0\alpha = 0$, we have $(-1)\,\alpha = 0 - \alpha = -\alpha = -(1\alpha)$; in like manner, since $(1)\,\alpha + (-2)\,\alpha = (1 - 2)\,\alpha = (-1)\,\alpha = -\alpha$, we infer that $(-2)\,\alpha = -\alpha - \alpha = -(2\alpha)$; and generally $(-m)\,\alpha = -(m\alpha)$, whatever whole number m may be: so that we may, without danger of confusion, *omit the parentheses* in these last symbols, and write simply, -1α, -2α, $-m\alpha$.

13. It follows that *whatever two whole numbers* (positive or negative, or

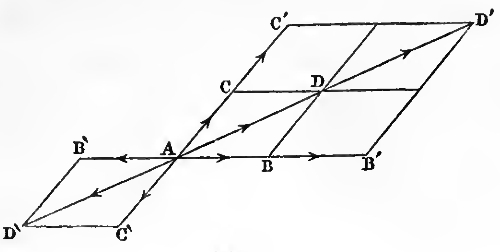

Fig. 12.

null) may be represented by m and n, and *whatever two vectors* may be denoted by α and β, we have always, as in algebra, the formulæ,

$$n\alpha \pm m\alpha = (n \pm m)\,\alpha, \qquad n\,(m\alpha) = (nm)\,\alpha = nm\alpha,$$

and (compare fig. 12),

$$m\,(\beta \pm \alpha) = m\beta \pm m\alpha;$$

so that the *multiplication of vectors by coefficients* is a *doubly distributive operation*, at least if the *multipliers* be *whole numbers*; a restriction which, however, will soon be removed.

14. If $m\alpha = \beta$, the coefficient m being still *whole*, the vector β is said to be a multiple of α; and conversely (at least if the integer m be different from *zero*), the vector α is said to be a *sub-multiple* of β. A *multiple of a sub-multiple* of a vector is said to be a *fraction* of that vector; thus, if $\beta = m\alpha$, and $\gamma = n\alpha$, then γ is a fraction of β, which is denoted as follows, $\gamma = \frac{n}{m}\beta$; also β is said to be *multiplied* by the fractional coefficient $\frac{n}{m}$, and γ is said to be the *product* of this multiplication. It follows that if x and y be *any two fractions*, (positive or negative or null, whole numbers being included), and if α and β be *any two vectors*, then

$$y\alpha \pm x\alpha = (y \pm x)\,\alpha, \quad y\,(x\alpha) = (yx)\,\alpha = yx\alpha, \quad x\,(\beta \pm \alpha) = x\beta \pm x\alpha;$$

results which include those of Art. 13, and may be extended to the case where x and y are *incommensurable coefficients*, considered as *limits* of *fractional* ones.

15. For any actual vector α, and for any coefficient x, of any of the foregoing kinds, the *product* $x\alpha$, interpreted as above, represents always a *vector* β, which has the *same direction* as the *multiplicand-line* α, if $x > 0$, but has the *opposite* direction if $x < 0$, becoming *null* if $x = 0$. Conversely, if α and β be *any two actual vectors*, with directions *either* similar *or* opposite, in *each* of which two cases we shall say that they are *parallel vectors*, and shall write $\beta \parallel \alpha$ (because *both* are then *parallel*, in the *usual* sense of the word, to *one common line*), we can always find, or conceive as found, a *coefficient* $x \gtrless 0$, which shall satisfy the equation $\beta = x\alpha$; or, as we shall also write it, $\beta = \alpha x$; and the positive or negative *number* x, so found, will bear to ± 1 the same *ratio*, as that which the *length* of the line β bears to the length of α.

16. Hence it is natural to say that this *coefficient* x is the *quotient* which results, from the *division of the vector* β, *by the parallel vector* α; and to write, accordingly,

$$x = \beta \div \alpha, \quad \text{or } x = \beta : \alpha, \quad \text{or } x = \frac{\beta}{\alpha};$$

so that we shall have, identically, as in algebra, at least if the *divisor-line* α be an *actual* vector, and if the *dividend-line* β be *parallel* thereto, the equations,

$$(\beta : \alpha) \,.\, \alpha = \frac{\beta}{\alpha}\alpha = \beta, \quad \text{and} \quad x\alpha : \alpha = \frac{x\alpha}{\alpha} = x;$$

which will afterwards be *extended*, by *definition*, to the case of *non-parallel* vectors. We may write also, under the same conditions, $a = \frac{\beta}{x}$, and may say that the *vector* a is the quotient of the *division* of the other vector β by the *number* x; so that we shall have these other identities,

$$\frac{\beta}{x} \cdot x = (ax =)\, \beta, \quad \text{and} \quad \frac{ax}{x} = a.$$

17. The positive or negative *quotient*, $x = \frac{\beta}{a}$, which is thus obtained by the *division* of one of two *parallel vectors* by another, including *zero* as a *limit*, may also be called a SCALAR; because it can always be found, and in a certain sense *constructed*, by the *comparison of positions upon one common scale* (or *axis*); or can be put under the form,

$$x = \frac{\text{C} - \text{A}}{\text{B} - \text{A}} = \frac{\text{AC}}{\text{AB}},$$

where the *three points*, A, B, C, are *collinear* (as in the figure annexed). Such *scalars* are, therefore, simply the REALS (or *real quantities*) of *Algebra*; but, in combination with the *not less real* VECTORS above considered, they form *one* of the *main elements* of the *System*, or *Calculus*, to which the present work relates. In fact it will be shown, at a later stage, that there is an important sense in which we can conceive a scalar to be *added* to a vector; and that the *sum* so obtained, or the combination "*Scalar plus Vector*" is a QUATERNION.

Fig. 13.

CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

SECTION 1.

On Linear Equations connecting two Co-initial Vectors.

18. When several vectors, OA, OB, . . are all drawn from one common point O, that point is said to be the *Origin of the System;* and each particular vector, such as OA, is said to be *the vector of its own term*, A. In the present and future sections we shall always suppose, if the contrary be not expressed, that all the vectors α, β, . . which we may have occasion to consider, are thus drawn from one *common origin.* But if it be desired to *change* that origin O, without changing the *term-points* A, . . we shall only have to *subtract*, from *each* of their *old* vectors α, . . one *common vector* ω, namely, the *old vector* OO′ of the *new origin* O′; since the *remainders*, $\alpha - \omega$, $\beta - \omega$, . . will be the *new vectors* α', β', . . of the *old points* A, B, . . . For example, we shall have

$$\alpha' = \mathrm{O'A} = \mathrm{A} - \mathrm{O'} = (\mathrm{A} - \mathrm{O}) - (\mathrm{O'} - \mathrm{O}) = \mathrm{OA} - \mathrm{OO'} = \alpha - \omega.$$

19. If *two* vectors α, β, or OA, OB, be thus drawn from a given origin O, and if their *directions* be either similar *or* opposite, so that the *three points*, O, A, B, are situated *on one right line* (as in the figure annexed), then (by 16, 17) their *quotient* $\frac{\beta}{\alpha}$ is some positive or negative *scalar*, such as x; and conversely, the equation $\beta = x\alpha$, interpreted with this reference to an *origin*, expresses the *condition of collinearity*, of the points O, A, B; the particular *values* $x = 0$, $x = 1$, corresponding to the particular *positions*, O and A, of the *variable point* B, whereof the *indefinite right line* OA is the *locus*.

Fig. 14.

20. The *linear equation*, connecting the *two* vectors α and β, acquires a more symmetric *form*, when we write it thus:

$$a\alpha + b\beta = 0;$$

where a and b are *two* scalars, of which however only the *ratio* is important.

The *condition of coincidence*, of the two points A and B, answering above to $x = 1$, is now $\frac{-a}{b} = 1$; or, more symmetrically,

$$a + b = 0.$$

Accordingly, when $a = -b$, the linear equation becomes

$$b(\beta - \alpha) = 0, \quad \text{or } \beta - \alpha = 0,$$

since we do not suppose that *both* the coefficients vanish; and the equation $\beta = \alpha$, or OB = OA, requires that the *point* B should *coincide* with the point A: a case which may also be conveniently expressed by the formula,

$$\text{B} = \text{A};$$

coincident points being thus treated (in *notation* at least) as *equal*. In general, the linear equation gives,

$$a\,.\,\text{OA} + b\,.\,\text{OB} = 0, \quad \text{and therefore} \quad a : b = \text{BO} : \text{OA}.$$

SECTION 2.

On Linear Equations between three Co-initial Vectors.

21. If two (actual and co-initial) vectors, α, β, be *not* connected by *any* equation of the form $a\alpha + b\beta = 0$, with *any two* scalar coefficients a and b whatever, their *directions* can *neither* be similar *nor* opposite to each other; they therefore *determine a plane* AOB, in which the (now actual) vector, represented by the sum $a\alpha + b\beta$, is situated. For if, for the sake of symmetry, we denote this sum by the symbol $-c\gamma$, where c is some *third scalar*, and $\gamma = \text{OC}$ is some *third vector*, so that the *three* co-initial vectors, α, β, γ, are connected by the *linear equation*,

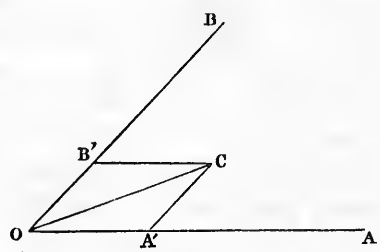

Fig. 15.

$$a\alpha + b\beta + c\gamma = 0;$$

and if we make

$$\text{OA}' = \frac{-a\alpha}{c} \qquad \text{OB}' = \frac{-b\beta}{c};$$

then the two auxiliary points, A′ and B′, will be situated (by 19) on the two indefinite right lines, OA, OB, respectively: and we shall have the equation,

$$\text{OC} = \text{OA}' + \text{OB}',$$

so that the figure A′OB′C is (by 6) a parallelogram, and consequently plane.

22. Conversely, if C be *any point in the plane* AOB, we can draw from it the *ordinates*, CA′ and CB′, to the lines OA and OB, and can determine the ratios of the three scalars, a, b, c, so as to satisfy the two equations,

$$\frac{a}{c} = -\frac{\text{OA}'}{\text{OA}}, \qquad \frac{b}{c} = -\frac{\text{OB}'}{\text{OB}};$$

after which we shall have the recent expressions for OA′, OB′, with the relation $\text{OC} = \text{OA}' + \text{OB}'$ as before; and shall thus be brought back to the linear equation $a\alpha + b\beta + c\gamma = 0$, which equation may therefore be said to express the *condition of complanarity* of the *four points*, O, A, B, C. And if we write it under the form,

$$x\alpha + y\beta + z\gamma = 0,$$

and consider the vectors α and β as *given*, but γ as a *variable vector*, while x, y, z *are variable scalars*, the *locus* of the *variable point* C will then be the *given plane*, OAB.

23. It may happen that the point C is situated *on the right line* AB, which is here considered as a *given* one. In that case (comp. Art. 17, fig. 13), the quotient $\frac{\text{AC}}{\text{AB}}$ must be equal to some scalar, suppose t; so that we shall have an equation of the form,

$$\frac{\gamma - \alpha}{\beta - \alpha} = t, \quad \text{or } \gamma = \alpha + t(\beta - \alpha), \quad \text{or } (1 - t)\,\alpha + t\beta - \gamma = 0;$$

by comparing which last form with the linear equation of Art. 21, we see that the *condition of collinearity* of the three points A, B, C, in the given plane OAB, is expressed by the formula,

$$a + b + c = 0.$$

This condition may also be thus written,

$$1 = \frac{-a}{c} + \frac{-b}{c}, \quad \text{or } \frac{\text{OA}'}{\text{OA}} + \frac{\text{OB}'}{\text{OB}} = 1;$$

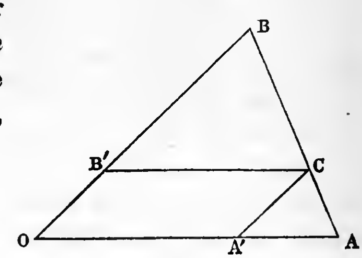

Fig. 16.

and under this last form it expresses a *geometrical relation*, which is otherwise known to exist.

24. When we have thus the *two* equations,

$$a\alpha + b\beta + c\gamma = 0, \quad \text{and} \quad a + b + c = 0,$$

so that the three co-initial vectors α, β, γ terminate on one right line, and may on that account be said to be *termino-collinear*, if we eliminate, successively and separately, each of the three scalars a, b, c, we are conducted to these three other equations, expressing certain *ratios of segments :*

$$b(\beta - \alpha) + c(\gamma - \alpha) = 0, \quad c(\gamma - \beta) + a(\alpha - \beta) = 0,$$
$$a(\alpha - \gamma) + b(\beta - \gamma) = 0;$$

or

$$0 = b \,.\, \text{AB} + c \,.\, \text{AC} = c \,.\, \text{BC} + a \,.\, \text{BA} = a \,.\, \text{CA} + b \,.\, \text{CB}.$$

Hence follows this *proportion*, between *coefficients* and *segments*,

$$a : b : c = \text{BC} : \text{CA} : \text{AB}.$$

We might also have observed that the proposed equations give,

$$\alpha = \frac{b\beta + c\gamma}{b + c}, \qquad \beta = \frac{c\gamma + a\alpha}{c + a}, \qquad \gamma = \frac{a\alpha + b\beta}{a + b};$$

whence

$$\frac{\text{AC}}{\text{AB}} = \frac{\gamma - \alpha}{\beta - \alpha} = \frac{b}{a + b} = -\frac{b}{c}, \text{ \&c.}$$

25. If we still treat α and β as *given*, but regard γ and $\frac{y}{x}$ as *variable*, the equation

$$\gamma = \frac{x\alpha + y\beta}{x + y}$$

will express that the *variable point* C is situated *somewhere* on the *indefinite right line* AB, or that it has this *line* for its *locus :* while it *divides* the *finite* line AB into *segments*, of which the *variable quotient* is,

$$\frac{\text{AC}}{\text{CB}} = \frac{y}{x}.$$

Let C′ be *another point* on the *same line*, and let its vector be,

$$\gamma' = \frac{x'\alpha + y'\beta}{x' + y'};$$

then, in like manner, we shall have this *other ratio of segments*,

$$\frac{\text{AC}'}{\text{C}'\text{B}} = \frac{y'}{x'}.$$

If, then, we agree to employ, generally, *for any group of four collinear points*, the *notation*,

$$(\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}} = \frac{\text{AB}}{\text{BC}} : \frac{\text{AD}}{\text{DC}};$$

so that this *symbol*,

$$(\text{ABCD}),$$

may be said to denote the *anharmonic function*, or *anharmonic quotient*, or simply the *anharmonic of the group*, A, B, C, D : we shall have, in the present case, the equation,

$$(\mathrm{ACBC}') = \frac{\mathrm{AC}}{\mathrm{CB}} : \frac{\mathrm{AC}'}{\mathrm{C'B}} = \frac{yx'}{xy'}.$$

26. When the *anharmonic quotient* becomes equal to *negative unity* the *group* becomes (as is well known) *harmonic*. If then we have the two equations,

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \qquad \gamma' = \frac{x\alpha - y\beta}{x - y},$$

the two points C and C′ are *harmonically conjugate* to each other, with respect to the *two given points*, A and B; and when they *vary together*, in consequence of the variation of the value of $\frac{y}{x}$, they form (in a well-known sense), on the indefinite right line AB, *divisions in involution*; the *double points* (or *foci*) of this involution, namely, the points of which each is *its own* conjugate, being the points A and B themselves. As a verification, if we denote by μ the vector of the middle point M of the given interval AB, so that $\beta - \mu = \mu - \alpha$, or $\mu = \frac{1}{2}(\alpha + \beta)$, we easily find that

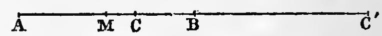

Fig. 17.

$$\frac{\gamma - \mu}{\beta - \mu} = \frac{y - x}{y + x} = \frac{\beta - \mu}{\gamma' - \mu}, \qquad \text{or } \frac{\mathrm{MC}}{\mathrm{MB}} = \frac{\mathrm{MB}}{\mathrm{MC}'};$$

so that the *rectangle* under the *distances* MC, MC′, of the two *variable* but *conjugate points*, C, C′, from the *centre* M of the involution, is equal to the *constant square* of *half* the *interval* between the two *double points*, A, B. More generally, if we write

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \qquad \gamma' = \frac{lx\alpha + my\beta}{lx + my},$$

where the anharmonic quotient $\frac{l}{m} = \frac{yx'}{xy'}$ is *any constant scalar*, then in another known and modern* phraseology, the points C and C′ will form, on the indefinite line AB, *two homographic divisions*, of which A and B are still the *double points*. More generally still, if we establish the two equations

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \text{and} \quad \gamma' = \frac{lx\alpha' + my\beta'}{lx + my},$$

$\frac{l}{m}$ being still constant, but $\frac{y}{x}$ variable, while $\alpha' = \mathrm{OA}'$, $\beta' = \mathrm{OB}'$, and $\gamma' = \mathrm{OC}'$, the *two given lines*, AB and A′B′, are then *homographically divided*, by the *two variable points* C and C′, not now supposed to move along one *common* line.

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* See the *Géométrie Supérieure* of M. Chasles, p. 107. (Paris, 1852.)

27. When the linear equation $a\alpha + b\beta + c\gamma = 0$ subsists, *without* the relation $a + b + c = 0$ between its coefficients, then the three co-initial vectors α, β, γ are still *complanar*, but they no longer terminate *on one right line;* their *term-points*, A, B, C being now the corners of a *triangle.*

In this more general case, we may propose to find the vectors α', β', γ' of the three points,

$$A' = OA \cdot BC, \quad B' = OB \cdot CA, \quad C' = OC \cdot AB;$$

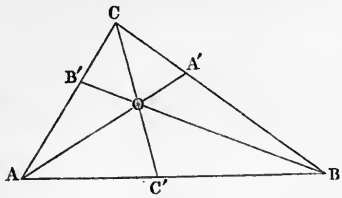

Fig. 18.

that is to say, of the points in which the lines drawn from the origin O to the three *corners* of the triangle *intersect* the three respectively opposite *sides*. The three *collineations* OAA′, &c., give (by 19) three expressions of the forms,

$$\alpha' = x\alpha, \qquad \beta' = y\beta, \qquad \gamma' = z\gamma,$$

where x, y, z are three scalars, which it is required to determine by means of the three *other* collineations, A′BC, &c., with the help of relations derived from the principle of Art 23. Substituting therefore for α its value $x^{-1}\alpha'$, in the *given* linear equation, and equating to zero the sum of the coefficients of the *new* linear equation which results, namely,

$$x^{-1}a\alpha' + b\beta + c\gamma = 0;$$

and eliminating similarly β, γ, each in its turn, from the original equation; we find the values,

$$x = \frac{-a}{b+c}, \qquad y = \frac{-b}{c+a}, \qquad z = \frac{-c}{a+b};$$

whence the sought vectors are expressed in either of the two following ways:

$$\text{I.} \ldots \alpha' = \frac{-a\alpha}{b+c}, \qquad \beta' = \frac{-b\beta}{c+a}, \qquad \gamma' = \frac{-c\gamma}{a+b};$$

or

$$\text{II.} \ldots \alpha' = \frac{b\beta + c\gamma}{b+c}, \qquad \beta' = \frac{c\gamma + a\alpha}{c+a}, \qquad \gamma' = \frac{a\alpha + b\beta}{a+b}.$$

In fact we see, by *one* of these expressions for α', that A′ is on the line OA; and by the *other* expression for the same vector α', that the same point A′ is on the line BC. As another verification, we may observe that the last expressions for α', β', γ', coincide with those which were found in Art. 24, for α, β, γ themselves, on the particular supposition that the three points A, B, C were collinear.

28. We may next propose to determine the *ratios* of the *segments* of the *sides* of the triangle ABC, made by the points A′, B′, C′. For this purpose, we may write the last equations for α', β', γ' under the form,

$$0 = b(\alpha' - \beta) - c(\gamma - \alpha') = c(\beta' - \gamma) - a(\alpha - \beta') = a(\gamma' - \alpha) - b(\beta - \gamma');$$

and we see that they then give the required ratios, as follows:

$$\frac{\text{BA}'}{\text{A}'\text{C}} = \frac{c}{b}, \quad \frac{\text{CB}'}{\text{B}'\text{A}} = \frac{a}{c}, \quad \frac{\text{AC}'}{\text{C}'\text{B}} = \frac{b}{a};$$

whence we obtain at once the known *equation of six segments*,

$$\frac{\text{BA}'}{\text{A}'\text{C}} \cdot \frac{\text{CB}'}{\text{B}'\text{A}} \cdot \frac{\text{AC}'}{\text{C}'\text{B}} = 1,$$

as the *condition of concurrence* of the three right lines AA′, BB′, CC′, in a *common point*, such as O. It is easy also to infer, from the same ratios of segments, the following *proportion* of *coefficients* and *areas*,

$$a : b : c = \text{OBC} : \text{OCA} : \text{OAB},$$

in which we must, in general, attend to algebraic *signs*; a *triangle* being conceived to *pass* (*through zero*) from *positive* to *negative*, or *vice versâ*, as *compared* with any *given* triangle in its own plane, when (in the course of any continuous change) its *vertex crosses* its *base*. It may be observed that *with this convention* (which is, in fact, a *necessary* one, for the establishment of *general formulæ*) we have, for *any three points*, the equation

$$\text{ABC} + \text{BAC} = 0,$$

exactly as we had (in Art. 5) for any *two* points, the equation

$$\text{AB} + \text{BA} = 0.$$

More fully, we have, on this plan, the formulæ,

$$\text{ABC} = -\text{BAC} = \text{BCA} = -\text{CBA} = \text{CAB} = -\text{ACB};$$

and *any two complanar triangles*, ABC, A′B′C′, bear *to each other* a *positive* or a *negative ratio*, according as the two *rotations*, which may be conceived to be denoted by the same *symbols* ABC, A′B′C′, are *similarly* or *oppositely directed*.

29. If A′ and B′ *bisect* respectively the sides BC and CA, then

$$a = b = c,$$

and C′ bisects AB; whence the known theorem follows, that *the three bisectors of the sides of a triangle concur*, in a point which is often called the *centre of gravity*, but which we prefer to call the *mean point* of the triangle, and which is here the *origin* O. At the same time, the first expressions in Art. 27 for α', β', γ' become,

$$\alpha' = -\frac{\alpha}{2}, \quad \beta' = -\frac{\beta}{2}, \quad \gamma' = -\frac{\gamma}{2};$$

whence this other known theorem results, that *the three bisectors trisect each other*.

30. The linear equation between α, β, γ reduces itself, in the case last considered, to the form,

$$\alpha + \beta + \gamma = 0, \quad \text{or OA + OB + OC} = 0;$$

the three vectors α, β, γ, or OA, OB, OC, are therefore, in this case, adapted (by Art. 10) to become the *successive sides* of a *triangle*, by *transports without rotation;* and accordingly, if we complete (as in fig. 19) the parallelogram AOBD, the triangle OAD will have the property in question. It follows (by 11) that if we *project* the *four points* O, A, B, C, by any system of *parallel ordinates*, into four *other* points, $\text{O}_,, \text{A}_,, \text{B}_,, \text{C}_,$ on any assumed *plane*, the *sum* of the three *projected vectors*, $\alpha_,, \beta_,, \gamma_,$ or $\text{O}_,\text{A}_,$, &c., will be *null;* so that we shall have the *new linear equation*,

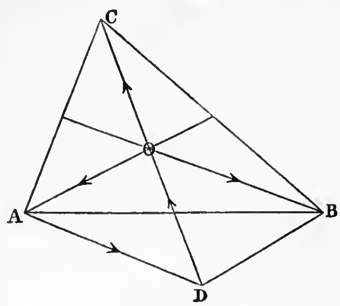

Fig. 19.

$$\alpha_, + \beta_, + \gamma_, = 0,$$

or,

$$\text{O}_,\text{A}_, + \text{O}_,\text{B}_, + \text{O}_,\text{C}_, = 0;$$

and in fact it is evident (see fig. 20) that the *projected mean point* $\text{O}_,$ will be the *mean point of the projected triangle*, $\text{A}_,, \text{B}_,, \text{C}_,$. We shall have also the equation,

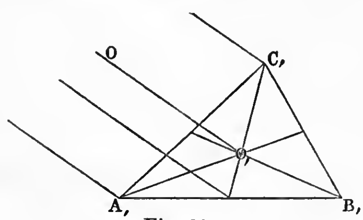

Fig. 20.

$$(\alpha_, - \alpha) + (\beta_, - \beta) + (\gamma_, - \gamma) = 0;$$

where $\qquad \alpha_, - \alpha = \text{O}_,\text{A}_, - \text{OA} = (\text{O}_,\text{A} + \text{AA}_,) - (\text{OO}_, + \text{O}_,\text{A}) = \text{AA}_, - \text{OO}_,;$

hence $\qquad \text{OO}_, = \frac{1}{3}(\text{AA}_, + \text{BB}_, + \text{CC}_,),$

or the *ordinate of the mean point of a triangle* is the *mean* of the *ordinates of the three corners*.

SECTION 3.

On Plane Geometrical Nets.

31. Resuming the more general case of Art. 27, in which the coefficients a, b, c are supposed to be *unequal*, we may next inquire, in what points A″, B″, C″ do the lines B′C′, C′A′, A′B′ meet respectively the sides BC, CA, AB, of the triangle; or may seek to assign the vectors α'', β'', γ'' of the points of intersection (comp. 27),

$$\text{A}'' = \text{B}'\text{C}' \cdot \text{BC}, \qquad \text{B}'' = \text{C}'\text{A}' \cdot \text{CA}, \qquad \text{C}'' = \text{A}'\text{B}' \cdot \text{AB}.$$

The first expressions in Art. 27 for β', γ', give the equations,

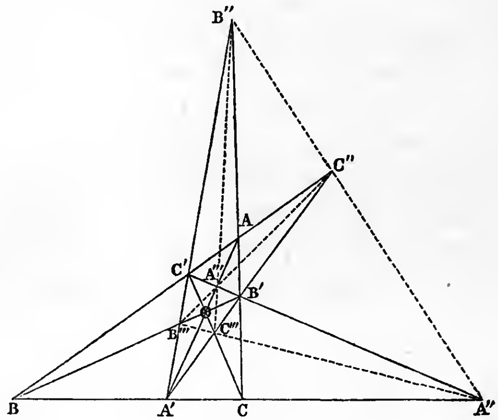

Fig. 21.

$$(c + a)\,\beta' + b\beta = 0, \qquad (a + b)\,\gamma' + c\gamma = 0\,;$$

whence

$$\frac{b\beta - c\gamma}{b - c} = \frac{(a + b)\,\gamma' - (c + a)\,\beta'}{(a + b) - (c + a)}\,;$$

but (by 25) one member is the vector of a point on BC, and the other of a point on B′C′; each therefore is a value for the vector α'' of A″, and similarly for β'' and γ''. We may therefore write,

$$\alpha'' = \frac{b\beta - c\gamma}{b - c}, \qquad \beta'' = \frac{c\gamma - a\alpha}{c - a}, \qquad \gamma'' = \frac{a\alpha - b\beta}{a - b}\,;$$

and by comparing these expressions with the second set of values of α', β', γ' in Art. 27, we see (by 26) that the points A″, B″, C″ are, respectively, the *harmonic conjugates* (as they are indeed known to be) of the points A′, B′, C′, with respect to the three pairs of points, B, C; C, A; A, B; so that, in the notation of Art. 25, we have the equations,

$$(\text{BA}'\text{CA}'') = (\text{CB}'\text{AB}'') = (\text{AC}'\text{BC}'') = -1.$$

And because the expressions for α'', β'', γ'' conduct to the following linear equation between those three vectors,

$$(b-c)\,\alpha'' + (c-a)\,\beta'' + (a-b)\,\gamma'' = 0,$$

with the relation

$$(b-c) + (c-a) + (a-b) = 0$$

between its coefficients, we arrive (by 23) at this other known theorem, that *the three points* A″, B″, C″ *are collinear*, as indicated by one of the *dotted lines* in the recent fig. 21.

32. The line A″B′C′ may represent *any rectilinear transversal*, cutting the sides of a triangle ABC; and because we have

$$\frac{\text{BA}''}{\text{A}''\text{C}} = \frac{\alpha'' - \beta}{\gamma - \alpha''} = -\frac{c}{b},$$

while $\frac{\text{CB}'}{\text{B}'\text{A}} = \frac{a}{c}$, and $\frac{\text{AC}'}{\text{C}'\text{B}} = \frac{b}{a}$, as before, we arrive at this *other equation of six segments*, for any triangle cut by a right line (comp. 28),

$$\frac{\text{BA}''}{\text{A}''\text{C}} \cdot \frac{\text{CB}'}{\text{B}'\text{A}} \cdot \frac{\text{AC}'}{\text{C}'\text{B}} = -1;$$

which again agrees with known results.

33. Eliminating β and γ between either set of expressions (27) for β' and γ', with the help of the given linear equation, we arrive at this other equation, connecting the three vectors α, β', γ':

$$0 = -a\alpha + (c+a)\,\beta' + (a+b)\,\gamma'.$$

Treating this on the same plan as the given equation between α, β, γ, we find that if (as in fig. 21) we make,

$$\text{A}''' = \text{OA} \cdot \text{B}'\text{C}', \qquad \text{B}''' = \text{OB} \cdot \text{C}'\text{A}', \qquad \text{C}''' = \text{OC} \cdot \text{A}'\text{B}',$$

the vectors of these three new points of intersection may be expressed in either of the two following ways, whereof the first is shorter, but the second is, for some purposes (comp. 34, 36), more convenient:

$$\text{I.} \ldots \alpha''' = \frac{a\alpha}{2a+b+c}, \qquad \beta''' = \frac{b\beta}{2b+c+a}, \qquad \gamma''' = \frac{c\gamma}{2c+a+b};$$

or

$$\text{II.} \ldots \alpha''' = \frac{2a\alpha + b\beta + c\gamma}{2a+b+c}, \qquad \beta''' = \frac{2b\beta + c\gamma + a\alpha}{2b+c+a}, \qquad \gamma''' = \frac{2c\gamma + a\alpha + b\beta}{2c+a+b}.$$

And the three equations, of which the following is one,

$$(b - c)\,\alpha'' - (2b + c + a)\,\beta''' + (2c + a + b)\,\gamma''' = 0,$$

with the relations between their coefficients which are evident on inspection, show (by 23) that we have the three additional *collineations*, A″B‴C‴, B″C‴A‴, C″A‴B‴, as indicated by three of the dotted lines in the figure. Also, because we have the two expressions,

$$\alpha''' = \frac{(a+b)\,\gamma' + (c+a)\,\beta'}{(a+b)+(c+a)}, \qquad \alpha'' = \frac{(a+b)\gamma' - (c+a)\beta'}{(a+b)-(c+a)},$$

we see (by 26) that the two points A″, A‴ are *harmonically conjugate* with respect to B′ and C′; and similarly for the two other pairs of points, B″, B‴, and C″, C‴, compared with C′, A′, and with A′, B′: so that, in a notation already employed (25, 31), we may write,

$$(\mathrm{B'A'''C'A''}) = (\mathrm{C'B'''A'B''}) = (\mathrm{A'C'''B'C''}) = -1.$$

34. If we *begin*, as above, with any *four complanar points*, O, A, B, C, of which no three are collinear, we can (as in fig. 18), by what may be called a *First Construction*, derive from them six lines, connecting them two by two, and intersecting each other in three new points, A′, B′, C′; and then by a *Second Construction* (represented in fig. 21), we may connect these by three new lines, which will give, by their intersections with the former lines, six new points, A″, . . C‴. We might proceed to connect these with each other, and with the given points, by sixteen new lines, or lines of a *Third Construction*, namely, the four dotted lines of fig. 21, and twelve other lines, whereof three should be drawn from each of the four given points: and these would be found to determine eighty-four new points of intersection, of which some may be seen, although they are not marked, in the figure.

But *however far* these processes of *linear construction* may be continued, so as to form what has been called* a *plane geometrical net*, the *vectors* of the points thus determined have all one *common property*: namely, that each can be represented by an expression of the form,

$$\rho = \frac{xa\alpha + yb\beta + zc\gamma}{xa + yb + zc};$$

where the *coefficients* x, y, z are some *whole numbers*. In fact we see (by 27, 31, 33) that such expressions can be assigned for the *nine* derived vectors,

* By Prof. A. F. Möbius, in page 274 of his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827).

$\alpha', \ldots \gamma'''$, which alone have been hitherto considered; and it is not difficult to perceive, from the nature of the calculations employed, that a similar result must hold good, for every vector subsequently deduced. But this and other connected results will become more completely evident, and their *geometrical signification* will be better understood, after a somewhat closer consideration of *anharmonic quotients*, and the introduction of a certain system of *anharmonic co-ordinates*, for points and lines in one plane, to which we shall next proceed: reserving, for a subsequent Chapter, any applications of the same theory to *space*.

SECTION 4.

On Anharmonic Co-ordinates and Equations of Points and Lines in one Plane.

35. If we compare the last equations of Art. 33 with the corresponding equations of Art. 31, we see that the *harmonic group* $BA'CA''$, on the side BC of the triangle ABC in fig. 21, has been simply *reflected* into another such group, $B'A'''C'A''$, on the line $B'C'$, by a *harmonic pencil* of four rays, all passing through the point O; and similarly for the other groups. More generally, let OA, OB, OC, OD, or briefly $O \,.\, ABCD$, be *any* pencil, with the point O for *vertex*; and let the *new ray* OD be cut, as in fig. 22, by the three sides of the triangle ABC, in the three points A_1, B_1, C_1; let also

$$OA_1 = \alpha_1 = \frac{yb\beta + zc\gamma}{yb + zc},$$

so that (by 25) we shall have the anharmonic quotients,

$$(BA'CA_1) = \frac{y}{z}, \qquad (CA'BA_1) = \frac{z}{y};$$

and let us seek to express the two other vectors of intersection, β_1 and γ_1, with a view to determining the anharmonic ratios of the groups on the two other sides. The given equation (27),

$$a\alpha + b\beta + c\gamma = 0,$$

shows us at once that these two vectors are,

$$OB_1 = \beta_1 = \frac{(y - z)\,c\gamma + ya\alpha}{(y - z)\,c + ya};$$

$$OC_1 = \gamma_1 = \frac{(z - y)\,b\beta + za\alpha}{(z - y)\,b + za};$$

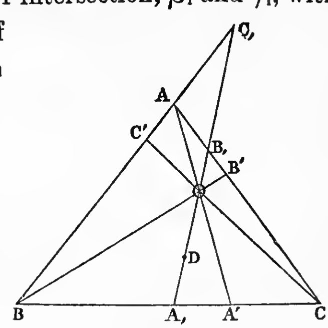

Fig. 22.

whence we derive (by 25) these two other anharmonics,

$$(\text{CB}'\text{AB}_1) = \frac{y-z}{y}; \qquad (\text{BC}'\text{AC}_1) = \frac{z-y}{z};$$

so that we have the relations,

$$(\text{CB}'\text{AB}_1) + (\text{CA}'\text{BA}_1) = (\text{BC}'\text{AC}_1) + (\text{BA}'\text{CA}_1) = 1.$$

But in general, for any four collinear points A, B, C, D, it is not difficult to prove that

$$\frac{\text{AB}}{\text{BC}} \cdot \text{CD} + \frac{\text{AC}}{\text{CB}} \cdot \text{BD} = \text{DA};$$

whence by the definition (25) of the signification of the symbol (ABCD), the following identity is derived,

$$(\text{ABCD}) + (\text{ACBD}) = 1.$$

Comparing this, then, with the recently found relations, we have, for fig. 22, the following anharmonic equations:

$$(\text{CAB}'\text{B}_1) = (\text{CA}'\text{BA}_1) = \frac{z}{y};$$

$$(\text{BAC}'\text{C}_1) = (\text{BA}'\text{CA}_1) = \frac{y}{z};$$

and we see that (as was to be expected from known principles) the anharmonic of the *group* does not change, when we pass from one side of the triangle, considered as a *transversal* of the pencil, to another such side, or transversal. We may therefore speak (as usual) of such an anharmonic of a *group*, as being at the same time the *Anharmonic of a Pencil;* and, with attention to the *order of the rays*, and to the definition (25), may *denote* the two last anharmonics by the two following reciprocal expressions:

$$(\text{O}\,.\,\text{CABD}) = \frac{z}{y}; \qquad (\text{O}\,.\,\text{BACD}) = \frac{y}{z};$$

with other resulting values, when the *order* of the rays is changed; it being understood that

$$(\text{O}\,.\,\text{CABD}) = (\text{C}'\text{A}'\text{B}'\text{D}'),$$

if the rays OC, OA, OB, OD be cut, in the points C', A', B', D', by any one right line.

36. The expression (34),

$$\rho = \frac{xa\alpha + yb\beta + zc\gamma}{xa + yb + zc},$$

may represent the vector of *any point* P *in the given plane*, by a suitable choice of the *coefficients* x, y, z, or simply of their *ratios*. For since (by 22) the three complanar vectors PA, PB, PC must be connected by some linear equation, of the form

$$a' \,.\, \text{PA} + b' \,.\, \text{PB} + c' \,.\, \text{PC} = 0,$$

or

$$a'(\alpha - \rho) + b'(\beta - \rho) + c'(\gamma - \rho) = 0,$$

which gives

$$\rho = \frac{a'\alpha + b'\beta + c'\gamma}{a' + b' + c'},$$

we have only to write

$$\frac{a'}{a} = x, \qquad \frac{b'}{b} = y, \qquad \frac{c'}{c} = z,$$

and the proposed expression for ρ will be obtained. Hence it is easy to infer, on principles already explained, that if we write (compare the annexed fig. 23),

$$P_1 = PA \cdot BC, \qquad P_2 = PB \cdot CA, \qquad P_3 = PC \cdot AB,$$

we shall have, with the same coefficients xyz, the following expressions for the vectors OP_1, OP_2, OP_3, or ρ_1, ρ_2, ρ_3, of these three points of intersection, P_1, P_2, P_3:

$$\rho_1 = \frac{yb\beta + zc\gamma}{yb + zc}, \qquad \rho_2 = \frac{zc\gamma + xa\alpha}{zc + xa},$$

$$\rho_3 = \frac{xa\alpha + yb\beta}{xa + yb};$$

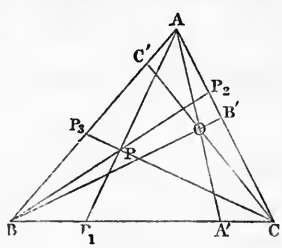

Fig. 23.

which give at once the following anharmonics of pencils, or of groups,

$$(\text{A} \,.\, \text{BOCP}) = (\text{BA}'\text{CP}_1) = \frac{y}{z};$$

$$(\text{B} \,.\, \text{COAP}) = (\text{CB}'\text{AP}_2) = \frac{z}{x};$$

$$(\text{C} \,.\, \text{AOBP}) = (\text{AC}'\text{BP}_3) = \frac{x}{y};$$

whereof we see that the product is unity. *Any two of these three pencils* suffice to *determine the position of the point* P, when the *triangle* ABC, and the *origin* O are given; and therefore it appears that the *three coefficients* x, y, z, or *any scalars proportional to them*, of which the *quotients* thus represent the *anharmonics* of those pencils, may be conveniently called the ANHARMONIC CO-ORDINATES *of that point*, P, *with respect to the given triangle and origin:* while the point P itself may be *denoted* by the *Symbol*,

$$\text{P} = (x, y, z).$$

With this notation, the thirteen points of fig. 21 come to be thus symbolized:

$$\begin{array}{llll} \text{A} = (1, 0, 0), & \text{B} = (0, 1, 0), & \text{C} = (0, 0, 1), & \text{O} = (1, 1, 1); \\ \text{A}' = (0, 1, 1) & \text{B}' = (1, 0, 1), & \text{C}' = (1, 1, 0); & \\ \text{A}'' = (0, 1, -1), & \text{B}'' = (-1, 0, 1), & \text{C}'' = (1, -1, 0); & \\ \text{A}''' = (2, 1, 1), & \text{B}''' = (1, 2, 1), & \text{C}''' = (1, 1, 2). & \end{array}$$

37. If P_1 and P_2 be any two points in the given plane,

$$\text{P}_1 = (x_1, y_1, z_1), \qquad \text{P}_2 = (x_2, y_2, z_2),$$

and if t and u be any two scalar coefficients, then the following *third point*,

$$\text{P} = (tx_1 + ux_2,\ ty_1 + uy_2,\ tz_1 + uz_2),$$

is *collinear* with the two former points, or (in other words) is situated *on the right line* P_1P_2. For, if we make

$$x = tx_1 + ux_2, \qquad y = ty_1 + uy_2, \qquad z = tz_1 + uz_2,$$

and

$$\rho_1 = \frac{x_1 a\alpha + . .}{x_1 a + . .}, \qquad \rho_2 = \frac{x_2 a\alpha + . .}{x_2 a + . .}, \qquad \rho = \frac{xa\alpha + . .}{xa + . .},$$

these *vectors* of the three points $\text{P}_1\text{P}_2\text{P}$ are connected by the *linear equation*,

$$t\,(x_1 a + . .)\,\rho_1 + u\,(x_2 a + . .)\,\rho_2 - (xa + . .)\,\rho = 0;$$

in which (comp. 23), the *sum* of the *coefficients* is *zero*. Conversely, the point P cannot be collinear with P_1, P_2, unless its co-ordinates admit of being thus expressed in terms of theirs. It follows that if a *variable point* P be obliged to *move along a given right line* P_1P_2, or if it have such a *line* (in the given plane) for its *locus*, its *co-ordinates* xyz must satisfy a *homogeneous equation of the first degree, with constant coefficients*; which, in the known notation of determinants, may be thus written,

$$0 = \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix};$$

or, more fully,

$$0 = x\,(y_1 z_2 - z_1 y_2) + y\,(z_1 x_2 - x_1 z_2) + z\,(x_1 y_2 - y_1 x_2);$$

or briefly

$$0 = lx + my + nz,$$

where l, m, n are *three constant scalars*, whereof the *quotients* determine the *position* of the right *line* Λ, which is thus the *locus* of the point P. It is natural to call the *equation*, which thus *connects* the *co-ordinates* of the *point* P, the *Anharmonic Equation of the Line* Λ; and we shall find it convenient also to speak of the *coefficients* l, m, n, in that equation, as being the *Anharmonic Co-ordinates of that Line:* which line may also be *denoted* by the *Symbol*,

$$\Lambda = [l, m, n].$$

38. For example, the three sides BC, CA, AB of the given triangle have thus for their *equations*,

$$x = 0, \qquad y = 0, \qquad z = 0,$$

and for their *symbols*,

$$[1, 0, 0], \qquad [0, 1, 0], \qquad [0, 0, 1].$$

The three additional lines OA, OB, OC, fig. 18, have, in like manner, for their equations and symbols,

$$y - z = 0, \qquad z - x = 0, \qquad x - y = 0,$$

$$[0, 1, -1], \qquad [-1, 0, 1], \qquad [1, -1, 0].$$

The lines B′C′A″, C′A′B″, A′B′C″, of fig. 21, are

$$y + z - x = 0, \qquad z + x - y = 0, \qquad x + y - z = 0,$$

or

$$[-1, 1, 1], \qquad [1, -1, 1], \qquad [1, 1, -1];$$

the lines A″B‴C‴, B″C‴A‴, C″A‴B‴, of the same figure, are in like manner represented by the equations and symbols,

$$y + z - 3x = 0, \qquad z + x - 3y = 0, \qquad x + y - 3z = 0,$$

$$[-3, 1, 1], \qquad [1, -3, 1], \qquad [1, 1, -3];$$

and the line A″B″C″ is

$$x + y + z = 0, \quad \text{or} \quad [1, 1, 1].$$

Finally, we may remark that on the same plan, the equation and the symbol of what is often called the *line at infinity*, or of the *locus of all the infinitely distant points in the given plane*, are respectively,

$$ax + by + cz = 0, \quad \text{and} \quad [a, b, c];$$

because the *linear function*, $ax + by + cz$, of the *co-ordinates* x, y, z of a point P in the plane, is the *denominator* of the expression (34, 36) for the *vector* ρ of that point: so that the *point* P is at an infinite distance from the origin O, when, and only when, this linear function *vanishes*.

39. These *anharmonic co-ordinates of a line*, although above interpreted (37) with reference to the *equation* of that line, considered as connecting the co-ordinates of a variable *point* thereof, are capable of receiving an independent geometrical interpretation. For the three points L, M, N, in which the line Λ, or $[l, m, n]$, or $lx + my + nz = 0$, intersects the three sides BC, CA, AB of the

given triangle ABC, or the three given lines $x = 0$, $y = 0$, $z = 0$ (38), may evidently (on the plan of 36) be thus denoted:

$$\text{L} = (0, n, -m); \qquad \text{M} = (-n, 0, l); \qquad \text{N} = (m, -l, 0).$$

But we had also (by 36),

$$\text{A}'' = (0, 1, -1); \qquad \text{B}'' = (-1, 0, 1); \qquad \text{C}'' = (1, -1, 0);$$

whence it is easy to infer, on the principles of recent articles, that

$$\frac{n}{m} = (\text{BA}''\text{CL}); \qquad \frac{l}{n} = (\text{CB}''\text{AM}); \qquad \frac{m}{l} = (\text{AC}''\text{BN});$$

with the resulting relation,

$$(\text{BA}''\text{CL}) \,.\, (\text{CB}''\text{AM}) \,.\, (\text{AC}''\text{BN}) = 1.$$

40. Conversely, this last equation is easily proved, with the help of the known and general relation between *segments* (32), applied to *any two transversals*, A''B''C'' and LMN, of any triangle ABC. In fact, we have thus the two equations,

$$\frac{\text{BA}''}{\text{A}''\text{C}} \cdot \frac{\text{CB}''}{\text{B}''\text{A}} \cdot \frac{\text{AC}''}{\text{C}''\text{B}} = -1, \quad \frac{\text{BL}}{\text{LC}} \cdot \frac{\text{CM}}{\text{MA}} \cdot \frac{\text{AN}}{\text{NB}} = -1;$$

on dividing the former of which by the latter, the last formula of the last article results. We might therefore in this way have been led, *without* any consideration of a *variable point* P, to introduce *three auxiliary scalars, l, m, n, defined* as having their *quotients* $\frac{n}{m}, \frac{l}{n}, \frac{m}{n}$ equal respectively, as in 39, to the three anharmonics of groups,

$$(\text{BA}''\text{CL}), \qquad (\text{CB}''\text{AM}), \qquad (\text{AC}''\text{BN});$$

and then it would have been evident that these three scalars, l, m, n (or any others proportional thereto), are sufficient to *determine the position of the right line* Λ, or LMN, considered as a *transversal* of the given triangle ABC: so that they might naturally have been called, on this account, as above, *the anharmonic co-ordinates* of that *line*. But although the anharmonic co-ordinates of a point and of a line may thus be *independently defined*, yet the *geometrical utility* of such definitions will be found to depend mainly on their *combination:* or on the formula $lx + my + nz = 0$ of 37, which may at pleasure be considered as expressing, either that the *variable point* (x, y, z) is *situated somewhere* upon the *given right line* $[l, m, n]$; or else that the *variable line* $[l, m, n]$ *passes, in some direction, through the given point* (x, y, z).

41. If Λ_1 and Λ_2 be any *two* right lines in the given plane,

$$\Lambda_1 = [l_1, m_1, n_1], \qquad \Lambda_2 = [l_2, m_2, n_2],$$

then any *third* right line Λ in the same plane, which passes *through the intersection* $\Lambda_1 \cdot \Lambda_2$, or (in other words) which *concurs* with them (at a finite or infinite distance), may be represented (comp. 37) by a symbol of the form,

$$\Lambda = [tl_1 + ul_2,\ tm_1 + um_2,\ tn_1 + un_2],$$

where t and u are scalar coefficients. Or, what comes to the same thing, if l, m, n be the anharmonic co-ordinates of the line Λ, then (comp. again 37), the equation

$$0 = l\,(m_1n_2 - n_1m_2) + \&c. = \begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix},$$

must be satisfied; because, if (X, Y, Z) be the supposed point *common* to the three lines, the three equations

$$lX + mY + nZ = 0, \quad l_1X + m_1Y + n_1Z = 0, \quad l_2X + m_2Y + n_2Z = 0,$$

must co-exist. Conversely, this co-existence will be possible, and the three lines will have a common point (which may be infinitely distant), if the recent *condition of concurrence* be satisfied. For example, because $[a, b, c]$ has been seen (in 38) to be the symbol of the line at infinity (at least if we still retain the same significations of the scalars a, b, c as in Articles 27, &c.), it follows that

$$\Lambda = [l, m, n], \quad \text{and} \quad \Lambda' = [l + ua,\ m + ub,\ n + uc],$$

are symbols of two *parallel lines*; because they *concur at infinity*. In general, all problems respecting intersections of right lines, collineations of points, &c., in the given plane, when treated by this *anharmonic method*, conduct to easy *eliminations* between *linear equations* (of the *scalar* kind), on which we need not here delay: the *mechanism* of such *calculations* being for the most part the *same* as in the known method of *trilinear co-ordinates*: although (as we have seen) the *geometrical interpretations* are altogether *different*.

SECTION 5.

On Plane Geometrical Nets, resumed.

42. If we now *resume*, for a moment, the consideration of those plane geometrical *nets*, which were mentioned in Art. 34; and agree to call those points and lines, in the given plane, *rational points* and *rational lines*, respectively, which have their *anharmonic co-ordinates equal* (or *proportional*) to *whole*

numbers; because then the *anharmonic quotients,* which were discussed in the last Section, are *rational;* but to say that a point or line is *irrational,* or that it is *irrationally related* to the given system of *four initial points* O, A, B, C, when its anharmonic co-ordinates are *not* thus *all* equal (or proportional) to *integers;* it is clear that *whatever four points* we may assume as *initial,* and *however far* the construction of the net may be carried, the *net-points* and *net-lines* which result will *all* be *rational,* in the sense just now defined. In fact, we *begin* with such; and the subsequent *eliminations* (41) can never afterwards conduct to any, that are of the contrary kind: the right line which *connects* two rational points being always a rational line; and the point of *intersection* of two rational lines being necessarily a rational point. The assertion made in Art. 34 is therefore fully justified.

43. Conversely, *every rational point* of the given plane, with respect to the four assumed initial points OABC, *is a point of the net* which those four points determine. To prove this, it is evidently sufficient to show that every rational point $A_1 = (0, y, z)$, on any one side BC of the given triangle ABC, can be so constructed. Making, as in fig. 22,

$$B_1 = OA_1 \cdot CA, \quad \text{and} \quad C_1 = OA_1 \cdot AB,$$

we have (by 35, 36) the expressions,

$$B_1 = (y, 0, y - z), \qquad C_1 = (z, z - y, 0);$$

from which it is easy to infer (by 36, 37), that

$$C'B_1 \cdot BC = (0, y, z - y), \qquad B'C_1 \cdot BC = (0, y - z, z);$$

and thus we can reduce the linear construction of the rational point $(0, y, z)$, in which the two *whole* numbers y and z may be supposed to be *prime* to each other, to depend on that of the point $(0, 1, 1)$, which has already been constructed as A′. It follows that although *no irrational point* Q of the plane *can be a net-point,* yet *every such point* can be *indefinitely approached* to, by continuing the linear construction; so that it can be *included within* a *quadrilateral interstice* $P_1P_2P_3P_4$, or even within a *triangular interstice* $P_1P_2P_3$, which interstice of the net can be made *as small* as we may desire. Analogous remarks apply to *irrational lines* in the plane, which can *never coincide* with *net-lines,* but may always be indefinitely *approximated* to by such.

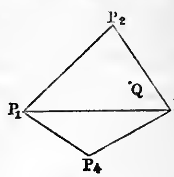

Fig. 24.

44. If P, P_1, P_2 be any three *collinear* points of the net, so that the formulæ of 37 apply, and if P′ be any *fourth* net-point (x', y', z') upon the same line, then writing $\quad x_1a + y_1b + z_1c = v_1, \qquad x_2a + y_2b + z_2c = v_2,$

we shall have two expressions of the forms,

$$\rho = \frac{tv_1\rho_1 + uv_2\rho_2}{tv_1 + uv_2}, \qquad \rho' = \frac{t'v_1\rho_1 + u'v_2\rho_2}{t'v_1 + u'v_2},$$

in which the coefficients $tut'u'$ are *rational*, because the co-ordinates xyz, &c., are such, whatever the constants abc may be. We have therefore (by 25) the following rational expression for the anharmonic of this *net-group :*

$$(\text{P}_1\text{PP}_2\text{P}') = \frac{ut'}{tu'} = \frac{(yx_1 - xy_1)\,(y'x_2 - x'y_2)}{(xy_2 - yx_2)\,(x'y_1 - y'x_1)};$$

and similarly for every other group of the same kind. Hence *every group* of four collinear net-points, and consequently also *every pencil* of four concurrent net-lines, has a *rational value* for its *anharmonic function;* which value depends *only* on the *processes* of *linear construction* employed, in arriving at that group or pencil, and is quite *independent* of the *configuration* or *arrangement* of the *four initial points:* because the *three initial constants, a, b, c, disappear* from the expression which results. It was thus that, in fig. 21, the *nine pencils*, which had the nine derived points A′ . . C′′′ for their vertices, were all *harmonic* pencils, in whatever manner the four points O, A, B, C might be arranged. In general, it may be said that plane geometrical nets are all *homographic figures ;** and conversely, in any two such plane *figures, corresponding points* may be considered as either *coinciding*, or at least (by 43) as indefinitely approaching to coincidence, with *similarly constructed points* of two plane *nets*: that is, with points of which (in their respective systems) the *anharmonic co-ordinates* (36) are *equal integers.*

45. Without entering here on any *general* theory of *transformation* of anharmonic co-ordinates, we may already see that if we select *any four net-points* O_1, A_1, B_1, C_1, of which no three are collinear, *every other point* P of the same net is *rationally related* (42) *to these;* because (by 44) the three new anharmonics of pencils, $(\text{A}_1 \,.\, \text{B}_1\text{O}_1\text{C}_1\text{P}) = \frac{y_1}{z_1}$, &c., are rational: and therefore (comp. 36) the *new co-ordinates* x_1, y_1, z_1 of the point P, as well its *old* co-ordinates xyz, are equal or proportional to *whole numbers.* It follows (by 43) that *every point* P of the net can be *linearly constructed*, if *any four* such points be *given* (no three being collinear, as above) ; or, in other words, that the *whole net* can be *reconstructed*,† if *any one* of its *quadrilaterals* (such as the

* Compare the *Géométrie Supérieure* of M. Chasles, p. 362.

† This theorem (45) of the possible *reconstruction of a plane net*, from any one of its *quadrilaterals*, and the theorem (43) respecting the possibility of indefinitely *approaching* by *net-lines* to the points above called *irrational* (42), without ever *reaching* such points by any processes of *linear construction*

interstice in fig. 24) be *known.* As an example, we may suppose that the four points OA′B′C′ in fig. 21 are given, and that it is required to *recover* from them the three points ABC, which had previously been among the *data* of the construction. For this purpose, it is only necessary to determine first the three auxiliary points A‴, B‴, C‴, as the intersections OA′ · B′C′, &c.; and next the three other auxiliary points A″, B″, C″, as B′C′ · B‴C‴, &c.: after which the formulæ, A = B′B″ · C′C″, &c., will enable us to return, as required, to the points A, B, C, as intersections of known right lines.

SECTION 6.

On Anharmonic Equations, and Vector Expressions, for Curves in a given Plane.

46. When, in the expressions 34 or 36 for a variable *vector* $\rho = \text{OP}$, the three variable *scalars* (or anharmonic co-ordinates) x, y, z are *connected* by any given algebraic *equation*, such as

$$f_p(x, y, z) = 0,$$

supposed to be rational and integral, and homogeneous of the p^{th} *degree*, then the *locus* of the *term* P (Art. 1) of that vector is a *plane curve* of the p^{th} *order;* because (comp. 37) it is *cut* in p *points* (distinct or coincident, and real or imaginary), by any *given right line*, $lx + my + nz = 0$, in the given plane.

For example, if we write

$$\rho = \frac{t^2 a\alpha + u^2 b\beta + v^2 c\gamma}{t^2 a + u^2 b + v^2 c},$$

where t, u, v are three new variable scalars, of which we shall suppose that the sum is zero, then, by eliminating these between the four equations,

$$x = t^2, \quad y = u^2, \quad z = v^2, \quad t + u + v = 0,$$

we are conducted to the following equation of the *second* degree,

$$0 = f_p = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy;$$

so that here $p = 2$, and the locus of P is a *conic section.* In fact, it is the conic which *touches* the *sides* of the *given triangle* ABC, at the points above called A′, B′, C′; for if we seek its *intersections* with the side BC, by making $x = 0$ (38),

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of the kind here considered, have been taken, as regards their substance (although investigated by a totally different analysis), from that highly original treatise of MÖBIUS, which was referred to in a former note (p. 22). Compare the remarks in the following Chapter, upon *nets in space.*

we obtain a *quadratic* with *equal* roots, namely, $(y - z)^2 = 0$; which shows that there is *contact* with this side at the point (0, 1, 1), or A′ (36): and similarly for the two other sides.

47. If the point O, in which the three right lines AA′, BB′, CC′ *concur*, be (as in fig. 18, &c.) *interior* to the triangle ABC, the sides of that triangle are then all cut *internally*, by the points A′, B′, C′ of contact with the conic; so that in this case (by 28) the ratios of the constants a, b, c are all *positive*, and the *denominator* of the recent expression (46) for ρ cannot *vanish*, for any *real* values of the variable scalars t, u, v; and consequently no *such* values can render *infinite* that *vector* ρ. The *conic* is therefore generally in this case, as in fig. 25, an *inscribed ellipse;* which becomes however the *inscribed circle*, when

$$a^{-1} : b^{-1} : c^{-1} = \mathrm{s - a : s - b : s - c};$$

a, b, c denoting here the lengths of the sides of the triangle, and s being their semi-sum.

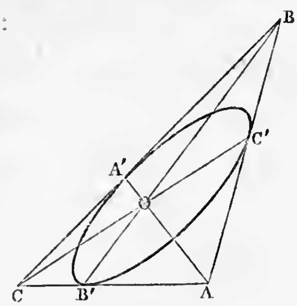

Fig. 25.

48. But if the point of concourse O be *exterior* to the *triangle of tangents* ABC, so that *two* of its sides are cut *externally*, then *two* of the three ratios of *segments* (28) are *negative;* and therefore *one* of the three *constants* a, b, c may be treated as < 0, but each of the two others as > 0. Thus if we suppose that

$$b > 0, \qquad c > 0, \qquad a < 0, \qquad a + b > 0, \qquad a + c > 0,$$

A′ will be a point on the side BC *itself*, but the points B′, C′, O will be on the lines AC, AB, AA′ *prolonged*, as in fig. 26; and then the conic A′B′C′ will be an *ellipse* (including the case of a *circle*), or a *parabola*, or an *hyperbola*, according as the *roots* of the *quadratic*,

$$(a + c)\,t^2 + 2ctu + (b + c)\,u^2 = 0,$$

obtained by equating the denominator (46) of the vector ρ to zero, are either, Ist, *imaginary;* or IInd, *real* and *equal;* or IIIrd, *real* and *unequal:* that is, according as we have

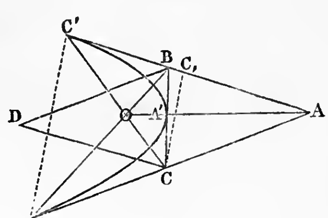

Fig. 26.

$$bc + ca + ab > 0, \quad \text{or} = 0, \quad \text{or} < 0;$$

or (because the product abc is here negative), according as

$$a^{-1} + b^{-1} + c^{-1} < 0, \quad \text{or} = 0, \quad \text{or} > 0.$$

For example, if the conic be what is often called the *exscribed circle*, the known ratios of segments give the proportion,

$$a^{-1} : b^{-1} : c^{-1} = -\mathrm{s} : \mathrm{s} - \mathrm{c} : \mathrm{s} - \mathrm{b};$$

and

$$-\mathrm{s} + \mathrm{s} - \mathrm{c} + \mathrm{s} - \mathrm{b} < 0.$$

49. More generally, if $\mathrm{C}_{\prime}$ be (as in fig. 26) a point upon the side AB, or on that side prolonged, such that $\mathrm{CC}_{\prime}$ is parallel to the chord $\mathrm{B'C'}$, then

$$\mathrm{C}_{\prime}\mathrm{C}' : \mathrm{AC}' = \mathrm{CB}' : \mathrm{AB}' = -a : c, \quad \text{and} \quad \mathrm{AB} : \mathrm{AC}' = a + b : b;$$

writing then the *condition* (48) *of ellipticity* (or circularity) under the form, $\frac{-a}{c} < \frac{a+b}{b}$, we see that the conic is an ellipse, parabola, or hyperbola, according as $\mathrm{C}_{\prime}\mathrm{C}' <$ or $=$ or $>$ AB; the arrangement being *still*, in other respects, that which is represented in fig. 26. Or, to express the same thing more symmetrically, if we complete the parallelogram CABD, then according as the point D falls, Ist, *beyond the chord* $\mathrm{B'C'}$, with respect to the point A; or IInd, *on that chord;* or IIIrd, *within the triangle* $\mathrm{AB'C'}$, the general arrangement of the same figure being retained, the curve is *elliptic*, or *parabolic*, or *hyperbolic*. In that *other* arrangement or configuration, which answers to the system of inequalities, $b > 0$, $c > 0$, $a + b + c < 0$, the point A′ is still upon the side BC *itself*, but O is on the line A′A prolonged *through* A; and *then* the inequality,

$$a(b + c) + bc < -(b^2 + bc + c^2) < 0,$$

shows that the conic is *necessarily* an hyperbola; whereof it is easily seen that *one branch* is touched by the side BC at A′, while the *other* branch is touched in B′ and C′, by the sides CA and BA prolonged through A. The curve is also hyperbolic, if either $a + b$ or $a + c$ be negative, while b and c are positive as before.

50. When the quadratic (48) has its roots real and unequal, so that the conic is an *hyperbola*, then the *directions* of the *asymptotes* may be found, by substituting those roots, or the values of t, u, v which correspond to them (or any scalars *proportional* thereto), in the *numerator* of the expression (46) for ρ; and similarly we can find the direction of the *axis* of the *parabola*, for the case when the roots are real but equal: for we shall thus obtain the directions, or direction, in which a right line OP must be drawn from O, so as to *meet the conic at infinity*. And the same *conditions* as before, for distinguishing the *species* of the conic, may be otherwise obtained by combining the *anharmonic equation*, $f = 0$ (46), of that conic, with the corresponding equation $ax + by + cz = 0$ (38) of the *line at infinity;* so as to inquire (on known principles of

modern geometry) whether that *line* meets that *curve* in *two imaginary points*, or *touches* it, or *cuts* it, in points which (although *infinitely distant*) are here to be considered as *real*.

51. In general, if $f(x, y, z) = 0$ be the anharmonic equation (46) of *any plane curve*, considered as the *locus* of a variable *point* P; and if the *differential** of this equation be thus denoted,

$$0 = \mathrm{d}f(x, y, z) = X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z;$$

then because, by the supposed *homogeneity* (46) of the function f, we have the relation

$$Xx + Yy + Zz = 0,$$

we shall have also this other but analogous relation,

$$Xx' + Yy' + Zz' = 0,$$

if

$$x' - x : y' - y : z' - z = \mathrm{d}x : \mathrm{d}y : \mathrm{d}z;$$

that is (by the principles of Art. 37), if $P' = (x', y', z')$ be *any point upon the tangent to the curve*, drawn at the point $P = (x, y, z)$, and regarded as the *limit of a secant*. The *symbol* (37) of this *tangent* at P may therefore be thus written,

$$[X, Y, Z], \quad \text{or} \quad [\mathrm{D}_x f, \mathrm{D}_y f, \mathrm{D}_z f];$$

where D_x, D_y, D_z are known *characteristics of partial derivation*.

52. For example, when f has the form assigned in 46, as answering to the conic lately considered, we have $\mathrm{D}_x f = 2(x - y - z)$, &c.; whence the tangent at any point (x, y, z) of this curve may be denoted by the symbol,

$$[x - y - z, \quad y - z - x, \quad z - x - y];$$

in which, as usual, the co-ordinates of the line may be replaced by any others proportional to them. Thus at the point A′, or (by 36) at (0, 1, 1), which is evidently (by the form of f) a point upon the curve, the tangent is the line $[-2, 0, 0]$, or $[1, 0, 0]$; that is (by 38), the side BC of the given triangle, as was otherwise found before (46). And in general it is easy to see that the recent symbol denotes the right line, which is (in a well known sense) the *polar* of the point (x, y, z), with respect to the same given conic; or that the line $[X', Y', Z']$ is the polar of the point (x', y', z'): because the equation

$$Xx' + Yy' + Zz' = 0,$$

* In the theory of *quaternions*, as distinguished from (although including) that of *vectors*, it will be found necessary to introduce a *new definition of differentials*, on account of the *non-commutative* property of *quaternion-multiplication*: but, for the present, the *usual* significations of the signs d and D are sufficient.

which for a *conic* may be written as $X'x + Y'y + Z'z = 0$, expresses (by 51) the condition requisite, in order that a point (x, y, z) of the curve* should belong to a tangent which passes through the point (x', y', z'). Conversely, the *point* (x, y, z) is (in the same well-known sense) the *pole* of the *line* $[X, Y, Z]$; so that the *centre* of the conic, which is (by known principles) the *pole of the line at infinity* (38), is the point which satisfies the conditions $a^{-1}X = b^{-1}Y = c^{-1}Z$; it is therefore, for the present conic, the point K $= (b + c, c + a, a + b)$, of which the *vector* OK is easily reduced, by the help of the linear equation, $a\alpha + b\beta + c\gamma = 0$ (27), to the form,

$$\kappa = -\frac{a^2\alpha + b^2\beta + c^2\gamma}{2(bc + ca + ab)};$$

with the verification that the *denominator vanishes*, by 48, when the conic is a *parabola*. In the more general case, when this denominator is different from zero, it can be shown that *every chord* of the curve, which is drawn *through the extremity* K of the vector κ, is *bisected* at that point K: which point would therefore in this way be seen again to be the *centre*.

53. Instead of the *inscribed conic* (46), which has been the subject of recent articles, we may, as another example, consider that *exscribed* (or *circumscribed*) conic, which passes through the three corners A, B, C of the given triangle, and touches there the lines AA″, BB″, CC″ of fig. 21. The anharmonic equation of this new conic is easily seen to be,

$$yz + zx + xy = 0;$$

the vector of a variable point P of the curve may therefore be expressed as follows,

$$\rho = \frac{t^{-1}a\alpha + u^{-1}b\beta + v^{-1}c\gamma}{t^{-1}a + u^{-1}b + v^{-1}c},$$

with the condition $t + u + v = 0$, as before. The vector of its centre K′ is found to be

$$\kappa' = \frac{2(a^2\alpha + b^2\beta + c^2\gamma)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab};$$

and it is an ellipse, a parabola, or an hyperbola, according as the denominator of this last expression is negative, or null, or positive. And because these two recent *vectors*, κ, κ', bear a *scalar ratio* to each other, it follows (by 19) that *the three points* O, K, K′ *are collinear;* or in other words, that the *line of*

* If the curve $f = 0$ were of a degree *higher* than the *second*, then the two equations above written would represent what are called the *first polar*, and the *last* or the *line-polar*, of the point (x', y', z'), with respect to the given curve.

centres K, K′, of the *two conics* here considered, *passes through the point of concourse* O *of the three lines* AA′, BB′, CC′. More generally, if L be the *pole* of *any given right line* $\Lambda = [l, m, n]$ (37), with respect to the *inscribed* conic (46), and if L′ be the pole of the *same line* Λ with respect to the *exscribed* conic of the present article, it can be shown that the *vectors* OL, OL′, or λ, λ', of these *two poles* are of the forms,

$$\lambda = k\,(la\alpha + mb\beta + nc\gamma), \qquad \lambda' = k'\,(la\alpha + mb\beta + nc\gamma),$$

where k and k' are *scalars;* the three points O, L, L′ are therefore ranged *on one right line.*

54. As an example of a *vector-expression* for a curve of an order *higher than the second,* the following may be taken :

$$\text{OP} = \rho = \frac{t^3a\alpha + u^3b\beta + v^3c\gamma}{t^3a + u^3b + v^3c};$$

with $t + u + v = 0$, as before. Making $x = t^3$, $y = u^3$, $z = v^3$, we find here by elimination of t, u, v the *anharmonic equation,*

$$(x + y + z)^3 - 27xyz = 0;$$

the locus of the point P is therefore, in this example, a *curve of the third order,* or briefly a *cubic curve.* The *mechanism* (41) of calculations with *anharmonic co-ordinates* is so much the same as that of the known *trilinear method,* that it may suffice to remark briefly here that the *sides* of the given triangle ABC are the *three* (real) *tangents of inflexion;* the *points of inflexion* being those which are marked as A″, B″, C″ in fig. 21 ; and the *origin* of vectors O being a *conjugate point.** If $a = b = c$, in which case (by 29) this origin O becomes (as in fig. 19) the *mean point* of the triangle, the *chord of inflexion* A″B″C″ is then the *line at infinity,* and the curve takes the form represented in fig. 27 ; having *three infinite branches,* inscribed within the angles vertically opposite to those of the given triangle ABC, of which the sides are the *three asymptotes.*

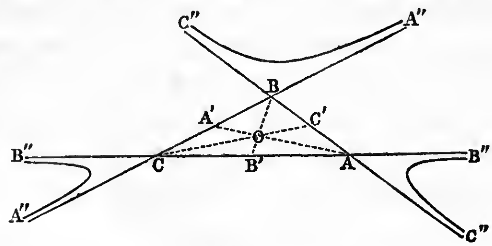

Fig. 27.

55. It would be improper to enter here into any details of discussion of such cubic curves, for which the reader will naturally turn to other works.†

* Answering to the values $t = 1$, $u = \theta$, $v = \theta^2$, where θ is one of the imaginary cube-roots of unity ; which values of t, u, v give $x = y = z$, and $\rho = 0$.

† Especially the excellent Treatise on *Higher Plane Curves*, by the Rev. George Salmon, F.T.C.D., &c. Dublin, 1852.

But it may be remarked, in passing, that because the *general cubic* may be represented, on the present plan, by combining the general expression of Art. 34 or 36 for the vector ρ, with the scalar equation

$$s^3 = 27\,kxyz, \quad \text{where} \quad s = x + y + z;$$

k denoting an arbitrary constant, which becomes equal to unity, when the origin is (as in 54) a conjugate point; it follows that if $\text{P} = (x, y, z)$ and $\text{P}' = (x', y', z')$ be *any two points of the curve*, and if we make $s' = x' + y' + z'$, we shall have the relation,

$$xyzs'^3 = x'y'z's^3, \quad \text{or} \quad \frac{xs'}{sx'} \cdot \frac{ys'}{sy'} \cdot \frac{zs'}{sz'} = 1:$$

in which it is not difficult to prove that

$$\frac{xs'}{sx'} = (\text{A}'' \,.\, \text{PBP}'\text{B}''); \quad \frac{ys'}{sy'} = (\text{B}'' \,.\, \text{PCP}'\text{C}''); \quad \frac{zs'}{sz'} = (\text{C}'' \,.\, \text{PAP}'\text{A}'');$$

the notation (35) of *anharmonics of pencils* being retained. We obtain therefore thus the following *Theorem* :—" *If the sides of any given plane* triangle* ABC *be cut* (as in fig. 21) *by any given rectilinear transversal,* $\text{A}''\text{B}''\text{C}''$, *and if any two points* P and P′ *in its plane be such as to satisfy the anharmonic relation*

$$(\text{A}'' \,.\, \text{PBP}'\text{B}'') \,.\, (\text{B}'' \,.\, \text{PCP}'\text{C}'') \,.\, (\text{C}'' \,.\, \text{PAP}'\text{A}'') = 1,$$

then these two points P, P′ *are on one common cubic curve, which has the three collinear points,* A″, B″, C″ *for its three real points of inflexion, and has the sides* BC, CA, AB *of the triangle for its three tangents at those points* "; a result which seems to offer a *new geometrical generation for curves of the third order.*

56. Whatever the *order* of a plane curve may be, or whatever may be the *degree* p of the *function* f in 46, we saw in 51 that the *tangent* to the curve at any point $\text{P} = (x, y, z)$ is the right line

$$\Lambda = [l, m, n], \quad \text{if} \quad l = \text{D}_x f, \quad m = \text{D}_y f, \quad n = \text{D}_z f;$$

expressions which, by the supposed *homogeneity* of f, give the relation $lx + my + nz = 0$, and therefore enable us to establish the system of the *two* following differential equations,

$$l\text{d}x + m\text{d}y + n\text{d}z = 0, \qquad x\text{d}l + y\text{d}m + z\text{d}n = 0.$$

If then, by elimination of the ratios of x, y, z, we arrive at a *new homogeneous equation* of the form,

$$0 = \text{F}(\text{D}_x f, \ \text{D}_y f, \ \text{D}_z f),$$

* This Theorem may be extended, with scarcely any modification, from *plane* to *spherical curves*, of the *third order*.

as one that is true for all values of x, y, z which render the function $f = 0$ (although it may require to be cleared of *factors*, introduced by this *elimination*), we shall have the equation

$$\text{F}(l, m, n) = 0,$$

as a *condition* that must be *satisfied by the tangent* Λ to the curve, *in all the positions which can be assumed by that right line.* And, by *comparing* the two differential equations.

$$d\text{F}(l, m, n) = 0, \qquad x dl + y dm + z dn = 0,$$

we see that we may write the *proportion*,

$x : y : z = \text{D}_l\text{F} : \text{D}_m\text{F} : \text{D}_n\text{F}$, and the *symbol* $\text{P} = (\text{D}_l\text{F}, \text{D}_m\text{F}, \text{D}_n\text{F})$, if (x, y, z) be, as above, the *point of contact* P of the *variable line* $[l, m, n,]$ in any one of its positions, *with the curve* which is its *envelope.* Hence we can pass (or return) *from the tangential equation* $\text{F} = 0$, of a curve *considered as the envelope of a right line* Λ, *to the local equation* $f = 0$, of the same curve considered (as in 46) as the *locus of a point* P: since, if we obtain, by elimination of the ratios of l, m, n, an equation of the form

$$0 = f(\text{D}_l\text{F}, \text{D}_m\text{F}, \text{D}_n\text{F}),$$

(cleared, if it be necessary, of foreign factors) as a consequence of the homogeneous equation $\text{F} = 0$, we have only to *substitute* for these *partial derivatives*, D_lF, &c., the *anharmonic co-ordinates* x, y, z, to which they are proportional. And when the functions f and F are not only *homogeneous* (as we shall *always* suppose them to be), but also *rational* and *integral* (which it is sometimes convenient *not* to assume them as being), then, while the *degree* of the function f, or of the *local equation*, marks (as before) the *order* of the curve, the degree of the *other homogeneous function* F, or of the *tangential equation* $\text{F} = 0$, is easily seen to denote, in this *anharmonic method* (as, from the analogy of *other* and older methods, it might have been expected to do), the *class* of the curve to which that equation belongs: or the *number of tangents* (distinct or coincident, and real and imaginary), which can be drawn *to that curve, from an arbitrary point* in its plane.

57. As an example (comp. 52), if we eliminate x, y, z between the equations,

$$l = x - y - z, \quad m = y - z - x, \quad n = z - x - y, \quad lx + my + nz = 0,$$

where l, m, n are the co-ordinates of the tangent to the *inscribed conic* of Art. 46, we are conducted to the following tangential equation of that conic, or *curve of the second class*,

$$\text{F}(l, m, n) = mn + nl + lm = 0;$$

with the verification that the sides [1, 0, 0], &c. (38), of the triangle ABC are among the lines which satisfy this equation. Conversely, if this *tangential equation* were *given* we might (by 56) *derive* from it expressions for the *co-ordinates of contact* x, y, z, as follows:

$$x = \mathrm{D}_l\mathrm{F} = m + n, \qquad y = n + l, \qquad z = l + m;$$

with the verification that the side [1, 0, 0] touches the conic, considered now as an *envelope*, in the point (0, 1, 1), or A′, as before: and then, by eliminating l, m, n, we should be brought back to the *local equation*, $f = 0$, of 46. In like manner, from the local equation $f = yz + zx + xy = 0$ of the *exscribed conic* (53), we can derive by differentiation the *tangential co-ordinates*,*

$$l = \mathrm{D}_x f = y + z, \qquad m = z + x, \qquad n = x + y,$$

and so obtain by elimination the tangential equation, namely,

$$\mathrm{F}(l, m, n) = l^2 + m^2 + n^2 - 2mn - 2nl - 2lm = 0;$$

from which we could in turn *deduce* the *local* equation. And (comp. 40), the very simple formula

$$lx + my + nz = 0,$$

which we have so often had occasion to employ, as *connecting two sets* of anharmonic co-ordinates, may not only be considered (as in 37) as the *local equation of a given right line* Λ, *along* which a *point* P *moves*, but also as the *tangential equation of a given point, round which a right line turns*: according as we suppose the set l, m, n, or the set x, y, z, to be *given*. Thus, while the right *line* A″B″C″, or [1, 1, 1], of fig. 21, was represented in 38 by the *equation* $x + y + z = 0$, the *point* O of the same figure, or the point (1, 1, 1), may be represented by the *analogous equation*,

$$l + m + n = 0;$$

because the *co-ordinates* l, m, n of *every line*, which passes *through this point* O, must satisfy this equation of the first degree, as may be seen exemplified, in the same Art. 38, by the lines OA, OB, OC.

58. To give an instance or two of the use of forms, which, although *homogeneous*, are yet not *rational* and *integral* (56) we may write the local equation of the *inscribed* conic (46) as follows:

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0;$$

* This *name* of "*tangential co-ordinates*" appears to have been first introduced by Dr. Booth in a Tract published in 1840, to which the author of the present Elements cannot now more particularly refer: but the *system* of Dr. Booth was entirely different from his own. See the reference in Salmon's *Higher Plane Curves*, note to page 16.

and then (suppressing the common numerical factor $\frac{1}{2}$), the partial derivatives are

$$l = x^{-\frac{1}{2}}, \qquad m = y^{-\frac{1}{2}}, \qquad n = z^{-\frac{1}{2}};$$

so that a form of the tangential equation for this conic is,

$$l^{-1} + m^{-1} + n^{-1} = 0;$$

which evidently, when cleared of fractions, agrees with the first form of the last Article: with the verification (48), that $a^{-1} + b^{-1} + c^{-1} = 0$ when the curve is a *parabola;* that is, when it is *touched* (50) by the *line at infinity* (38). For the *exscribed* conic (53), we may write the *local* equation thus,

$$x^{-1} + y^{-1} + z^{-1} = 0;$$

whence it is allowed to write also,

$$l = x^{-2}, \qquad m = y^{-2}, \qquad n = z^{-2},$$

and

$$l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0;$$

a form of the tangential equation which, when cleared of radicals, agrees again with 57. And it is evident that we could return, with equal ease, from these tangential to these local equations.

59. For the *cubic curve* with a *conjugate point* (54), the local equation may be thus written,*

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0;$$

we may therefore assume for its tangential co-ordinates the expressions,

$$l = x^{-\frac{2}{3}}, \qquad m = y^{-\frac{2}{3}}, \qquad n = z^{-\frac{2}{3}};$$

and a form of its tangential equation is thus found to be,

$$l^{-\frac{1}{2}} + m^{-\frac{1}{2}} + n^{-\frac{1}{2}} = 0.$$

Conversely, if this tangential form were *given*, we might return to the local equation, by making

$$x = l^{-\frac{3}{2}}, \qquad y = m^{-\frac{3}{2}}, \qquad z = n^{-\frac{3}{2}},$$

which would give $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, as before. The tangential equation just now found becomes, when it is cleared of *radicals*,

$$0 = l^{-2} + m^{-2} + n^{-2} - 2m^{-1}n^{-1} - 2n^{-1}l^{-1} - 2l^{-1}m^{-1};$$

or, when it is also cleared of *fractions*,

$$0 = \mathrm{F} = m^2n^2 + n^2l^2 + l^2m^2 - 2nl^2m - 2lm^2n - 2mn^2l;$$

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* Compare Salmon's *Higher Plane Curves*, page 172 [Art. 216, new ed.].

of which the *biquadratic* form shows (by 56) that *this cubic* is a *curve of the fourth class*, as indeed it is known to be. The *inflexional* character (54) of the points A″, B″, C″ upon this curve is here recognised by the circumstance, that when we make $m - n = 0$, in order to find the four tangents from $A'' = (0, 1, -1)$ (36), the resulting biquadratic, $0 = m^4 - 4lm^3$, has *three equal roots;* so that the line [1, 0, 0], or the side BC, *counts as three*, and is therefore a *tangent of inflexion :* the *fourth tangent* from A″ being the line [1, 4, 4], which touches the cubic at the point (− 8, 1, 1).

60. In general, the two equations (56),

$$n\mathrm{D}_x f - l\mathrm{D}_z f = 0, \qquad n\mathrm{D}_y f - m\mathrm{D}_z f = 0,$$

may be considered as expressing that the homogeneous equation,

$$f(nx, ny, -lx - my) = 0,$$

which is obtained by eliminating z with the help of the relation $lx + my + nz = 0$, from $f(x, y, z) = 0$, and which we may denote by $\phi(x, y) = 0$, has *two equal roots* $x : y$, if l, m, n be still the co-ordinates of a *tangent* to the curve f; an *equality* which obviously corresponds to the *coincidence* of *two intersections* of that line with that curve. Conversely, if we seek by the usual methods the *condition of equality* of two roots $x : y$ of the homogeneous equation of the p^{th} degree,

$$0 = \phi(x, y) = f(nx, ny, -lx - my),$$

by eliminating the ratio $x : y$ between the two *derived* homogeneous equations, $0 = \mathrm{D}_x\phi$, $0 = \mathrm{D}_y\phi$, we shall in general be conducted to a result of the *dimension* $2p(p-1)$ in l, m, n, and of the *form*,

$$0 = n^{p(p-1)}\,\mathrm{F}(l, m, n);$$

and so, by the rejection of the *foreign factor* $n^{p(p-1)}$, introduced by this *elimination*,* we shall obtain the *tangential equation* $\mathrm{F} = 0$, which will be in general of the *degree* $p(p-1)$; such being generally the known *class* (56) of the curve of which the *order* (46) is denoted by p: with (of course) a similar mode of passing, reciprocally, from a tangential to a local equation.

61. As an example, when the function f has the *cubic form* assigned in 54, we are thus led to investigate the condition for the existence of two equal roots in the *cubic equation*,

$$0 = \phi(x, y) = \{(n - l)\,x + (m - l)\,y\}^3 + 27n^2xy\,(lx + my),$$

* Compare the method employed in Salmon's *Higher Plane Curves*, page 98 [Art. 91, new ed.] to find the equation of the *reciprocal* of a given curve, with respect to the imaginary conic, $x^2 + y^2 + z^2 = 0$. In general, if the function F be deduced from f as above, then $\mathrm{F}(xyz) = 0$, and $f(xyz) = 0$ are equations of *two reciprocal curves.*

by eliminating $x : y$ between *two derived* and *quadratic equations;* and the result presents itself, in the first instance, as of the *twelfth dimension* in the tangential co-ordinates l, m, n; but it is found to be *divisible by* n^6, and when this division is effected, it is reduced to the *sixth degree,* thus appearing to imply that the curve is of the *sixth class,* as in fact the *general cubic* is well known to be. A *further reduction* is however possible in the present case, on account of the *conjugate point* O (54), which introduces (comp. 57) the *quadratic factor,*

$$(l + m + n)^2 = 0;$$

and when *this factor also* is *set aside,* the tangential equation is found to be *reduced to the biquadratic form** already assigned in 59; the *algebraic division,* last performed, corresponding to the known *geometric depression* of a *cubic curve* with a *double point,* from the *sixth* to the *fourth class.* But it is time to close this Section on *Plane Curves;* and to proceed, as in the next Chapter we propose to do, to the consideration and comparison of *vectors of points in space.*

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* If we multiply that form $F = 0$ (59) by z^2, and then change nz to $-lx - my$, we obtain a biquadratic equation in $l : m$, namely,

$$0 = \psi(l, m) = (l - m)^2 (lx + my)^2 + 2lm (l + m) (lx + my) z + l^2 m^2 z^2;$$

and if we then eliminate $l : m$ between the two derived cubics, $0 = D_l\psi$, $0 = D_m\psi$, we are conducted to the following equation of the twelfth degree, $0 = x^3y^3z^3 f(x, y, z)$, where f has the same cubic form as in 54. We are therefore thus brought *back* (comp. 59) from the *tangential* to the *local* equation of the cubic curve (54); complicated, however, as we see, with the *factor* $x^3y^3z^3$, which corresponds to the system of the three real tangents of inflexion to that curve, each tangent being taken three times. The reason why we have not here been obliged to reject *also* the foreign factor, z^{12}, as by the general theory (60) we might have expected to be, is that we multiplied the biquadratic function F *only* by z^2, and *not* by z^4.

CHAPTER III.

APPLICATIONS OF VECTORS TO SPACE.

SECTION 1.

On Linear Equations between Vectors not Complanar.

62. When three given and actual vectors OA, OB, OC, or α, β, γ, are *not* contained in any common plane, and when the three scalars a, b, c do not *all* vanish, then (by 21, 22) the expression $a\alpha + b\beta + c\gamma$ cannot become equal to *zero;* it must therefore represent *some actual vector* (1), which we may, for the sake of symmetry, denote by the symbol $-d\delta$: where the *new* (actual) *vector* δ, or OD, is not contained in any one of the three given and distinct planes, BOC, COA, AOB, unless some one, at least, of the three given coefficients a, b, c, vanishes; and where the *new scalar*, d, is either greater or less than zero. We shall thus have a *linear equation between four vectors,*

$$a\alpha + b\beta + c\gamma + d\delta = 0;$$

which will give

$$\delta = \frac{-a\alpha}{d} + \frac{-b\beta}{d} + \frac{-c\gamma}{d}, \quad \text{or} \quad \mathrm{OD} = \mathrm{OA}' + \mathrm{OB}' + \mathrm{OC}';$$

where OA′, OB′, OC′, or $\frac{-a\alpha}{d}$, $\frac{-b\beta}{d}$, $\frac{-c\gamma}{d}$, are the vectors of the three points A′, B′, C′, into which the point D is *projected, on the three given lines* OA, OB, OC, by planes drawn parallel to the three given planes, BOC, &c.; so that they are the three *co-initial edges* of a *parallelepiped,* whereof the *sum,* OD or δ, is the *internal* and *co-initial diagonal* (comp 6). Or we may project D *on the three planes,* by lines DA″, DB″, DC″ parallel to the three given lines, and then shall have

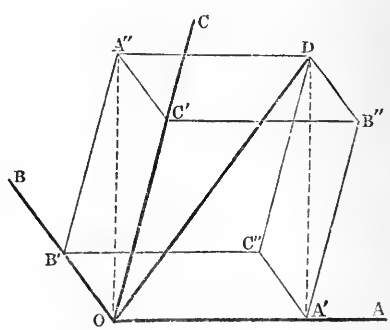

Fig. 28.

$$\mathrm{OA}'' = \mathrm{OB}' + \mathrm{OC}' = \frac{b\beta + c\gamma}{-d}, \text{ \&c., and } \delta = \mathrm{OD} = \mathrm{OA}' + \mathrm{OA}'' = \mathrm{OB}' + \mathrm{OB}'' = \mathrm{OC}' + \mathrm{OC}''.$$

And it is evident that this construction will apply to *any fifth point* D *of space*, if the *four* points OABC be still supposed to be *given*, and *not complanar:* but that some at least of the *three ratios* of the *four scalars* a, b, c, d (which last letter is not *here* used as a mark of *differentiation*) will *vary* with the *position* of the point D, or with the *value* of its vector δ. For example, we shall have $a = 0$, if D be situated in the plane BOC; and similarly for the two other given planes through O.

63. We may inquire (comp. 23), *what relation* between these scalar *coefficients* must exist, in order that the point D may be situated *in the fourth given plane* ABC; or what is the *condition of complanarity* of the *four points*, A, B, C, D. Since the *three vectors* DA, DB, DC are now supposed to be *complanar*, they must (by 22) be connected by a *linear equation*, of the form

$$a(\alpha - \delta) + b(\beta - \delta) + c(\gamma - \delta) = 0;$$

comparing which with the recent and more general form (62), we see that the required *condition* is,

$$a + b + c + d = 0.$$

This equation may be written (comp. again 23) as

$$\frac{-a}{d} + \frac{-b}{d} + \frac{-c}{d} = 1, \quad \text{or} \quad \frac{\text{OA}'}{\text{OA}} + \frac{\text{OB}'}{\text{OB}} + \frac{\text{OC}'}{\text{OC}} = 1;$$

and, under this last form, it expresses a known *geometrical property* of a *plane* ABCD, referred to three *co-ordinate axes* OA, OB, OC, which are drawn from any common *origin* O, and *terminate* upon the plane. We have also, in this *case of complanarity* (comp. 28), the following *proportion* of *coefficients* and *areas:*

$$a : b : c : -d = \text{DBC} : \text{DCA} : \text{DAB} : \text{ABC};$$

or, more symmetrically, with attention to *signs* of areas,

$$a : b : c : d = \text{BCD} : -\text{CDA} : \text{DAB} : -\text{ABC};$$

where fig. 18 may serve for illustration, if we conceive O in that figure to be replaced by D.

64. When we have thus at once the *two* equations,

$$a\alpha + b\beta + c\gamma + d\delta = 0, \quad \text{and} \quad a + b + c + d = 0,$$

so that the *four co-initial vectors*, α, β, γ, δ *terminate* (as above) *on one common plane*, and may therefore be said (comp. 24) to be *termino-complanar*, it is evident that the *two right lines*, DA and BC, which connect *two pairs* of the *four complanar points*, must *intersect* each other in some point A′ of the plane, at a finite or infinite distance. And there is no difficulty in perceiving, on the plan of 31, that the *vectors* of the *three points*, A′, B′, C′ of intersection, which thus result, are the following:

$$\left\{\begin{array}{ll} \text{for } A' = BC \cdot DA, & \alpha' = \dfrac{b\beta + c\gamma}{b + c} = \dfrac{a\alpha + d\delta}{a + d}; \\[2ex] \text{for } B' = CA \cdot DB, & \beta' = \dfrac{c\gamma + a\alpha}{c + a} = \dfrac{b\beta + d\delta}{b + d}; \\[2ex] \text{for } C' = AB \cdot DC, & \gamma' = \dfrac{a\alpha + b\beta}{a + b} = \dfrac{c\gamma + d\delta}{c + d}; \end{array}\right.$$

expressions which are *independent* of the *position* of the arbitrary *origin* O, and which accordingly coincide with the corresponding expressions in 27, when we place that origin in the point D, or make $\delta = 0$. Indeed, these last results hold good (comp. 31), even when the *four vectors*, α, β, γ, δ, or the *five points* O, A, B, C, D, are *all* complanar. For, although there *then* exist *two* linear equations between those four vectors, which may in general be written thus,

$$a'\alpha + b'\beta + c'\gamma + d'\delta = 0, \qquad a''\alpha + b''\beta + c''\gamma + d''\delta = 0,$$

without the relations, $a' + \&c. = 0$, $a'' + \&c. = 0$, between the coefficients, yet if we form from these *another* linear equation, of the form,

$$(a'' + ta')\,\alpha + (b'' + tb')\,\beta + (c'' + tc')\,\gamma + (d'' + td')\,\delta = 0,$$

and determine t by the condition,

$$t = -\frac{a'' + b'' + c'' + d''}{a' + b' + c' + d'},$$

we shall only have to make $a = a'' + ta'$, &c., and the two equations written at the commencement of the present article will then both be satisfied; and will conduct to the expressions assigned above, for the three vectors of intersection: which *vectors* may thus be found, without its being *necessary* to employ those processes of *scalar elimination*, which were treated of in the foregoing Chapter.

As an *Example*, let the two given equations be (comp. 27, 33),

$$a\alpha + b\beta + c\gamma = 0, \qquad (2a + b + c)\,\alpha''' - a\alpha = 0;$$

and let it be required to determine the vectors of the intersections of the three pairs of lines BC, AA‴; CA, BA‴; and AB, CA‴. Forming the combination,

$$(2a + b + c)\,\alpha''' - a\alpha + t\,(a\alpha + b\beta + c\gamma) = 0,$$

and determining t by the condition,

$$(2a + b + c) - a + t\,(a + b + c) = 0,$$

which gives $t = -1$, we have for the three sought vectors the expressions,

$$\frac{b\beta + c\gamma}{b + c}, \quad \frac{c\gamma + 2a\alpha}{c + 2a}, \quad \frac{2a\alpha + b\beta}{2a + b};$$

whereof the first $= \alpha'$, by 27. Accordingly, in fig. 21, the line AA‴ intersects BC in the point A′; and although the two *other* points of intersection here considered, which belong to what has been called (in 34) *a Third Construction*, are not marked in that figure, yet their *anharmonic symbols* (36), namely, (2, 0, 1) and (2, 1, 0), might have been otherwise found by combining the equations $y = 0$ and $x = 2z$ for the two lines CA, BA‴; and by combining $z = 0$, $x = 2y$ for the remaining pair of lines.

65. In the more general case, when the *four given points* A, B, C, D, are *not* in any *common plane*, let E be *any fifth given point* of space, not situated on any one of the *four faces* of the *given pyramid* ABCD, nor on any such face prolonged; and let its vector OE $= \varepsilon$. Then the *four co-initial vectors*, EA, EB, EC, ED, whereof (by supposition) no three are complanar, and which do not terminate upon one plane, must be (by 62) connected by some equation of the form

$$a\,.\,\mathrm{EA} + b\,.\,\mathrm{EB} + c\,.\,\mathrm{EC} + d\,.\,\mathrm{ED} = 0\,;$$

where the *four scalars*, a, b, c, d, and their *sum*, which we shall denote by $-e$, are *all different from zero*. Hence, because $\mathrm{EA} = \alpha - \varepsilon$, &c., we may establish the following *linear equation between five co-initial vectors*, $\alpha, \beta, \gamma, \delta, \varepsilon$, whereof *no four are termino-complanar* (64),

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon = 0\,;$$

with the *relation*, $a + b + c + d + e = 0$, between the *five scalars* a, b, c, d, e, whereof no one now separately vanishes. Hence also,

$$\varepsilon = (a\alpha + b\beta + c\gamma + d\delta) : (a + b + c + d), \text{ \&c.}$$

66. Under these conditions, if we write

$$\mathrm{D}_1 = \mathrm{DE} \cdot \mathrm{ABC}, \quad \text{and} \quad \mathrm{OD}_1 = \delta_1,$$

that is, if we denote by δ_1 the vector of the point D_1 in which the right line DE intersects the plane ABC, we shall have

$$\delta_1 = \frac{a\alpha + b\beta + c\gamma}{a + b + c} = \frac{d\delta + e\varepsilon}{d + e}.$$

In fact, these two expressions are *equivalent*, or represent one *common* vector, in virtue of the given equations; but the first shows (by 63) that this vector

δ_1 terminates on the *plane* ABC, and the second shows (by 25) that it terminates on the *line* DE; its extremity D_1 must therefore be, as required, the *intersection* of this line with that plane. We have therefore the two equations,

$$\text{I.} \ldots a(\alpha - \delta_1) + b(\beta - \delta_1) + c(\gamma - \delta_1) = 0;$$

$$\text{II.} \ldots d(\delta - \delta_1) + e(\varepsilon - \delta_1) = 0;$$

whence (by 28 and 24) follow the two proportions,

$$\text{I}'. \ldots a : b : c = \text{D}_1\text{BC} : \text{D}_1\text{CA} : \text{D}_1\text{AB};$$

$$\text{II}'. \ldots d : e = \text{ED}_1 : \text{D}_1\text{D};$$

the arrangement of the points, in the annexed fig. 29, answering to the case where all the four coefficients a, b, c, d are *positive* (or have one *common sign*), and when therefore the remaining coefficient e is *negative* (or has the *opposite* sign).

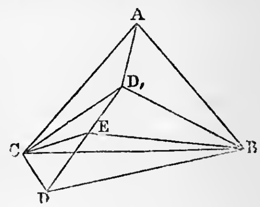

Fig. 29.

67. For the three *complanar triangles*, in the first proportion, we may substitute any three *pyramidal volumes*, which *rest* upon those triangles as their *bases*, and which have one *common vertex*, such as D or E; and because the collineation DED_1 gives $\text{DD}_1\text{BC} - \text{ED}_1\text{BC} = \text{DEBC}$, &c., we may write this other proportion,

$$\text{I}''. \ldots a : b : c = \text{DEBC} : \text{DECA} : \text{DEAB}.$$

Again, the same collineation gives

$$\text{ED}_1 : \text{DD}_1 = \text{EABC} : \text{DABC};$$

we have therefore, by II′., the proportion,

$$\text{II}''. \ldots d : -e = \text{EABC} : \text{DABC}.$$

But
$$\text{DEBC} + \text{DECA} + \text{DEAB} + \text{EABC} = \text{DABC},$$
and
$$a + b + c + d = -e;$$

we may therefore establish the following *fuller* formula of proportion, between *coefficients* and *volumes*:

$$\text{III.} \ldots a : b : c : d : -e = \text{DEBC} : \text{DECA} : \text{DEAB} : \text{EABC} : \text{DABC};$$

the *ratios* of all these five *pyramids* to each other being considered as *positive*, for the particular *arrangement* of the *points* which is represented in the recent figure.

68. The *formula* III. may however be regarded as perfectly *general*, if we agree to say that a *pyramidal volume changes sign*, or rather that it *changes its*

algebraical character, as *positive* or *negative*, in *comparison* with a *given* pyramid, and with a given *arrangement* of points, *in passing through zero* (comp. 28); namely *when*, in the course of any continuous change, *any one of its vertices crosses the corresponding base.* With this *convention** we shall have, generally,

$$\text{DABC} = -\,\text{ADBC} = \text{ABDC} = -\,\text{ABCD},\quad \text{DEBC} = \text{BCDE},\quad \text{DECA} = \text{CDEA};$$

the proportion III. may therefore be expressed in the following more *symmetric*, but equally *general form :*

$$\text{III}' \ldots\ a : b : c : d : e = \text{BCDE} : \text{CDEA} : \text{DEAB} : \text{EABC} : \text{ABCD};$$

the *sum* of *these five pyramids* being *always* equal to *zero*, when *signs* (as above) are attended to.

69. We saw (in 24) that the two equations,

$$a\alpha + b\beta + c\gamma = 0,\qquad a + b + c = 0,$$

gave the proportion of *segments*,

$$a : b : c = \text{BC} : \text{CA} : \text{AB},$$

whatever might be the position of the *origin* O. In like manner we saw (in 63) that the two other equations,

$$a\alpha + b\beta + c\gamma + d\delta = 0,\qquad a + b + c + d = 0,$$

gave the proportion of *areas*,

$$a : b : c : d = \text{BCD} : -\,\text{CDA} : \text{DAB} : -\,\text{ABC};$$

where again the origin is arbitrary. And we have just deduced (in 68) a corresponding proportion of *volumes* from the two analogous equations (65),

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon = 0,\qquad a + b + c + d + e = 0,$$

with an equally arbitrary origin. If then we conceive these *segments*, *areas*, and *volumes* to be replaced by the *scalars* to which they are thus *proportional*, we may establish the *three general formulæ :*

$$\text{I.}\quad \text{OA} \,.\, \text{BC} + \text{OB} \,.\, \text{CA} + \text{OC} \,.\, \text{AB} = 0;$$

$$\text{II.}\quad \text{OA} \,.\, \text{BCD} - \text{OB} \,.\, \text{CDA} + \text{OC} \,.\, \text{DAB} - \text{OD} \,.\, \text{ABC} = 0;$$

$$\text{III.}\quad \text{OA} \,.\, \text{BCDE} + \text{OB} \,.\, \text{CDEA} + \text{OC} \,.\, \text{DEAB} + \text{OD} \,.\, \text{EABC} + \text{OE} \,.\, \text{ABCD} = 0;$$

where in I., A, B, C are *any three collinear points;*
in II., A, B, C, D are *any four complanar points;*
and in III., A, B, C, D, E are *any five points of space;*

---

* Among the consequences of this convention respecting *signs* of *volumes*, which has already been adopted by some modern geometers, and which indeed is *necessary* (comp. 28) for the establishment of *general formulæ*, one is that *any two pyramids*, ABCD, A′B′C′D′, bear to each other a *positive* or a *negative ratio*, according as the two *rotations*, BCD and B′C′D′, supposed to be *seen* respectively from the points A and A′, have *similar* or *opposite directions*, as right-handed or left-handed.

while O is, in *each* of the three formulæ, an entirely *arbitrary point.* It must, however, be remembered, that the *additions* and *subtractions* are supposed to be performed according to the *rules of vectors,* as stated in the First Chapter of the present Book; the segments, or areas, or volumes, which the equations indicate, being treated as *coefficients* of those vectors. We might still further *abridge* the *notations,* while retaining the *meaning* of these formulæ, by *omitting the symbol* of the *arbitrary origin* O; and by thus writing,*

I′. $$\mathrm{A}\,.\,\mathrm{BC} + \mathrm{B}\,.\,\mathrm{CA} + \mathrm{C}\,.\,\mathrm{AB} = 0,$$

for any three collinear points; with corresponding formulæ II′. and III′., for any four complanar points, and for any five points of space.

SECTION 2.

On Quinary Symbols for Points and Planes in Space.

70. The equations of Art. 65 being still supposed to hold good, the vector ρ of any point P of space may, in indefinitely many ways, be expressed (comp. 36) under the form:

$$\text{I.}\ \ldots\ \mathrm{OP} = \rho = \frac{xa\alpha + yb\beta + zc\gamma + wd\delta + ve\varepsilon}{xa + yb + zc + wd + ve};$$

in which the *ratios of the differences* of the *five coefficients, xyzwv, determine the position of the point.* In fact, because the four points ABCD are not in any common plane, there necessarily exists (comp. 65) a determined *linear relation* between the *four vectors* drawn to them from the point P, which may be written thus,

$$x'a\,.\,\mathrm{PA} + y'b\,.\,\mathrm{PB} + z'c\,.\,\mathrm{PC} + w'd\,.\,\mathrm{PD} = 0,$$

giving the expression,

$$\text{II.}\ \ldots\ \rho = \frac{x'a\alpha + y'b\beta + z'c\gamma + w'd\delta}{x'a + y'b + z'c + w'd},$$

in which the *ratios* of the *four scalars* $x'y'z'w'$, depend upon, and conversely determine, the *position* of P; writing, then,

$$x = tx' + v, \qquad y = ty' + v, \qquad z = tz' + v, \qquad w = tw' + v,$$

where t and v are *two* new and *arbitrary* scalars, and remembering that $a\alpha + \ldots + e\varepsilon = 0$, and $a + \ldots + e = 0$ (65), we are conducted to the form for ρ, assigned above.

* We should thus have some of the *notations* of the *Barycentric Calculus,* but employed here with different *interpretations.*

71. When the *vector* ρ is thus expressed, the *point* P may be denoted by the *Quinary Symbol* (x, y, z, w, v); and we may write the *equation*,

$$\text{P} = (x, y, z, w, v).$$

But we see that the *same point* P may *also* be denoted by this *other symbol*, of the same kind, (x', y', z', w', v'), provided that the following *proportion* between *differences of coefficients* (70) holds good:

$$x' - v' : y' - v' : z' - v' : w' - v' = x - v : y - v : z - v : w - v.$$

Under this condition, we shall therefore write the following *formula of congruence*,

$$(x', y', z', w', v') \equiv (x, y, z, w, v),$$

to express that these *two quinary symbols*, although *not identical in composition*, have yet the *same geometrical signification*, or denote one *common point*. And we shall reserve the symbolic *equation*,

$$(x', y', z', w', v') = (x, y, z, w, v),$$

to express that the *five coefficients*, $x' \ldots v'$, of the one symbol, are *separately equal* to the *corresponding* coefficients of the other, $x' = x, \ldots v' = v$.

72. Writing also, generally,

$$(tx, ty, tz, tw, tv) = t\,(x, y, z, w, v),$$

$$(x' + x, \ldots v' + v) = (x', \ldots v') + (x, \ldots v), \text{ \&c.},$$

and abridging the particular symbol* $(1, 1, 1, 1, 1)$ to (U), while (Q), (Q'), .. may briefly denote the quinary symbols $(x, \ldots v)$, $(x', \ldots v')$, .. we may thus establish the *congruence* (71),

$$(Q') \equiv (Q), \text{ if } (Q) = t\,(Q') + u\,(U);$$

in which t and u are arbitrary coefficients. For example,

$$(0, 0, 0, 0, 1) \equiv (1, 1, 1, 1, 0), \quad \text{and} \quad (0, 0, 0, 1, 1) \equiv (1, 1, 1, 0, 0);$$

each symbol of the first pair denoting (65) the given point E; and each symbol of the second pair denoting (66) the derived point D_1. When the coefficients are *so simple* as in these last expressions, we may occasionally *omit the commas*, and thus write, still more briefly,

$$(00001) \equiv (11110); \qquad (00011) \equiv (11100).$$

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* *This quinary symbol* (U) denotes *no determined point*, since it corresponds (by 70, 71) to the *indeterminate vector* $\rho = \frac{0}{0}$; but it admits of useful *combinations* with *other* quinary symbols, as above.

73. If *three vectors*, ρ, ρ', ρ'', expressed each under the *first form* (70), be *termino-collinear* (24) and if we denote their denominators, $xa + \ldots$, $x'a + \ldots$, $x''a + \ldots$, by m, m', m'', they must then (23) be connected by a *linear equation* with a *null sum* of coefficients, which may be written thus:

$$tm\rho + t'm'\rho' + t''m''\rho'' = 0; \qquad tm + t'm' + t''m'' = 0.$$

We have, therefore, the two *equations of condition*,

$$t(xa\alpha + \ldots + ve\varepsilon) + t'(x'a\alpha + \ldots + v'e\varepsilon) + t''(x''a\alpha + \ldots + v''e\varepsilon) = 0;$$
$$t(xa + \ldots + ve) + t'(x'a + \ldots + v'e) + t''(x''a + \ldots + v''e) = 0;$$

where t, t', t'' are three new scalars, while the five vectors $\alpha \ldots \varepsilon$, and the five scalars $a \ldots e$, are subject only to the two equations (65): but these equations of condition are satisfied by supposing that

$$tx + t'x' + t''x'' = \ldots = tv + t'v' + t''v'' = -u,$$

where u is some new scalar, and they cannot be satisfied otherwise. Hence the *condition of collinearity* of the *three points* P, P′, P″, in which the three vectors ρ, ρ', ρ'' terminate, and of which the quinary symbols are (Q), (Q'), (Q''), may briefly be expressed by the equation,

$$t(Q) + t'(Q') + t''(Q'') = -u(U);$$

so that *if any four scalars, t, t', t'', u, can be found, which satisfy this last symbolic equation*, then, but not in any other case, *those three points* PP′P″ *are ranged on one right line*. For example, the three points D, E, D_1, which are denoted (72) by the quinary symbols, (00010), (00001), (11100), are *collinear;* because the *sum* of these three *symbols* is (U). And if we have the equation,

$$(Q'') = t(Q) + t'(Q') + u(U),$$

where t, t', u are *any three scalars*, then (Q'') is a symbol for a point P″, *on the right line* PP′. For example, the symbol $(0, 0, 0, t, t')$ may denote *any point on the line* DE.

74. By reasonings precisely similar it may be proved, that if (Q) (Q') (Q'') (Q''') be quinary symbols for any *four points* PP′P″P‴ in any *common plane*, so that the *four vectors* $\rho\rho'\rho''\rho'''$ are *termino-complanar* (64), then an equation, of the form

$$t(Q) + t'(Q') + t''(Q'') + t'''(Q''') = -u(U),$$

must hold good; and conversely, that if the *fourth symbol* can be expressed as follows,

$$(Q''') = t(Q) + t'(Q') + t''(Q'') + u(U),$$

with any scalar values of t, t', t'', u, then the *fourth point* P‴ is situated *in the plane* PP′P″ of the *other three*. For example, the four points,

$$(10000), \quad (01000), \quad (00100), \quad (11100),$$

or A, B, C, D_1 (66), are *complanar;* and the symbol $(t, t', t'', 0, 0)$ may represent *any point in the plane* ABC.

75. When a point P is thus *complanar with three given points*, P_0, P_1, P_2, we have therefore expressions of the following forms, for the *five coefficients* $x, .. v$ of its quinary symbol, in terms of the *fifteen* given coefficients of *their* symbols, and of *four* new and arbitrary scalars:

$$x = t_0x_0 + t_1x_1 + t_2x_2 + u; \ldots \quad v = t_0v_0 + t_1v_1 + t_2v_2 + u.$$

And hence, by *elimination* of these four scalars, $t_0 .. u$, we are conducted to a *linear equation* of the form

$$l(x - v) + m(y - v) + n(z - v) + r(w - v) = 0,$$

which may be called the *Quinary Equation of the Plane* $P_0P_1P_2$, or of the supposed *locus of the point* P: because it expresses a *common property* of all the points of that locus; and because the *three ratios* of the *four new coefficients* l, m, n, r, *determine the position of the plane* in space. It is, however, more *symmetrical*, to write the quinary equation of a plane Π as follows,

$$lx + my + nz + rw + sv = 0,$$

where the *fifth coefficient*, s, is connected with the others by the relation,

$$l + m + n + r + s = 0;$$

and then we may say that $[l, m, n, r, s]$ is (comp. 37) the *Quinary Symbol of the Plane* Π, and may write the *equation*,

$$\Pi = [l, m, n, r, s].$$

For example, the coefficients of the symbol for a point P in the plane ABC may be thus expressed (comp. 74):

$$x = t_0 + u, \quad y = t_1 + u, \quad z = t_2 + u, \quad w = u, \quad v = u;$$

between which the only relation, *independent of the four arbitrary scalars* $t_0 .. u$, is $w - v = 0$; this therefore is the *equation* of the plane ABC, and the *symbol* of that plane is $[0, 0, 0, 1, -1]$; which may (comp. 72) be sometimes written more briefly, without commas, as $[0001\bar{1}]$. It is evident that, in any such symbol, the *coefficients* may all be multiplied by any *common factor*.

76. The symbol of the *plane* $P_0P_1P_2$ having been thus determined, we may next propose to find a symbol for the *point*, P, in which that plane is *intersected* by a given *line* P_3P_4: or to *determine the coefficients* $x \,..\, v$, or at least the *ratios* of their *differences* (70), in the quinary symbol of that point,

$$(x, y, z, w, v) = P = P_0P_1P_2 \cdot P_3P_4.$$

Combining, for this purpose, the expressions,

$$x = t_3x_3 + t_4x_4 + u', \,.\,. \quad v = t_3v_3 + t_4v_4 + u',$$

(which are included in the symbolical equation (73),

$$(Q) = t_3\,(Q_3) + t_4\,(Q_4) + u'\,(U),$$

and express the *collinearity* PP_3P_4), with the equations (75),

$$lx + \,.\,.\, + sv = 0, \qquad l + \,.\,.\, + s = 0,$$

(which express the *complanarity* $PP_0P_1P_2$), we are conducted to the formula,

$$t_3\,(lx_3 + \,.\,.\, + sv_3) + t_4\,(lx_4 + \,.\,.\, + sv_4) = 0\,;$$

which determines the *ratio* $t_3 : t_4$, and contains the solution of the problem. For example, if P be a point *on the line* DE, then (comp. 73),

$$x = y = z = u', \qquad w = t_3 + u', \qquad v = t_4 + u'\,;$$

but if it be *also* a point *in the plane* ABC, then $w - v = 0$ (75), and therefore $t_3 - t_4 = 0$; hence

$$(Q) = t_3\,(00011) + u'\,(11111), \quad \text{or} \quad (Q) \equiv (00011)\,;$$

which last symbol had accordingly been found (72) to represent the *intersection* (66), $D_1 = ABC \cdot DE$.

77. When the five coefficients, *xyzwv*, of any given quinary *symbol* (*Q*) for a point P, or those of any *congruent* symbol (71), are any *whole numbers* (positive or negative, or zero), we shall say (comp. 42) that the *point* P is *rationally related to the five given points*, A . . E; or briefly, that it is a *Rational Point of the System*, which those five points determine. And in like manner, when the five coefficients, *lmnrs*, of the quinary symbol (75) of a *plane* Π are either *equal* or *proportional* to *integers*, we shall say that the plane is a *Rational Plane* of the same *System;* or that it is *rationally related* to the same five points. On the contrary, when the quinary symbol of a point, or of a plane, has *not* thus already *whole coefficients*, and cannot be *transformed* (comp. 72) so as to have them, we shall say that the point or plane is *irrationally related* to the given points; or briefly, that it is *irrational.* A *right line* which *connects* two

rational points, or is the *intersection* of two *rational planes*, may be called, on the same plan, a *Rational Line;* and lines which cannot in either of these two ways be constructed, may be said by contrast to be *Irrational Lines.* It is evident from the nature of the *eliminations* employed (comp. again 42), that a *plane*, which is *determined* as *containing three rational points*, is necessarily a *rational plane;* and in like manner, that a *point*, which is determined as the common *intersection of three rational planes*, is always a *rational point:* as is also every point which is obtained by the intersection of a *rational line* with a *rational plane;* or of *two rational lines* with *each other* (when they happen to be complanar).

78. Finally, when *two points*, or *two planes*, differ only by the *arrangement* (or *order*) of the *coefficients* in their quinary *symbols*, those points or planes may be said to have one *common type;* or briefly to be *syntypical.* For example, the *five given points*, A, . . E, are thus syntypical, as being represented by the quinary symbols (10000), . . (00001); and the *ten planes*, obtained by taking all the *ternary combinations* of those five points, have in like manner one common type. Thus, the quinary symbol of the plane ABC has been seen (75) to be $[0001\bar{1}]$; and the analogous symbol $[1\bar{1}000]$ represents the plane CDE, &c. Other examples will present themselves, in a shortly subsequent Section, on the subject of *Nets in Space.* But it seems proper to say here a few words, respecting those *Anharmonic Co-ordinates, Equations, Symbols,* and *Types*, for *Space*, which are obtained from the theory and expressions of the present Section, by *reducing* (as we are allowed to do) the *number* of the *coefficients*, in each symbol or equation, from *five to four*.

SECTION 3.

On Anharmonic Co-ordinates in Space.

79. When we adopt the *second form* (70) for ρ, or suppose (as we may) that the *fifth coefficient* in the *first* form *vanishes*, we get this *other general expression* (comp. 34, 36), for the *vector of a point in space:*

$$\text{OP} = \rho = \frac{xa\alpha + yb\beta + zc\gamma + wd\delta}{xa + yb + zc + wd};$$

and may then write the symbolic *equation* (comp. 36, 71),

$$\text{P} = (x, y, z, w),$$

and call this last the *Quaternary Symbol of the Point* P: although we shall soon see cause for calling it also the *Anharmonic Symbol* of that point. Meanwhile we may remark, that the *only congruent symbols* (71), of this *last form*, are those which differ merely by the introduction of a *common factor*: the *three ratios* of the *four coefficients*, $x \,.\,.\, w$, being *all* required, in order to *determine the position of the point*; whereof those four coefficients may accordingly be said (comp. 36) to be the *Anharmonic Co-ordinates in Space.*

80. When we thus suppose that $v = 0$, in the quinary *symbol* of the *point* P, we may *suppress the fifth term* sv, in the quinary *equation* of a *plane* Π, $lx + \,.\,.\, + sv = 0$ (75); and therefore may *suppress* also (as here unnecessary) the *fifth coefficient*, s, in the *quinary symbol* of that plane, which is thus reduced to the *quaternary form*,

$$\Pi = [l, m, n, r].$$

This last may also be said (37, 79), to be the *Anharmonic Symbol of the Plane*, of which the *Anharmonic Equation* is

$$lx + my + nz + rw = 0;$$

the *four coefficients*, $lmnr$, which we shall call also (comp. again 37) the *Anharmonic Co-ordinates of that Plane* Π, being not connected *among themselves* by any *general relation* (such as $l + \,.\,.\, + s = 0$): since their *three ratios* (comp. 79) are *all* in general necessary, in order to determine the *position of the plane in space.*

81. If we suppose that the *fourth coefficient*, w, *also vanishes*, in the recent symbol of a point, that *point* P is *in the plane* ABC; and may then be sufficiently represented (as in 36) by the *Ternary Symbol* (x, y, z). And if we attend only to the points in which an *arbitrary plane* Π *intersects* the *given plane* ABC, we may *suppress its fourth coefficient*, r, as being for *such* points unnecessary. In this manner, then, we are reconducted to the *equation*, $lx + my + nz = 0$, and to the *symbol*, $\Lambda = [l, m, n]$, for a *right line* (37) *in the plane* ABC, considered here as the *trace*, on that plane, of an *arbitrary plane* Π *in space.* If this plane Π be given by its *quinary symbol* (75), we thus obtain the *ternary symbol* for its *trace* Λ, by simply *suppressing the two last coefficients*, r and s.

82. In the more general case, when the point P is *not* confined to the plane ABC, if we denote (comp. 72) its *quaternary symbol* by (Q), the lately established formulæ of *collineation* and *complanarity* (73, 74) will still hold good: provided that we now *suppress* the symbol (U), or suppose its *coefficient* to be *zero.* Thus, the formula,

$$(Q) = t'(Q') + t''(Q'') + t'''(Q'''),$$

expresses that the point P is *in the plane* P′P″P‴; and if the coefficient t''' *vanish*, the equation which then remains, namely,

$$(Q) = t'(Q') + t''(Q''),$$

signifies that P is thus *complanar* with the *two given points* P′, P″, and with an *arbitrary third point;* or, in other words, that it is *on the right line* P′P″ whence (comp. 76) problems of *intersections* of *lines* with *planes* can easily be resolved. In like manner, if we denote briefly by $[R]$ the quaternary symbol $[l, m, n, r]$ for a *plane* Π, the formula

$$[R] = t'[R'] + t''[R''] + t'''[R''']$$

expresses that the plane Π passes *through the intersection* of the *three planes* Π′, Π″, Π‴; and if we suppose $t''' = 0$, so that

$$[R] = t'[R'] + t''[R''],$$

the formula thus found denotes that the plane Π passes through the point of intersection of the *two* planes, Π′, Π″, with *any third plane;* or (comp. 41), that this plane Π *contains the line of intersection* of Π′, Π″; in which case the *three planes*, Π, Π′, Π″, may be said to be *collinear*. Hence it appears that either of the *two expressions*,

$$\text{I.} \ldots t'(Q') + t''(Q''), \qquad \text{II.} \ldots t'[R'] + t''[R''],$$

may be used as a *Symbol of a Right Line in Space:* according as we consider that *line* Λ either, Ist, as *connecting two given points*, or IInd, as being the *intersection of two given planes*. The remarks (77) on *rational* and *irrational points, planes*, and *lines* require no modification here; and those on *types* (78) adapt themselves as easily to *quaternary* as to *quinary* symbols.

83. From the foregoing general formulæ of collineation and complanarity, it follows that the point P′, in which the line AB intersects the plane CDP through CD and any proposed point $P = (xyzw)$ of space, may be denoted thus:

$$P' = AB \cdot CDP = (xy00);$$

for example, $E = (1111)$, and $C' = AB \cdot CDE = (1100)$. In general, if ABCDEF be *any six points of space*, the *four collinear planes* (82), ABC, ABD, ABE, ABF, are said to form a *pencil* through AB; and if this be *cut* by any rectilinear *transversal*, in *four points*, C′, D′, E′, F′, then (comp. 35) the *anharmonic function* of this *group* of *points* (25) is called also the *Anharmonic of the Pencil of Planes:* which may be thus denoted,

$$(AB \,.\, CDEF) = (C'D'E'F').$$

Hence (comp. again 25, 35), by what has just been shown respecting C′ and P′, we may establish the important formula:

$$(\text{CD} \,.\, \text{AEBP}) = (\text{AC}'\text{BP}') = \frac{x}{y};$$

so that this *ratio of coefficients*, in the symbol $(xyzw)$ for a *variable point* P (79), represents the *anharmonic of a pencil of planes*, of which the *variable plane* CDP is *one;* the *three other* planes of this pencil being *given*. In like manner,

$$(\text{AD} \,.\, \text{BECP}) = \frac{y}{z}, \quad \text{and} \quad (\text{BD} \,.\, \text{CEAP}) = \frac{z}{x};$$

so that (comp. 36) the *product* of these three last anharmonics is *unity*. On the same plan we have also,

$$(\text{BC} \,.\, \text{AEDP}) = \frac{x}{w}, \qquad (\text{CA} \,.\, \text{BEDP}) = \frac{y}{w}, \qquad (\text{AB} \,.\, \text{CEDP}) = \frac{z}{w};$$

so that the *three ratios*, of the three first coefficients xyz to the fourth coefficient w, suffice to *determine the three planes*, BCP, CAP, ABP, whereof the *point* P is the common *intersection*, by means of the *anharmonics of three pencils of planes*, to which the three planes respectively belong. And thus we see a *motive* (besides that of *analogy* to expressions already used for *points in a given plane*), for calling the *four coefficients*, $xyzw$, in the *quaternary symbol* (79) for a *point in space*, the *Anharmonic Co-ordinates of that Point*.

84. In general, if there be *any four collinear points*, $\text{P}_0, \ldots \text{P}_3$, so that (comp. 82) their *symbols* are connected *by two linear equations*, such as the following,

$$(Q_1) = t\,(Q_0) + u\,(Q_2), \qquad (Q_3) = t'\,(Q_0) + u'\,(Q_2),$$

then the anharmonic of their *group* may be expressed (comp. 25, 44) as follows:

$$(\text{P}_0\text{P}_1\text{P}_2\text{P}_3) = \frac{ut'}{tu'};$$

as appears by considering the *pencil* $(\text{CD} \,.\, \text{P}_0\text{P}_1\text{P}_2\text{P}_3)$, and the *transversal* AB (83). And in like manner, if we have (comp. again 82) the two other symbolic equations, connecting *four collinear planes* $\Pi_0 \ldots \Pi_3$,

$$[R_1] = t\,[R_0] + u\,[R_2], \qquad [R_3] = t'\,[R_0] + u'\,[R_2],$$

the anharmonic of their *pencil* (83) is expressed by the precisely similar formula,

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = \frac{ut'}{tu'};$$

as may be proved by supposing the pencil to be cut by the same transversal line AB.

85. It follows that if $f(xyzw)$ and $f_1(xyzw)$ be any two homogeneous and linear functions of x, y, z, w; and if we determine four collinear planes $\Pi_0 \ldots \Pi_3$ (82), by the four equations,

$$f = 0, \qquad f_1 = f, \qquad f_1 = 0, \qquad f_1 = kf,$$

where k is any scalar; we shall have the following value of the anharmonic function, of the pencil of planes thus determined:

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = k = \frac{f_1}{f}.$$

Hence we derive this *Theorem*, which is important in the application of the present system of co-ordinates to space:—

"*The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates* (79) *of a variable Point* P *in space, may be expressed as the Anharmonic* $(\Pi_0\Pi_1\Pi_2\Pi_3)$ *of a Pencil of Planes;* whereof *three* are *given*, while the *fourth* passes *through the variable point* P, and *through a given right line* A which is *common* to the *three former planes*."

86. And in like manner may be proved this other but analogous Theorem:—

"*The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates* (80) *of a variable Plane* Π, *may be expressed as the Anharmonic* $(P_0P_1P_2P_3)$ *of a Group of Points;* whereof *three* are *given* and *collinear;* and the *fourth* is the *intersection*, $\Lambda \cdot \Pi$, of their *common* and *given right line* Λ, *with the variable plane* Π."

More fully, if the two given functions of $lmnr$ be F and F_1, and if we determine three points $P_0P_1P_2$ by the equations (comp. 57) $F = 0$, $F_1 = F$, $F_1 = 0$, and denote by P_3 the intersection of their common line Λ with Π, we shall have the quotient,

$$\frac{F_1}{F} = (P_0P_1P_2P_3).$$

For example, if we suppose that

$$A_2 = (1001), \qquad B_2 = (0101), \qquad C_2 = (0011),$$
$$A'_2 = (100\bar{1}), \qquad B'_2 = (010\bar{1}), \qquad C'_2 = (001\bar{1}),$$

so that $\quad A_2 = DA \cdot BCE$, &c., and $(DA_2AA'_2) = -1$, &c.,

we find that the three ratios of l, m, n to r, in the symbol $\Pi = [lmnr]$, may be expressed (comp. 39) under the form of anharmonics of groups, as follows:

$$\frac{l}{r} = (DA'_2AQ); \qquad \frac{m}{r} = (DB'_2BR); \qquad \frac{n}{r} = (DC'_2CS);$$

where Q, R, S denote the intersections of the plane Π with the three given right lines, DA, DB, DC. And thus we have a *motive* (comp. 83) besides that of *analogy* to *lines* in a given plane (37), for calling (as above) the *four coefficients* l, m, n, r, in the *quaternary symbol* (80) for a *plane* Π, the *Anharmonic Co-ordinates of that Plane in Space.*

87. It may be added, that if we denote by L, M, N the points in which the same plane Π is cut by the three given lines BC, CA, AB, and retain the notations A″, B″, C″ for those other points on the same three lines which were so marked before (in 31, &c.), so that we may now write (comp. 36)

$$\mathrm{A}'' = (01\bar{1}0), \qquad \mathrm{B}'' = (\bar{1}010), \qquad \mathrm{C}'' = (1\bar{1}00),$$

we shall have (comp. 39, 83) these three other anharmonics of groups, with their product equal to unity:

$$\frac{m}{n} = (\mathrm{CA}''\mathrm{BL})\,; \quad \frac{n}{l} = (\mathrm{AB}''\mathrm{CM})\,; \quad \frac{l}{m} = (\mathrm{BC}''\mathrm{AN})\,;$$

and the *six given points*, A″, B″, C″, A'_2, B'_2, C'_2, are all *in one given plane* [E], of which the *equation* and *symbol* are:

$$x + y + z + w = 0\,; \qquad [\mathrm{E}] = [1111].$$

The *six groups* of points, of which the anharmonic functions thus represent the *six ratios* of the four anharmonic co-ordinates, *lmnr*, of a *variable plane* Π, are therefore situated *on the six edges* of the *given pyramid*, ABCD; *two points* in each group being *corners* of that pyramid, and the *two others* being the *intersections* of the *edge* with the *two planes*, [E] and Π. Finally, the *plane* [E] is (in a known modern sense) the *plane of homology*,* and the point E is the *centre of homology*, of the *given* pyramid ABCD, and of an *inscribed pyramid* $\mathrm{A_1B_1C_1D_1}$, where $\mathrm{A_1 = EA \cdot BCD}$, &c.; so that $\mathrm{D_1}$ retains its recent signification (66, 76), and we may write the anharmonic symbols,

$$\mathrm{A_1} = (0111), \qquad \mathrm{B_1} = (1011), \qquad \mathrm{C_1} = (1101), \qquad \mathrm{D_1} = (1110).$$

And if we denote by $\mathrm{A'_1B'_1C'_1D'_1}$ the harmonic conjugates to these last points, with respect to the lines EA, EB, EC, ED, so that

$$(\mathrm{EA_1AA'_1}) = \ldots = (\mathrm{ED_1DD'_1}) = -1,$$

we have the corresponding symbols,

$$\mathrm{A'_1} = (2111), \qquad \mathrm{B'_1} = (1211), \qquad \mathrm{C'_1} = (1121), \qquad \mathrm{D'_1} = (1112).$$

Many other relations of position exist, between these various points, lines, and planes, of which some will come naturally to be noticed, in that theory of *nets in space* to which in the following Section we shall proceed.

* See Poncelet's *Traité des Propriétés Projectives* (Paris, 1822).

SECTION 4.

On Geometrical Nets in Space.

88. When we have (as in 65) five given points A . . E, whereof no four are complanar, we can *connect* any *two* of them by a right *line*, and the *three* others by a *plane*, and determine the *point* in which these last *intersect* one another: *deriving* thus a system of *ten lines* Λ_1, *ten planes* Π_1, and *ten points* P_1, from the *given* system of *five points* P_0, by what may be called (comp. 34) a *First Construction.* We may next propose to determine all the *new* and *distinct lines*, Λ_2, and *planes*, Π_2, which connect the ten derived points P_1 with the five given points P_0, and with each other; and may then inquire what *new* and *distinct points* P_2 arise (at this stage) as intersections of *lines with planes*, or of *lines in one plane* with *each other:* all *such* new lines, planes, and points being said (comp. again 34) to belong to a *Second Construction.* And then we might proceed to a *Third Construction* of the same kind, and so on for ever: building up thus what has been called* a *Geometrical Net in Space.* To *express* this geometrical process by *quinary symbols* (71, 75, 82) of *points*, *planes*, and *lines*, and by *quinary types* (78), so far at least as to the end of the *second construction*, will be found to be an useful exercise in the application of principles lately established: and therefore ultimately in that METHOD OF VECTORS, which is the subject of the present Book. And the *quinary form* will *here* be more convenient than the *quaternary*, because it will exhibit more clearly the geometrical dependence of the *derived* points and planes on the *five* given points, and will thereby enable us, through a principle of *symmetry*, to *reduce* the *number* of distinct *types*.

89. Of the *five given points*, P_0, the *quinary type* has been seen (78) to be (10000); while of the *ten derived points* P_1, of *first* construction, the corresponding *type* may be taken as (00011); in fact, considered as *symbols*, these two represent the points A and D_1. The nine other points P_1 are $A'B'C'A_1B_1C_1A_2B_2C_2$; and we have now (comp. 83, 87, 86) the symbols,

$$A' = BC \cdot ADE = (01100), \qquad A_1 = EA \cdot BCD = (10001),$$

$$A_2 = DA \cdot BCE = (10010);$$

also, in any symbol or equation of the present form, it is permitted to change A, B, C to B, C, A, provided that we at the same time write the third, first,

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* By Möbius, in p. 291 of his already cited *Barycentric Calculus.*

and second co-efficients, in the places of the first, second, and third: thus, B$'$ = CA · BDE = (10100), &c. The symbol $(xy000)$ represents an *arbitrary point on the line* AB; and the symbol $[00nrs]$, with $n + r + s = 0$, represents an *arbitrary plane through that line: each* therefore may be regarded (comp. 82) as a *symbol* also of the *line* AB *itself*, and at the same time as a *type* of the *ten lines* Λ_1; while the symbol $[0001\bar{1}]$, of the plane ABC (75), may be taken (78) as a type of the *ten planes* Π_1. Finally, the *five pyramids*,

BCDE, CADE, ABDE, ABCE, ABCD,

and the *ten triangles*, such as ABC, whereof each is a *common face* of *two* such pyramids, may be called *pyramids* R_1, and *triangles* T_1, of the *First Construction.*

90. Proceeding to a *Second Construction* (88), we soon find that the *lines* Λ_2 may be arranged in *two distinct groups; one* group consisting of *fifteen lines* $\Lambda_{2,1}$, such as the line* AA$'$D$_1$, whereof each *connects two points* P$_1$, and passes also *through one point* P$_0$, being the *intersection* of *two planes* Π_1 through that point, as here of ABC, ADE; while the *other group* consists of *thirty lines* $\Lambda_{2,2}$, such as B$'$C$'$, each connecting two points P$_1$, but *not* passing through any point P$_0$, and being one of the *thirty edges* of *five new pyramids* R_2, namely,

C$'$B$'$A$_2$A$_1$, A$'$C$'$B$_2$B$_1$, B$'$A$'$C$_2$C$_1$, A$_2$B$_2$C$_2$D$_1$, A$_1$B$_1$C$_1$D$_1$:

which pyramids R_2 may be said (comp. 87) to be *inscribed homologues* of the five former pyramids R_1, the *centres of homology* for these *five pairs of pyramids* being the five given points A . . E; and the *planes of homology* being five planes [A] . . [E], whereof the last has been already mentioned (87), but which belong properly to a *third* construction (88). The *planes* Π_2, of *second* construction, form in like manner *two groups; one* consisting of *fifteen planes* $\Pi_{2,1}$, such as the plane of the *five points*, AB$_1$B$_2$C$_1$C$_2$, whereof each passes through *one point* P$_0$, and through *four points* P$_1$, and contains *two lines* $\Lambda_{2,1}$, as here the lines AB$_1$C$_2$, AC$_1$B$_2$, besides containing *four lines* $\Lambda_{2,2}$, as here B$_1$B$_2$, &c.; while the *other group* is composed of *twenty planes* $\Pi_{2,2}$, such as A$_1$B$_1$C$_1$, namely, the *twenty faces* of the five recent pyramids R_2, whereof each contains *three points* P$_1$, and *three lines* $\Lambda_{2,2}$, but does not pass through any point P$_0$. It is now required to *express* these *geometrical conceptions*† of the *forty-five lines* Λ_2; the *thirty-five planes* Π_2; and the *five planes of homology* of pyramids, [A] . . . [E], by

* AB$_1$C$_2$, AB$_2$C$_1$, DA$'$A$_1$, EA$'$A$_2$, are other lines of this group.

† Möbius (in his *Barycentric Calculus*, p. 284, &c.) has very clearly pointed out the existence and chief properties of the foregoing *lines* and *planes;* but besides that his *analysis* is altogether different from ours, he does not appear to have aimed at *enumerating*, or even at *classifying*, all the *points* of what has been above called (88) the *second construction*, as we propose shortly to do.

quinary *symbols* and *types*, before proceeding to determine the *points* P_2 of *second* construction.

91. An arbitrary *point* on the right line $AA'D_1$ (90) may be represented by the symbol $(tuu00)$; and an arbitrary *plane* through that line by this other symbol, $[0m\bar{m}r\bar{r}]$, where $\bar{m}$ and $\bar{r}$ are written (to save commas) instead of $-m$ and $-r$; hence these two symbols may also (comp. 82) denote the *line* $AA'D_1$ itself, and may be used as types (78) to represent the *group* of lines $\Lambda_{2,1}$. The particular symbol $[01\bar{1}1\bar{1}]$, of the last form, represents that particular plane through the last-mentioned line, which contains also the line AB_1C_2 of the same group; and may serve as a type for the group of planes $\Pi_{2,1}$. The line $B'C'$, and the group $\Lambda_{2,2}$, may be represented by $(stu00)$ and $[\bar{t}ttu\bar{s}]$, if we agree* to write $s = t + u$, and $\bar{s} = -s$; while the plane $B'C'A_2$, and the group $\Pi_{2,2}$, may be denoted by $[\bar{1}111\bar{2}]$. Finally, the plane [E] has for its symbol $[1111\bar{4}]$; and the four other planes [A], &c., of homology of pyramids (90), have this last for their common type.

92. The *points* P_2, of *second* construction (88), are more numerous than the *lines* Λ_2 and *planes* Π_2 of that construction: yet with the help of *types*, as above, it is not difficult to classify and to enumerate them. It will be sufficient here to write down these types, which are found to be *eight*, and to offer some remarks respecting them; in doing which we shall avail ourselves of the eight following *typical points*, whereof the two first have already occurred, and which are all situated in the plane of ABC:

$$A'' = (01\bar{1}00); \quad A''' = (21100); \quad A^{IV} = (\bar{2}1100); \quad A^{V} = (02100);$$

$$A^{VI} = (02\bar{1}00); \quad A^{VII} = (12\bar{1}00); \quad A^{VIII} = (32100); \quad A^{IX} = (23\bar{1}00);$$

the second and third of these having $(\bar{1}0011)$ and (30011) for *congruent symbols* (71). It is easy to see that these *eight types* represent, respectively, ten, thirty, thirty, twenty, twenty, sixty, sixty, and sixty distinct points, belonging to *eight groups*, which we shall mark as $P_{2,1}, \ldots P_{2,8}$; so that the total number of the points P_2 is 290. If then we consent (88) to *close* the present inquiry, at the end of what we have above *defined* to be the *Second Construction*, the *total number of* the *net points*, P_1, P_2, which are thus *derived* by *lines* and *planes* from the *five given points* P_0, is found to be exactly *three hundred:* while the *joint* number of the *net-lines*, Λ_1, Λ_2, and of the *net-planes*, Π_1, Π_2, has been seen to be *one hundred*, so far.

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* With this convention, the line AB, and the group Λ_1, may be denoted by the *plane-symbol* $[00tu\bar{s}]$, their *point-symbol* being $(tu000)$.

(1.) To the type $P_{2,1}$ belong the *ten points*,

$$A''B''C'', \quad A'_2B'_2C'_2, \quad A'_1B'_1C'_1D'_1,$$

with the quinary symbols,

$$A'' = (01\bar{1}00), \ldots \quad A'_2 = (100\bar{1}0), \ldots \quad A'_1 = (1000\bar{1}), \ldots \quad D'_1 = (0001\bar{1}),$$

which are the *harmonic conjugates* of the ten points P_1, namely, of

$$A'B'C', \quad A_2B_2C_2, \quad A_1B_1C_1D_1,$$

with respect to the ten lines Λ_1, on which those points are situated; so that we have ten harmonic equations, $(BA'CA'') = -1$, &c., as already seen (31, 86, 87). Each point $P_{2,1}$ is the *common intersection* of a line Λ_1 with *three lines* $\Lambda_{2,2}$; thus we may establish the four following *formulæ of concurrence* (equivalent, by 89, to *ten* such formulæ):

$$A'' = BC \cdot B'C' \cdot B_1C_1 \cdot B_2C_2; \qquad A'_2 = DA \cdot D_1A_1 \cdot B'C_2 \cdot C'B_2;$$

$$A'_1 = EA \cdot D_1A_2 \cdot B'C_1 \cdot C'B_1; \qquad D'_1 = DE \cdot A_1A_2 \cdot B_1B_2 \cdot C_1C_2.$$

Each point $P_{2,1}$ is also situated in *three planes* Π_1; in *three* other planes, of the group $\Pi_{2,1}$; and in *six* planes $\Pi_{2,2}$; for example, A'' is a point common to the *twelve* planes,

$$ABC, \ BCD, \ BCE; \qquad AB_1C_2C_1B_2, \qquad DB'B_1C'C_1, \qquad EB'B_2C'C_2;$$

$$B'C'A_1, \quad B_1C_1A_1, \quad B_2C_2A_2, \quad B'C'A_2, \quad B_1C_1D_1, \quad B_2C_2D_1.$$

Each line, Λ_1, or $\Lambda_{2,2}$, contains *one* point $P_{2,1}$; but no line $\Lambda_{2,1}$ contains any. Each plane, Π_1 or $\Pi_{2,2}$, contains *three* such points; and each plane $\Pi_{2,1}$ contains *two*, which are the *intersections* of *opposite sides* of a *quadrilateral* Q_2 in that plane, whereof the *diagonals* intersect in a point P_0: for example, the diagonals B_1C_2, B_2C_1 of the quadrilateral $B_1B_2C_2C_1$, which is (by 90) in one of the planes $\Pi_{2,1}$, intersect* each other in the point A; while the opposite sides C_1B_1, B_2C_2 intersect in A''; and the two other opposite sides, B_1B_2, C_2C_1 have the point D'_1 for their intersection. The *ten points* $P_{2,1}$ are also ranged, *three by three*, on *ten lines* of *third construction* Λ_3, namely, on the *axes of homology*,

$$A''B'_1C'_1, \ldots A''B'_2C'_2, \ldots A'_1A'_2D'_1, \ldots A''B''C'',$$

of *ten pairs of triangles* T_1, T_2, which are situated in the ten planes Π_1, and of which the *centres* of homology are the ten points P_1: for example, the dotted line $A''B''C''$, in fig. 21, is the axis of homology of the two triangles, ABC, $A'B'C'$, whereof the latter is *inscribed* in the former, with the point O in that figure (replaced by D_1 in fig. 29), to represent their centre of homology. The same ten *points* $P_{2,1}$ are also ranged *six by six*, and the ten last *lines* Λ_3 are ranged

* Compare the first Note to page 62.

four by four, in *five planes* Π_3, namely in the *planes of homology* of *five pairs of pyramids*, R_1, R_2, already mentioned (90): for example, the plane [E] contains (87) the six points $A''B''C''A'_2B'_2C'_2$, and the four right lines,

$$A''B'_2C'_2, \quad B''C'_2A'_2, \quad C''A'_2B'_2, \quad A''B''C'';$$

which latter are the intersections of the *four faces*,

$$DCB, \quad DAC, \quad DBA, \quad ABC,$$

of the pyramid ABCD, with the corresponding faces,

$$D_1C_1B_1, \quad D_1A_1C_1, \quad D_1B_1A_1, \quad A_1B_1C_1,$$

of its *inscribed homologue* $A_1B_1C_1D_1$; and are contained, besides, in the four other planes,

$$A_2B'C', \quad B_2C'A', \quad C_2A'B', \quad A_2B_2C_2:$$

the three triangles, ABC, $A_1B_1C_1$, $A_2B_2C_2$, for instance, being *all homologous*, although *in different planes*, and having the line $A''B''C''$ for their *common axis* of homology. We may also say, that this line $A''B''C''$ is the *common trace* (81) of *two planes* $\Pi_{2,2}$, namely of $A_1B_1C_1$ and $A_2B_2C_2$, on the plane ABC; and in like manner, that the *point* A'' is the common *trace*, on that plane Π_1, of *two lines* $\Lambda_{2,2}$, namely of B_1C_1 and B_2C_2: being also the common trace of the two lines $B'_1C'_1$ and $B'_2C'_2$, which belong to the *third* construction.

(2.) On the whole, *these ten points*, of *second* construction, A'' . . ., may be considered to be already well known to geometers, in connexion with the theory of *transversal* lines and planes* in space: but it is important here to observe, with what simplicity and clearness their geometrical relations are *expressed* (88), by the *quinary symbols* and quinary *types* employed. For example, the *collinearity* (82) of the *four planes*, ABC, $A_1B_1C_1$, $A_2B_2C_2$, and [E], becomes evident from mere *inspection* of their *four symbols*,

$$[0001\bar{1}], \quad [111\bar{2}1], \quad [111\bar{1}\bar{2}], \quad [1111\bar{4}],$$

which represent (75) the *four quinary equations*,

$$w-v=0, \quad x+y+z-2w-v=0, \quad x+y+z-w-2v=0, \quad x+y+z+w-4v=0;$$

with this additional consequence, that the *ternary symbol* (81) of the *common trace*, of the three latter on the former, is [111]: so that this trace is (by 38) the line $A''B''C''$ of fig. 21, as above. And if we briefly denote the quinary symbols of the four planes, taken in the same form and order as above, by

* The collinear, complanar, and harmonic relations between the ten points, which we have above marked as $P_{2,1}$, and which have been considered by Möbius also, in connexion with his theory of *nets in space*, appear to have been first noticed by Carnot, in a Memoir upon *transversals*.

$[R_0]$ $[R_1]$ $[R_2]$ $[R_3]$, we see that they are connected by the two relations,

$$[R_1] = -[R_0] + [R_2]; \qquad [R_3] = 2[R_0] + [R_2];$$

whence if we denote the planes themselves by Π_1, Π_2, Π'_2, Π_3, we have (comp. 84) the following value for the *anharmonic of their pencil*,

$$(\Pi_1\Pi_2\Pi'_2\Pi_3) = -2;$$

a result which can be very simply verified, for the case when ABCD is a *regular pyramid*, and E (comp. 29) is its *mean point:* the plane Π_3, or [E], becoming in this case (comp. 38) the *plane at infinity*, while the three other planes, ABC, $A_1B_1C_1$, $A_2B_2C_2$, are *parallel;* the *second* being *intermediate* between the other two, but *twice as near* to the *third* as to the *first.*

(3.) We must be a little more concise in our remarks on the *seven other types* of points P_2, which indeed, if not so well known,* are perhaps also, on the whole, not quite so interesting: although it seems that some circumstances of their arrangement in space may deserve to be noted here, especially as affording an additional *exercise* (88), in the present system of symbols and types. The *type* $P_{2,2}$ represents, then, a *group* of *thirty points*, of which A‴, in fig. 21, is an example; each being the intersection of a line $\Lambda_{2,1}$ with a line $\Lambda_{2,2}$ as A‴ is the point in which AA′ intersects B′C′: but each belonging to no other line, among those which have been hitherto considered. But without aiming to describe here *all* the lines, planes, and points, of what we have called the *third* construction, we may already see that they must be expected to be numerous: and that the planes Π_3, and the lines Λ_3, of *that* construction, as well as the pyramids R_2, and the triangles T_2, of the *second* construction, above noticed, can only be regarded as *specimens*, which in a closer study of the subject, it becomes necessary to mark more fully, on the present plan, as $\Pi_{3,1}$, . . $T_{2,1}$. Accordingly it is found that not only is each point $P_{2,2}$ one of the corners of a *triangle* $T_{3,1}$ of *third* construction (as A‴ is of A‴B‴C‴ in fig. 21), the *sides* of which new triangle are lines $\Lambda_{3,2}$, passing each through one point $P_{2,1}$ and through two points $P_{2,2}$ (like the dotted line A″B‴C‴ of fig. 21); but also each such point $P_{2,2}$ is the intersection of *two new lines* of

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* It does not appear that any of these *other types*, or *groups*, of points P_2, have hitherto been noticed, in connexion with the *net in space*, except the one which we have ranked as the *fifth*, $P_{2,5}$, and which represents *two points* on each line Λ_1, as the type $P_{2,1}$ has been seen to represent *one* point on each of those ten lines of first construction: but that *fifth group*, which may be exemplified by the intersections of the line DE with the two planes $A_1B_1C_1$ and $A_2B_2C_2$, has been indicated by Möbius (in page 290 of his already cited work), although with a *different notation*, and as the result of a *different analysis.*

third construction, $\Lambda_{3,3}$, whereof each connects a point P_0 with a point $P_{2,1}$. For example, the point A''' is the *common trace* (on the plane ABC) of the two new lines, DA'_1, EA'_2: because, if we adopt for this point A''' the second of its two congruent symbols, we have (comp. 73, 82) the expressions,

$$A''' = (\bar{1}0011) = (D) - (A'_1) = (E) - (A'_2).$$

We may therefore establish the *formula of concurrence* (comp. the first sub-article):

$$A''' = AA' \cdot B'C' \cdot DA'_1 \cdot EA'_2;$$

which represents a system of *thirty* such formulæ.

(4.) It has been remarked that the point A''' may be represented, not only by the quinary symbol (21100), but also by the congruent symbol, $(\bar{1}0011)$; if then we write,

$$A_0 = (\bar{1}1100), \qquad B_0 = (1\bar{1}100), \qquad C_0 = (11\bar{1}00),$$

these three new points $A_0B_0C_0$, in the plane of ABC, must be considered to be *syntypical*, in the *quinary* sense (78), with the three points $A'''B'''C'''$, or to belong to the *same group* $P_{2,2}$, although they have (comp. 88) a different *ternary type*. It is easy to see that, while the triangle $A'''B'''C'''$ is (comp. again fig. 21) an *inscribed homologue* $T_{3,1}$ of the triangle $A'B'C'$, which is *itself* (comp. sub-article 1) an inscribed homologue $T_{2,1}$ of a triangle T_1, namely of ABC, with $A''B''C''$ for their *common axis* of homology, the new triangle $A_0B_0C_0$ is on the contrary an *exscribed homologue* $T_{3,2}$, with the *same axis* $\Lambda_{3,1}$, of the same given triangle T_1. But from the *syntypical relation* existing as above for *space* between the points A''' and A_0, we may expect to find that these two points $P_{2,2}$ admit of being *similarly constructed*, when the *five* points P_0 are treated as entering *symmetrically* (or similarly), as *geometrical elements*, into the constructions. The point A_0 must therefore be situated, not only on a line $\Lambda_{2,1}$, namely, on AA', but also on a line $\Lambda_{2,2}$, which is easily found to be A_1A_2, and on two lines $\Lambda_{3,3}$, each connecting a point P_0 with a point $P_{2,1}$; which latter lines are soon seen to be BB'' and CC''. We may therefore establish the formula of *concurrence* (comp. the last sub-article):

$$A_0 = AA' \cdot A_1A_2 \cdot BB'' \cdot CC'';$$

and may consider the three points A_0, B_0, C_0 as the *traces* of the three lines A_1A_2, B_1B_2, C_1C_2: while the three new lines AA'', BB'', CC'', which coincide in position with the sides of the exscribed triangle $A_0B_0C_0$, are the traces $\Lambda_{3,3}$ of three *planes* $\Pi_{2,1}$, such as $AB_1C_2B_2C_1$, which pass through the three given points A, B, C, but do not contain the lines $\Lambda_{2,1}$ whereon the six points $P_{2,2}$ in their plane Π_1 are situated. Every *other* plane Π_1 contains, in like manner, *six* points P_2 of the present group; every plane $\Pi_{2,1}$ contains *eight* of them; and every plane $\Pi_{2,2}$ contains *three;* each *line* $\Lambda_{2,1}$ passing through *two* such points, but each line $\Lambda_{2,2}$ only through *one.* But besides being (as above) the intersection of *two lines* Λ_2, each point of this group $P_{2,2}$ is common to *two planes* Π_1, *four planes* $\Pi_{2,1}$, and *two planes* $\Pi_{2,2}$; while each of these thirty points is also a *common corner* of *two* different *triangles* of *third* construction, of the lately mentioned kinds $T_{3,1}$ and $T_{3,2}$, situated respectively in the two planes of *first* construction which contain the point itself. It may be added that each of the two points $P_{2,2}$, on a line $\Lambda_{2,1}$, is the *harmonic conjugate* of *one* of the two points P_1, with respect to the point P_0, and to the *other* point P_1 on that line; thus we have here the two harmonic equations,

$$(AA'D_1A''') = (AD_1A'A_0) = -1,$$

by which the positions of the two points A''' and A_0 might be determined.

(5.) A *third group*, $P_{2,3}$, of *second construction*, consists (like the preceding group) of *thirty points*, ranged *two by two* on the fifteen lines $\Lambda_{2,1}$, and *six by six* on the ten planes Π_1, but so that each is common to *two* such planes; each is also situated in *two planes* $\Pi_{2,1}$, in *two* planes $\Pi_{2,2}$, and on *one line* $\Lambda_{3,1}$, in which (by sub-art. 1) these two last planes intersect each other, and two of the five planes $\Pi_{3,1}$; each plane $\Pi_{2,1}$ contains *four* such points, and each plane $\Pi_{2,2}$ contains *three* of them; but no point of this group is on any line Λ_1, or $\Lambda_{2,2}$. The *six points* $P_{2,3}$, which are *in the plane* ABC, are represented (like the corresponding points of the last group) by *two ternary types*, namely by $(\bar{2}11)$ and (311); and may be exemplified by the two following points, of which these last are the ternary symbols:

$$A^{IV} = AA' \cdot A''B''C'' = AA' \cdot A_1B_1C_1 \cdot A_2B_2C_2;$$

$$A_1^{IV} = AA' \cdot D'_1A'_2A_1 = AA' \cdot B'C_1C_2 \cdot C'B_1B_2.$$

The three points of the first sub-group A^{IV}. . are collinear; but the three points A_1^{IV}. . of the second sub-group are the corners of a *new triangle*, $T_{3,3}$, which is homologous to the triangle ABC, and to all the other triangles in its plane which have been hitherto considered, as well as to the two triangles $A_1B_1C_1$ and $A_2B_2C_2$; the line of the three former points being their *common axis* of homology; and the *sides* of the new triangle, $A_1^{IV}B_1^{IV}C_1^{IV}$, being the *traces* of the *three planes* (comp. 90) of homology of pyramids, [A], [B], [C]; as (comp. sub-art. 2) the line $A^{IV}B^{IV}C^{IV}$ or $A''B''C''$ is the *common trace* of the *two other planes* of the same group $\Pi_{2,1}$, namely of [D] and [E]. We may also say that the *point* A_1^{IV} is the *trace* of the *line* $A'_1A'_2$; and because the *lines* $B'C_0$, $C'B_0$ are the traces of the two *planes* $\Pi_{2,2}$ in which that point is contained, we may write the formula of concurrence,

$$A_1^{IV} = AA' \cdot A'_1A'_2 \cdot B'C_0 \cdot C'B_0.$$

(6.) It may be also remarked, that each of the two points $P_{2,3}$, on any line $\Lambda_{2,1}$, is the harmonic conjugate of a point $P_{2,2}$, with respect to the point P_0, and to *one* of the two points P_1 on that line; being *also* the harmonic conjugate of this last point, with respect to the same point P_0, and the *other* point $P_{2,2}$: thus, on the line $AA'D_1$, we have the *four harmonic equations*, which are not however all *independent*, since two of them can be deduced from the two others, with the help of the two analogous equations of the fourth sub-article:

$$(AA'''A'A^{IV}) = (AA'A_0A^{IV}) = (AA_0D_1A_1^{IV}) = (AD_1A'''A_1^{IV}) = -1.$$

And the *three pairs of derived points* P_1, $P_{2,2}$, $P_{2,3}$, on any such line $\Lambda_{2,1}$, will be found (comp. 26) to compose an *involution*, with the *given point* P_0 on the line for *one* of its two *double points* (or *foci*): the *other* double point of this involution being a point P_3 of *third* construction; namely, the point in which the line $\Lambda_{2,1}$ meets *that one* of the *five planes* of homology $\Pi_{3,1}$, which *corresponds* (comp. 90) to the particular point P_0 as *centre*. Thus, in the present example, if we denote by A^X the point in which the line AA' meets the plane [A], of which (by 81, 91) the trace on ABC is the line $[\bar{4}11]$, and therefore is (as has been stated) the side $B_1^{IV}C_1^{IV}$ of the lately mentioned triangle $T_{3,3}$, so that

$$A^X = (122) = AA' \cdot BC''' \cdot CB''' \cdot B_1^{IV}C_1^{IV},$$

we shall have the three harmonic equations,

$$(AA'A^XD_1) = (AA'''A^XA_0) = (AA^{IV}A^XA_1^{IV}) = -1;$$

which express that this new point A^X is the *common harmonic conjugate* of the given point A, with respect to the *three pairs of points*, $A'D_1$, $A'''A_0$, $A^{IV}A_1^{IV}$; and therefore that *these three pairs* form (as has been said) an *involution*, with A and A^X for its two *double points*.

(7.) It will be found that we have now exhausted all the types of points of second construction, which are situated upon lines $\Lambda_{2,1}$; there being only *four* such *points* on each such *line*. But there are still to be considered two new groups of points P_2 on lines Λ_1, and three others on lines $\Lambda_{2,2}$. Attending first to the former set of lines, we may observe that *each* of the *two new types*, $P_{2,4}$, $P_{2,5}$, represents *twenty points*, situated *two by two* on the ten lines of *first* construction, but not on any line Λ_2; and therefore *six by six* in the *ten planes* Π_1, each point however being common to *three* such planes: also each point $P_{2,4}$ is common to *three* planes $\Pi_{2,2}$, and each point $P_{2,5}$ is situated in *one* such plane; while each of these last planes contains *three* points $P_{2,4}$, but only *one* point $P_{2,5}$. If we attend only to points in the plane ABC, we can represent these *two new groups* by the *two ternary types* (021) and $(02\bar{1})$, which as *symbols* denote the two typical points,

$$A^{V} = BC \cdot C'A_1A_2 \cdot D_1A_1B_1 \cdot D_1A_2B_2; \quad A^{VI} = BC \cdot C'B_1B_2 = BC \cdot C'B_0;$$

we have also the concurrence,

$$A^{V} = BC \cdot C'A_0 \cdot D_1C'' \cdot AB'''.$$

It may be noted that A^{V} is the harmonic conjugate of C', with respect to A_0 and B_1^{IV}, which last point is on the same trace $C'A_0$, of the plane $C'A_1A_2$; and that A^{VI} is harmonically conjugate to B_1^{V}, with respect to C' and B_0, on the trace of the plane $C'B_1B_2$, where B_1^{V} denotes (by an analogy which will soon become more evident) the intersection of that trace with the line CA: so that we have the two equations,

$$(A_0C'B_1^{IV}A^{V}) = (B_0B_1^{V}C'A^{VI}) = -1.$$

(8.) *Each line* Λ_1 contains thus two points P_2, of each of the two last new groups, besides the point $P_{2,1}$, the point P_1, and the two points P_0, which had been previously considered: it contains therefore *eight points* in all, if we still abstain (88) from proceeding beyond the *Second Construction*. And it is easy to prove that these *eight points* can, *in two distinct modes*, be so arranged as to form (comp. sub-art 6) an *involution*, with *two* of them for the two *double* points thereof. Thus, if we attend only to points on the line BC, and represent them by ternary symbols, we may write,

$$B = (010), \qquad C = (001), \qquad A' = (011), \qquad A'' = (01\bar{1});$$
$$A^{V} = (021), \qquad A^{VI} = (02\bar{1}), \qquad A_1^{V} = (012), \qquad A_1^{VI} = (0\bar{1}2);$$

and the resulting harmonic equations

$$\text{I.} \ldots (BA'CA'') = (BA^{V}CA^{VI}) = (BA_1^{V}CA_1^{VI}) = -1,$$
$$\text{II.} \ldots (A'BA''C) = (A'A^{V}A''A_1^{V}) = (A'A^{VI}A''A_1^{VI}) = -1,$$

will then suffice to show: Ist, that *the two points* P_0, *on any line* Λ_1, are the *double points of an involution, in which the points* P_1, $P_{2,1}$ *form one pair of conjugates*, while the *two other pairs* are of the *common form*, $P_{2,4}$, $P_{2,5}$; and IInd, that *the two points* P_1 and $P_{2,1}$, *on any such line* Λ_1, *are the double points of a second involution, obtained by pairing the two points of each of the three other* *groups.* Also each of the two points P_0, on a line Λ_1, is the harmonic conjugate of one of the two points $P_{2,5}$ on that line, with respect to the other point of the same group, and to the point P_1 on the same line; thus,

$$(BA'A_1^{VI}A^{VI}) = (CA'A^{VI}A_1^{VI}) = -1.$$

(9.) It remains to consider briefly *three other groups* of points P_2, *each group* containing *sixty points*, which are situated, two by two, on the thirty lines $\Lambda_{2,2}$, and six by six in the ten planes Π_1. Confining our attention to those which are in the plane ABC, and denoting them by their ternary symbols, we have thus, on the line $B'C'$, the three new typical points, of the three remaining groups, $P_{2,6}$, $P_{2,7}$, $P_{2,8}$:

$$A^{VII} = (12\bar{1})\,; \qquad A^{VIII} = (321)\,; \qquad A^{IX} = (23\bar{1})\,;$$

with which may be combined these three others, of the same three types, and on the same line $B'C'$:

$$A_1^{VII} = (1\bar{1}2)\,; \qquad A_1^{VIII} = (312)\,; \qquad A_1^{IX} = (2\bar{1}3).$$

Considered as intersections of a line $\Lambda_{2,2}$ with lines Λ_3 in the same plane Π_1, or with planes Π_2 (in which *latter* character alone they belong to the *second* construction), the three points A^{VII}, &c., may be thus denoted:

$$A^{VII} = B'C' \cdot BB'' \cdot CB''' \cdot AA^{VI} = B'C' \cdot BC_1A_2A_1C_2\,;$$

$$A^{VIII} = B'C' \cdot D_1B'' \cdot AB'''A^{V} = B'C' \cdot D_1C_1A_1 \cdot D_1C_2A_2\,;$$

$$A^{IX} = B'C' \cdot A'C_0B_1^{IV}C_1^{V}B^{VI} \cdot BA^{IV}B_1^{VI}B_1^{VII} = B'C' \cdot A'C_1C_2\,;$$

with the harmonic equation,

$$(C_0A'C_1^{V}A^{IX}) = -1,$$

and with analogous expressions for the three other points, A_1^{VII}, &c. The line $B'C'$ thus intersects *one* plane $\Pi_{2,1}$ (or its trace BB'' on the plane ABC), in the point A^{VII}; it intersects *two* planes $\Pi_{2,2}$ (or their common trace D_1B'') in A^{VIII}; and *one* other plane $\Pi_{2,2}$ (or its trace $A'C_0$) in A^{IX}: and similarly for the other points, A_1^{VII}, &c., of the same three groups. *Each plane* $\Pi_{2,1}$ contains *twelve points* $P_{2,6}$, *eight* points $P_{2,7}$, and *eight* points $P_{2,8}$; while every plane $\Pi_{2,2}$ contains *six* points $P_{2,6}$, *twelve* points $P_{2,7}$, and *nine* points $P_{2,8}$. *Each point* $P_{2,6}$ is contained in *one plane* Π_1; in *three* planes $\Pi_{2,1}$; and in *two* planes $\Pi_{2,2}$. Each point $P_{2,7}$ is in *one* plane Π_1, in *two* planes $\Pi_{2,1}$, and in *four* planes $\Pi_{2,2}$. And each point $P_{2,8}$ is situated in *one* plane Π_1, in *two* planes $\Pi_{2,1}$, and in *three* planes $\Pi_{2,2}$.

(10.) The points of the three last groups are situated *only* on lines $\Lambda_{2,2}$; but, on each such line, *two* points of *each* of those three groups are situated;

which, along with *one* point of each of the *two* former groups, $P_{2,1}$ and $P_{2,2}$, and with the *two* points P_1, whereby the line itself is determined, make up a system of *ten points* upon that line. For example, the line $B'C'$ contains, besides the *six* points mentioned in the last sub-article, the *four* others:

$$B' = (101); \quad C' = (110); \quad A'' = (01\bar{1}); \quad A''' = (211).$$

Of these *ten* points, the *two* last mentioned, namely the points $P_{2,1}$ and $P_{2,2}$ upon the line $\Lambda_{2,2}$, are the *double points* (comp. sub-art. 8) of a *new involution*, in which the *two points* of *each* of the *four other groups* compose a conjugate pair, as is expressed by the harmonic equations,

$$(A''B'A'''C') = (A''A^{VII}A'''A_1^{VII}) = (A''A^{VIII}A'''A_1^{VIII}) = (A''A^{IX}A'''A_1^{IX}) = -1.$$

And the analogous equations,

$$(B'A''C'A''') = (B'A^{VII}C'A^{VIII}) = (B'A_1^{VII}C'A_1^{VIII}) = -1,$$

show that the two points P_1 on any line $\Lambda_{2,2}$ are the double points of *another involution* (comp. again sub-art. 8), whereof the two points $P_{2,1}$, $P_{2,2}$ on that line form one conjugate pair, while each of the two points $P_{2,6}$ is paired with one of the points $P_{2,7}$ as its conjugate. In fact, the *eight-rayed pencil* $(A \,.\, C'B'A'''A''A^{VIII}A^{VII}A_1^{VIII}A_1^{VII})$ coincides in position with the pencil $(A \,.\, BCA'A'' A^{V}A^{VI}A_1^{V}A_1^{VI})$, and may be said to be a *pencil in double involution;* the third and fourth, the fifth and sixth, and the seventh and eighth rays forming *one* involution, whereof the first and second are the two *double* rays;* while the first and second, the fifth and seventh, and the sixth and eighth rays compose *another* involution, whereof the double rays are the third and fourth of the pencil.

(11.) If we proceeded to connect systematically the points P_2 among themselves, and with the points P_1 and P_0, we should find many remarkable lines and planes of *third* construction (88), besides those which have been incidentally noticed above; for example, we should have a group $\Pi_{3,2}$ of *twenty new planes*, exemplified by the two following,

$$[E_D] = [1110\bar{3}], \qquad [D_E] = [11\bar{3}0],$$

which have the same common trace $\Lambda_{3,1}$, namely the line $A''B''C''$, on the plane ABC, as the two planes $A_1B_1C_1$, $A_2B_2C_2$, and the two planes $[D]$, $[E]$, of the groups $\Pi_{2,2}$ and $\Pi_{3,1}$, which have been considered in former sub-articles; and each of these new planes $\Pi_{3,2}$ would be found to contain *one* point P_0, *three* points

* Compare page 172 of the *Géom. Supérieure* of M. Chasles.

$P_{2,1}$, *six* points $P_{2,2}$, and *three* points $P_{2,3}$. It might be proved also that these twenty new planes are the *twenty faces* of *five new pyramids* R_3, which are the *exscribed homologues* of the five old pyramids R_1 (89), with the five given points P_0 for the corresponding *centres* of homology. But it would lead us beyond the proposed limits, to pursue this discussion further: although a few additional remarks may be useful, as serving to establish the *completeness* of the *enumeration* above given, of the lines, planes, and points of *second* construction.

93. In general, if there be any n *given points*, whereof no four are situated in any common plane, the *number* N of the *derived* points, which are immediately obtained from them, as *intersections* $\Lambda \cdot \Pi$ of *line* with *plane* (each *line* being drawn through *two* of the given points, and each *plane* through *three others*), or the number of points of the *form* AB · CDE, is easily seen to be,

$$N = f(n) = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \,.\, 2 \,.\, 3};$$

so that $N = 10$, as before, when $n = 5$. But if we were to apply this formula to the case $n = 15$, we should find, for that case, the value,

$$N = f(15) = 15 \,.\, 14 \,.\, 13 \,.\, 11 = 30030;$$

and thus *fifteen given and independent points of space* would conduct, by what might (relatively to *them*) be called a *First Construction* (comp. 88), to a system of *more* than *thirty thousand* points. Yet it has been lately stated (92), that from the fifteen points above called P_0, P_1, there can be derived, in this way, *only* two hundred and ninety points P_2, as intersections of the form* $\Lambda \,.\, \Pi$; and therefore *fewer* than *three hundred*. That this *reduction* of the *number* of *derived points*, at the end of what has been called (88) the *Second Construction* for the *net in space*, arising from the *dependence* of the *ten* points P_1 on the *five* points P_0, would be found to be *so considerable*, might not perhaps have been anticipated; and although the foregoing examination proves that *all* the *eight types* (92) do really represent points P_2, it may appear *possible*, at this stage, that some *other type* of such points has been *omitted*. A study of the manner in which the *types of points* result, from those of the *lines* and *planes* of which they are the intersections, would indeed decide this question; and it was, in fact, in that way that the eight types, or groups, $P_{2,1}$, . . $P_{2,8}$, of points of second construction for space, were investigated, and found to be sufficient: yet it

* The *definition* (88) of the points P_2 admits, indeed, *intersections* $\Lambda \,.\, \Lambda$ *of complanar lines*, when they are not already points P_0 or P_1; but all *such* intersections are *also* points of the form $\Lambda \,.\, \Pi$; so that *no generality* is *lost*, by confining ourselves to this *last form*, as in the present discussion we propose to do.

may be useful (compare the last sub-art.) to *verify*, as below, the *completeness* of the foregoing *enumeration.*

(1.) The *fifteen points*, P_0, P_1, admit of 105 *binary*, and of 455 *ternary combinations;* but these are far from determining so many distinct *lines* and *planes*. In fact, those 15 points are connected by 25 *collineations*, represented by the 25 lines Λ_1, $\Lambda_{2,1}$; which *lines* therefore *count* as 75, among the 105 *binary* combinations of points: and there *remain* only 30 combinations of this sort, which are constructed by the 30 *other* lines, $\Lambda_{2,2}$. Again, there are 25 *ternary* combinations of points, which are represented (as above) by *lines*, and therefore do not *determine* any *plane*. Also, in *each* of the *ten* planes Π_1, there are 29 $(= 35 - 6)$ *triangles* T_1, T_2, because each of those planes contains 7 *points* P_0, P_1, connected by 6 *relations* of collinearity. In like manner, each of the *fifteen* planes $\Pi_{2,1}$ contains 8 $(= 10 - 2)$ *other triangles* T_2, because it contains 5 *points* P_0, P_1, connected by *two* collineations. There remain therefore only 20 $(= 455 - 25 - 290 - 120)$ ternary combinations of points to be accounted for; and these are represented by the 20 planes $\Pi_{2,2}$. The *completeness* of the enumeration of the *lines* and *planes* of the *second* construction is therefore *verified;* and it only remains to verify that the 305 *points*, P_0, P_1, P_2, above considered, *represent all* the *intersections* $\Lambda \,.\, \Pi$, of the 55 lines Λ_1, Λ_2, with the 45 planes Π_1, Π_2.

(2.) Each plane Π_1 contains three lines of each of the three groups, Λ_1, $\Lambda_{2,1}$, $\Lambda_{2,2}$; each plane $\Pi_{2,1}$ contains two lines $\Lambda_{2,1}$, and four lines $\Lambda_{2,2}$; and each plane $\Pi_{2,2}$ contains three lines $\Lambda_{2,2}$. Hence (or because each line Λ_1 is *contained* in three planes Π_1; each line $\Lambda_{2,1}$ in two planes Π_1, and in two planes $\Pi_{2,1}$; and each line $\Lambda_{2,2}$ in one plane Π_1, in two planes $\Pi_{2,1}$, and in two planes $\Pi_{2,2}$), it follows that, without going beyond the *second* construction, there are 240 $(= 30 + 30 + 30 + 30 + 60 + 60)$ *cases of coincidence* of line and plane; so that the number of *cases of intersection* is *reduced*, hereby, from $55 \,.\, 45 = 2475$, to 2235 $(= 2475 - 240)$.

(3.) Each point P_0 represents *twelve* intersections of the form $\Lambda_1 \cdot \Pi_1$; because it is common to *four lines* Λ_1, and to *six planes* Π_1, each plane *containing two* of those four lines, but being *intersected* by the *two others* in that point P_0; as the plane ABC, for example, is intersected in A by the two lines, AD and AE. Again, each point P_0 is common to *three planes* $\Pi_{2,1}$, no one of which contains any of the four lines Λ_1 through that point; it represents therefore a system of *twelve other intersections*, of the form $\Lambda_1 \cdot \Pi_{2,1}$. Again, each point P_0 is common to *three lines* $\Lambda_{2,1}$, each of which is *contained in two* of the six planes Π_1, but *intersects the four others* in that point P_0; which therefore counts as *twelve* intersections, of the form $\Lambda_{2,1} \cdot \Pi_1$. Finally, each of the points P_0 represents *three* intersections, $\Lambda_{2,1} \cdot \Pi_{2,1}$; and it represents *no other* intersection, of the form $\Lambda \cdot \Pi$, within the limits of the present inquiry. Thus, *each* of the *five given points* is to be considered as representing, or constructing, *thirty-nine* (= 12 + 12 + 12 + 3) intersections of line with plane; and there remain only 2040 (= 2235 − 195) *other* cases of such intersection $\Lambda \cdot \Pi$, to be *accounted for* (in the present verification) by the 300 *derived points*, P_1, P_2.

(4.) For this purpose, the *nine columns*, headed as I. to IX. in the following *Table*, contain the *numbers* of such *intersections* which belong respectively to the *nine forms*,

$$\Lambda_1 \cdot \Pi_1, \quad \Lambda_1 \cdot \Pi_{2,1}, \quad \Lambda_1 \cdot \Pi_{2,2}; \qquad \Lambda_{2,1} \cdot \Pi_1, \quad \Lambda_{2,1} \cdot \Pi_{2,1}, \quad \Lambda_{2,1} \cdot \Pi_{2,2};$$

$$\Lambda_{2,2} \cdot \Pi_1, \quad \Lambda_{2,2} \cdot \Pi_{2,1}, \quad \Lambda_{2,2} \cdot \Pi_{2,2},$$

for each of the *nine typical derived points*, $A' \ldots A^{IX}$, of the *nine groups* P_1, $P_{2,1}$, .. $P_{2,8}$. Column X. contains, for each point, the *sum* of the *nine numbers*, thus tabulated in the preceding columns; and expresses therefore the entire number of intersections, which any *one* such *point* represents. Column XI. states the *number of the points* for each *type;* and column XII. contains the *product* of the two last numbers, or the number of intersections $\Lambda \,.\, \Pi$ which are represented (or constructed) by the *group*. Finally, the *sum* of the numbers in each of the *two last columns* is written at its foot; and because the 300 *derived points*, of first and second constructions, are thus found to represent the 2040 *intersections* which were to be accounted for, the *verification* is seen to be *complete:* and *no new type*, of points P_2, *remains to be discovered.*

(5.) TABLE OF INTERSECTIONS $\Lambda \cdot \Pi$.

TYPE.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
A'	1	6	6	6	12	18	18	24	24	115	10	1150
A''	0	3	6	0	0	0	6	3	12	30	10	300
A'''	0	0	0	0	2	2	1	2	0	7	30	210
A^{IV}	0	0	0	0	0	2	0	0	0	2	30	60
A^{V}	0	0	3	0	0	0	0	0	0	3	20	60
A^{VI}	0	0	1	0	0	0	0	0	0	1	20	20
A^{VII}	0	0	0	0	0	0	0	1	0	1	60	60
A^{VIII}	0	0	0	0	0	0	0	0	2	2	60	120
A^{IX}	0	0	0	0	0	0	0	0	1	1	60	60
											300	2040

(6.) It is to be remembered that we have *not admitted*, by our *definition* (88), any *points* which can *only* be determined by *intersections of three planes* Π_1, Π_2, as belonging to the *second construction:* nor have we counted, as *lines* Λ_2 of *that* construction, any lines which can *only* be found as intersections of *two* such planes. For example, we do not regard the *traces* AA$''$, &c., of certain *planes* $\Lambda_{2,1}$ considered in recent sub-articles, as among the lines of *second* construction, although they would present themselves early in an enumeration of the lines Λ_3 of the *third.* And any point in the plane ABC, which can *only* be determined (at the present stage) as the intersection of *two* such *traces*, is not regarded as a point P_2. A student might find it however to be not useless, as an exercise, to investigate the expressions for *such* intersections; and for that reason it may be noted here, that the *ternary types* (comp. 81) of the *forty-four traces* of *planes* Π_1, Π_2, on the plane ABC which are found to compose a system of only *twenty-two distinct lines* in that plane, whereof *nine* are lines Λ_1, Λ_2, are the seven following (comp. 38):

$$[100], \quad [01\bar{1}], \quad [\bar{1}11], \quad [111], \quad [011], \quad [\bar{2}11], \quad [\bar{2}1\bar{1}];$$

which, as ternary *symbols*, represent the *seven lines*,

$$BC, \quad AA', \quad B'C', \quad A''B''C'', \quad AA'', \quad D_1A'', \quad A'C_0.$$

(7.) Again, on the same principle, and with reference to the same definition, that new point, say F, which may be denoted by *either* of the two *congruent quinary symbols* (71),

$$F = (43210) \equiv (01234),$$

and which, as a *quinary type* (78), represents a *new group* of *sixty points of space* (and of *no more*, on account of this last *congruence*, whereas a quinary type, with *all* its *five* coefficients *unequal*, represents *generally* a group of 120 distinct points), is *not* regarded by us as a point P_2; although this new point F is easily seen to be the *intersection of three planes of second construction*, namely, of the three following, which all belong to the group $\Pi_{2,1}$:

$$[01\bar{1}\bar{1}1], \quad [1\bar{1}0\bar{1}1], \quad [1\bar{1}\bar{1}10],$$

or $AA'D_1C_1B_2$, $CC'D_1B_1A_2$, $EB'B_2C'C_2$. It may, however, be remarked in passing, that *each plane*, $\Pi_{2,1}$ contains *twelve points* P_3 of this new group: every such point being common (as is evident from what has been shown) to *three* such planes.

94. From the foregoing discussion it appears that the *five given points* P_0, and the *three hundred derived points* P_1, P_2, are *arranged in space*, upon the *fifty-*

five lines Λ_1, Λ_2, and in the *forty-five planes* Π_1, Π_2, as follows. Each *line* Λ_1 contains *eight* of the 305 points, forming on it what may be called (see the sub-article (8.) to 92) a *double involution.* Each line $\Lambda_{2,1}$ contains *seven points*, whereof *one*, namely the *given* point, P_0, has been seen (in the earlier sub-art. (6.)) to be a *double point* of *another* involution, to which the *three derived pairs* of points, P_1, P_2, on the same line belong. And each line $\Lambda_{2,2}$ contains *ten* points, forming on it a *new involution;* while *eight* of these ten points, with a different *order* of succession, compose still *another* involution* (92, (10.)). Again, each *plane* Π_1 contains *fifty-two points*, namely three *given* points, four points of *first*, and 45 points of *second* construction. Each plane $\Pi_{2,1}$ contains *forty-seven* points, whereof *one* is a given point, four are points P_1, and 42 are points P_2: of which last, 38 are situated *on the six lines* Λ_2 in the plane,

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* These theorems respecting the *relations of involution*, of given and derived points on lines of *first* and *second* constructions, for a *net in space*, are perhaps new; although some of the *harmonic relations*, above mentioned, have been noticed under other forms by Möbius: to whom, indeed, as has been stated, the *conception* of such a *net* is due. Thus, if we consider (compare the note to page 66) the two intersections,

$$E_1 = DE \cdot A_1B_1C_1, \qquad E_2 = DE \cdot A_2B_2C_2,$$

we easily find that they may be denoted by the quinary symbols,

$$E_1 = (000\bar{1}2), \qquad E_2 = (0002\bar{1});$$

they are, therefore, by Art. 92, the two points $P_{2,5}$ on the line DE: and consequently, by the theorem stated at the end of sub-art. (8.), the harmonic conjugate of *each*, taken with respect to the *other* and to the point D_1, must be one of the two points D, E on that line. Accordingly, we soon derive, by comparison of the symbols of these *five points*, $DED_1E_1E_2$, the two following harmonic equations, which belong to the *same type* as the *two last* of that sub-art. (8.):

$$(D_1DE_2E_1) = -1; \qquad (D_1EE_1E_2) = -1;$$

but *these* two equations have been assigned (with notations slightly different) in the formerly cited page 290 of the Barycentric Calculus. (Comp. again the recent note to page 66.) The *geometrical meaning* of the last equation may be illustrated, by conceiving that ABCD is a *regular pyramid*, and that E is its *mean point;* for then (comp. 92, sub-art. (2.)), D_1 is the mean point of the *base* ABC; D_1D is the *altitude* of the pyramid; and the *three segments* D_1E, D_1E_1, D_1E_2 are, respectively, the *quarter*, the *third part*, and the *half* of that altitude; they compose therefore (as the formula expresses) a *harmonic progression;* or D_1 and E_1 are *conjugate points*, with respect to E and E_2. But in order to exemplify the *double involution* of the same sub-art. (8.), it would be necessary to consider *three other points* P_2, on the *same line* DE; whereof *one*, above called D'_1, belongs to a *known group* $P_{2,1}$ (92, (2.)); but the *two others* are of the group $P_{2,4}$, and do not seem to have been previously noticed. As an example of an involution on a line of *third* construction, it may be remarked that on each line of the group $\Lambda_{3,3}$, or on each of the sides of any one of the ten triangles $T_{3,2}$, in addition to one given point P_0, and one derived point $P_{2,1}$, there are two points $P_{2,2}$, and two points $P_{2,6}$; and that the two first points are the double points of an involution, to which the two last pairs belong: thus, on the side A_0BC_0 of the exscribed triangle $A_0B_0C_0$, or on the trace of the plane $BC_1A_2A_1C_2$, we have the two harmonic equations,

$$(BA_0B''C_0) = (BA^{VII}B''C_1^{VII}) = -1.$$

Again, on the trace $A'C_0$ of the plane $A'C_1C_2$ (which latter trace is a line not passing through any one of the given points), C_0 and B_1^{IV} are the double points of an involution, wherein A' is conjugate to C_1^{V} and A^{IX} to B^{VI}. But it would be tedious to multiply such instances.

but four are *intersections* of that plane $\Pi_{2,1}$ with *four other* lines of second construction. Finally, each plane $\Pi_{2,2}$ passes through *no given point*, but contains *forty-three derived points*, whereof 40 are points of second construction. And because the planes of *first* construction *alone* contain *specimens* of *all the ten groups of points*, $P_0, P_1, P_{2,1}, \ldots P_{2,8}$, given or derived, and of *all* the *three groups of lines*, $\Lambda_1, \Lambda_{2,1}, \Lambda_{2,2}$, at the close of that *second* construction (since the types $P_{2,4}$, $P_{2,5}$, Λ_1 are not represented by any points or lines in any plane $\Pi_{2,1}$, nor are the types P_0, Λ_1, $\Lambda_{2,1}$ represented in a plane $\Pi_{2,2}$), it has been thought convenient to prepare the annexed *diagram* (fig. 30), which may serve to illustrate, by some selected instances, the *arrangement* of the *fifty-two points* P_0, P_1, P_2 in a plane Π_1, namely, in the plane ABC; as well as the arrangement of the *nine lines* Λ_1, Λ_2 in that plane, and the *traces* Λ_3 of other planes upon it.

View of the Arrangement of the Principal Points and Lines in a Plane of First Construction.

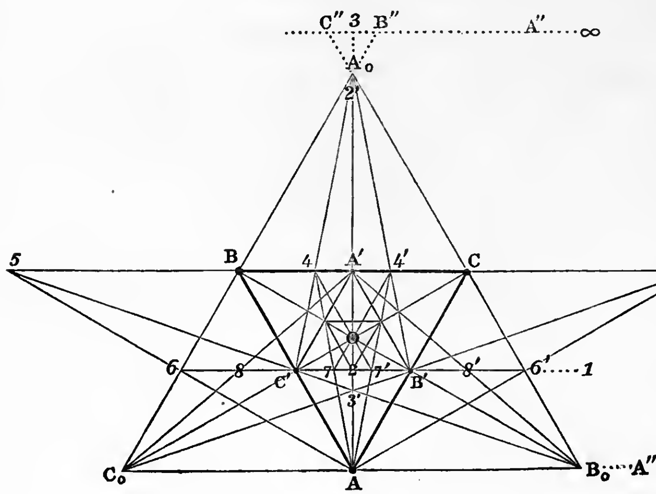

Fig. 30.

In this figure, the triangle ABC is supposed, for simplicity, to be the *equilateral base* of a *regular pyramid* ABCD (comp. sub-art. (2.) to 92); and D_1, again replaced by O, is supposed to be its *mean point* (29). The *first inscribed triangle*, A′B′C′, therefore, *bisects* the three sides; and the *axis of homology* A″B″C″ is the *line at infinity* (38): the number 1, on the line C′B′ prolonged, being designed to suggest that the *point* A″, to which that *line tends*, is of the type $P_{2,1}$, or belongs to the *first group* of points of *second construction*. A *second* inscribed *triangle*, $A'''B'''C'''$, for which fig. 21 may be consulted, is only *indicated* by the number 2 placed at the middle of the side $B'C'$, to suggest that *this* bisecting point A''' belongs to the *second group* of P_2. The *same number* 2, but with an *accent*, 2′, is placed near the corner A_0 of the *exscribed* triangle $A_0B_0C_0$, to remind us that this corner *also* belongs (by a syntypical relation in space) to the group $P_{2,2}$. The point A^{IV}, which is now infinitely distant, is indicated by the number 3, on the dotted line at the top; while the same number with an accent, lower down, marks the position of the point A_1^{IV}. Finally, the ten other numbers, unaccented or accented, 4, 4′, 5, 5′, 6, 6′, 7, 7′, 8, 8′, denote the places of the ten points, A^{V}, A_1^{V}, A^{VI}, A_1^{VI}, A^{VII}, A_1^{VII}, A^{VIII}, A_1^{VIII}, A^{IX}, A_1^{IX}. And the principal *harmonic relations*, and relations of *involution*, above mentioned, may be verified by inspection of this Diagram.

95. However far the series of construction of the net in space may be continued, we may now regard it as evident, at least on comparison with the analogous property (42) of the *plane* net, that *every point, line*, or *plane*, to which such constructions can conduct, must necessarily be *rational* (77) ; or that it must be *rationally related* to the system of the *five given points :* because the *anharmonic co-ordinates* (79, 80) of every *net-point*, and of every *net-plane*, are equal or proportional to *whole numbers*. Conversely (comp. 43) *every point, line*, or *plane*, in space, which is thus *rationally related* to the system of points ABCDE, is a point, line, or plane of the *net*, which those five points determine. Hence (comp. again 43), every *irrational* point, line, or plane (77), is indeed incapable of being *rigorously constructed*, by any processes of the kind above described: but it admits of being *indefinitely approximated to*, by points, lines, or planes of the net. *Every anharmonic ratio*, whether of a *group of net-points*, or of a *pencil of net-lines*, or of *net-planes*, has a *rational value* (comp. 44), which depends *only* on the *processes* of *linear construction* employed, in the generation of that group or pencil, and is entirely *independent* of the *arrangement*, or *configuration*, of the five given points in space. Also, all *relations* of *collineation*, and of *complanarity*, are *preserved*, in the passage from *one* net to *another*, by a change of the given system of points: so that it may be briefly said (comp. again 44) that *all geometrical nets in space are homographic figures*. Finally, *any five points** of such a net, of which no four are in one plane, are

* These *general properties* (95) of the *space-net* are in substance taken from Möbius, although (as has been remarked before) the *analysis* here employed appears to be new : as do also most of the *theorems* above given, respecting the *points* of *second construction* (92), at least after we pass beyond the *first group* $P_{2,1}$ of *ten* such points, which (as already stated) have been known comparatively long.

sufficient (comp. 45) for the determination of the *whole net*, or for the *linear construction* of *all* its points, including the five *given* ones.

(1.) As an Example, let the five points $A_1B_1C_1D_1$ and E be now supposed to be *given;* and let it be required to *derive* the four points ABCD, by linear constructions, from these new data. In other words, we are now required to *exscribe a pyramid* ABCD to a given pyramid $A_1B_1C_1D_1$, so that it may be *homologous* thereto, with the point E for their *given centre* of homology. An obvious process is (comp. 45) to *inscribe* another homologous pyramid, $A_3B_3C_3D_3$, so as to have $A_3 = EA_1 \cdot B_1C_1D_1$, &c.; and then to determine the *intersections* of *corresponding faces*, such as $A_1B_1C_1$ and $A_3B_3C_3$; for these *four lines* of intersection will be in the *common plane* [E] of homology of the *three pyramids*, and will be the *traces* on that plane of the *four sought planes*, ABC, &c., drawn through the four given points D_1, &c. If it were only required to construct *one corner* A of the exscribed pyramid, we might find the point above called A^{IV} as the common intersection of *three planes*, as follows,

$$A^{IV} = A_1B_1C_1 \cdot A_1D_1E \cdot A_3B_3C_3\,;$$

and then should have this other formula of intersection,

$$A = EA_1 \cdot D_1A^{IV}.$$

Or the point A might be determined by the anharmonic equation,

$$(EAA_1A_3) = 3,$$

which for a regular pyramid is easily verified.

(2.) As regards the general *passage* from *one* net in space to *another*, let the symbols $P_1 = (x_1 \,.\,.\, v_1), \,.\,.\, P_5 = (x_5 \,.\,.\, v_5)$ denote *any five given points*, whereof no four are complanar; and let $a'b'c'd'e'$ and u' be six coefficients, of which the five ratios are such as to satisfy the symbolical equation (comp. 71, 72),

$$a'\,(P_1) + b'\,(P_2) + c'\,(P_3) + d'\,(P_4) + e'\,(P_5) = -\,u'\,(U):$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \,.\,.\, + e'x_5 = \,.\,.\, a'v_1 + \,.\,.\, + e'v_5 = -\,u'.$$

Let P' be any sixth point of space, of which the quinary symbol satisfies the equation,

$$(P') = xa'\,(P_1) + yb'\,(P_2) + zc'\,(P_3) + wd'\,(P_4) + ve'\,(P_5) + u'\,(U)\,;$$

then it will be found that this last point P' can be *derived* from the five points $P_1 \,.\,.\, P_5$ by precisely the *same constructions*, as those by which the point $P = (xyzwv)$ is derived from the five points ABCDE. As an example, if $v' = x + y + z + w - 3v$, then the point $(xyzwv')$ is derived from $A_1B_1C_1D_1E$, by the same constructions as $(xyzwv)$ from ABCDE; thus A itself may be constructed from $A_1 \ldots E$, as the point $P = (30001)$ is from $A \ldots E$; which would conduct anew to the anharmonic equation of the last sub-article.

(3.) It may be briefly added here, that instead of *anharmonic ratios*, as connected with a net in space, or indeed generally in relation to *spatial problems*, we are permitted (comp. 68) to substitute products (or quotients) of *quotients of volumes* of pyramids; as a *specimen* of which substitution, it may be remarked, that the anharmonic relation, just referred to, admits of being replaced by the following equation, involving *one* such quotient of pyramids, but introducing *no auxiliary point:*

$$EA : A_1A = 3EB_1C_1D_1 : A_1B_1C_1D_1.$$

In general, if $xyzw$ be (as in 79, 83) the *anharmonic co-ordinates* of a point P in space, we may write,

$$\frac{x}{y} = \frac{PBCD}{PCDA} : \frac{EBCD}{ECDA};$$

with other equations of the same type, on which we cannot here delay.

SECTION 5.

On Barycentres of Systems of Points; and on Simple and Complex Means of Vectors.

96. In general, when the *sum* $\Sigma\alpha$ of any number of co-initial vectors,

$$\alpha_1 = OA_1, \ldots \quad \alpha_m = OA_m,$$

is *divided* (16) by their *number*, m, the resulting *vector*,

$$\mu = OM = \frac{1}{m}\Sigma\alpha = \frac{1}{m}\Sigma OA,$$

is said to be the *Simple Mean* of those m vectors; and the *point* M, in which this *mean vector* terminates, and of which the *position* (comp. 18) is easily seen to be *independent* of the position of the common *origin* O, is said to be the *Mean Point* (comp. 29), of the *system* of the m *points*, $A_1, \ldots A_m$. It is evident that we have the equation,

$$0 = (\alpha_1 - \mu) + \ldots + (\alpha_m - \mu) = \Sigma(\alpha - \mu) = \Sigma MA;$$

or that the *sum* of the m vectors, drawn *from the mean point* M, *to* the points A of the system, is equal to *zero*. And hence (comp. 10, 11, 30), it follows Ist, that these m vectors are equal to the m *successive sides* of a *closed polygon;* IInd, that if the system and its mean point be *projected*, by any *parallel* *ordinates*, on any assumed *plane* (or *line*), the *projection* M′, of the *mean point* M, is the *mean point of the projected system*: and IIIrd, that the *ordinate* MM′, of the *mean point*, is the *mean* of all the *other ordinates*, $A_1A'_1, \ldots A_mA'_m$. It follows, also, that if N be the mean point of *another system*, $B_1, \ldots B_n$; and if S be the mean point of the *total system*, $A_1 \ldots B_n$, of the $m + n = s$ points obtained by *combining* the *two* former, considered as *partial systems;* while ν and σ may denote the *vectors*, ON and OS, of these two last mean points: then we shall have the equations,

$$m\mu = \Sigma\alpha, \qquad n\nu = \Sigma\beta, \qquad s\sigma = \Sigma\alpha + \Sigma\beta = m\mu + n\nu,$$

$$m(\sigma - \mu) = n(\nu - \sigma), \qquad m \,.\, \text{MS} = n \,.\, \text{SN};$$

so that the *general mean point*, S, is situated *on the right line* MN, which connects the two *partial mean points*, M and N; and *divides that line* (internally), into *two segments* MS and SN, which are *inversely proportional* to the two *whole numbers*, m and n.

(1.) As an Example, let ABCD be a *gauche quadrilateral*, and let E be its *mean point;* or more fully, let

$$\text{OE} = \tfrac{1}{4}(\text{OA} + \text{OB} + \text{OC} + \text{OD}),$$

or

$$\varepsilon = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta);$$

that is to say, let $a = b = c = d$, in the equations of Art. 65. Then, with notations lately used, for certain *derived points* D_1, &c., if we write the *vector formulæ*,

$$\text{OA}_1 = \alpha_1 = \tfrac{1}{3}(\beta + \gamma + \delta), \ldots \qquad \delta_1 = \tfrac{1}{3}(\alpha + \beta + \gamma),$$

$$\text{OA}_2 = \alpha_2 = \tfrac{1}{2}(\alpha + \delta), \ldots \qquad \gamma_2 = \tfrac{1}{2}(\gamma + \delta),$$

$$\text{OA}' = \alpha' = \tfrac{1}{2}(\beta + \gamma), \ldots \qquad \gamma' = \tfrac{1}{2}(\alpha + \beta),$$

we shall have *seven* different expressions for the *mean vector*, ε; namely, the following:

$$\varepsilon = \tfrac{1}{4}(\alpha + 3\alpha_1) = \ldots \tfrac{1}{4}(\delta + 3\delta_1)$$
$$= \tfrac{1}{2}(\alpha' + \alpha_2) = \ldots \tfrac{1}{2}(\gamma' + \gamma_2).$$

And these conduct to the seven equations between *segments*,

$$\text{AE} = 3\text{EA}_1 \ldots \qquad \text{DE} = 3\text{ED}_1;$$

$$\text{A}'\text{E} = \text{EA}_2, \ldots \qquad \text{C}'\text{E} = \text{EC}_2;$$

which prove (what is otherwise known) that the *four right lines*, here denoted by $AA_1, \ldots DD_1$, whereof each connects a *corner* of the *pyramid* ABCD with the mean point of the opposite *face*, *intersect* and *quadrisect* each other, in one *common point*, E; and that the *three common bisectors* $A'A_2$, $B'B_2$, $C'C_2$, of *pairs of* *opposite edges*, such as BC and DA, intersect and *bisect* each other, in the *same* mean point: so that the *four middle points*, C', A', C_2, A_2, of the four successive *sides* AB, &c., of the gauche quadrilateral ABCD, are situated in one *common plane*, which *bisects also* the *common bisector*, $B'B_2$, of the *two diagonals*, AC and BD.

(2.) In this example, the number s of the *points* A . . D being *four*, the number of the derived *lines*, which thus cross each other in their general mean point E is seen to be *seven*; and the number of the derived *planes* through that point is *nine*: namely, in the notation lately used for the *net* in space, four lines Λ_1, three lines $\Lambda_{2,1}$, six planes Π_1, and three planes $\Pi_{2,1}$. Of these *nine planes*, the *six former* may (in the present connexion) be called *triple planes*, because each contains *three lines* (as the plane ABE, for instance, contains the lines AA_1, BB_1, $C'C_2$), all passing through the mean point E; and the *three latter* may be said, by contrast, to be *non-triple planes*, because each contains only *two lines* through that point, determined on the foregoing principles.

(3.) In general, let $\phi(s)$ denote the *number* of the *lines*, through the *general mean points* S of a *total system* of s given points, which is thus, in all possible ways, decomposed into *partial systems*; let $f(s)$ denote the number of the *triple planes*, obtained by grouping the given points into *three* such partial systems; let $\psi(s)$ denote the number of *non-triple* planes, each determined by grouping those s points in two different ways into *two* partial systems; and let $F(s) = f(s) + \psi(s)$ represent the entire number of distinct planes through the point S: so that

$$\phi(4) = 7, \qquad f(4) = 6, \qquad \psi(4) = 3, \qquad F(4) = 9.$$

Then it is easy to perceive that if we introduce a *new point* C, each *old line* MN furnishes *two new lines*, according as we group the new point with one or other of the two old partial systems, (M) and (N); and that there is, besides, *one other new line*, namely CS: we have, therefore, the *equation in finite differences*,

$$\phi(s+1) = 2\phi(s) + 1;$$

which, with the *particular value* above assigned for $\phi(4)$, or even with the simpler and more obvious value, $\phi(2) = 1$, conducts to the *general expression*,

$$\phi(s) = 2^{s-1} - 1.$$

(4.) Again, if (M) (N) (P) be any *three partial systems*, which jointly make up the old or given *total* system (S); and if, by grouping a *new point* a with each of these in turn, we form *three new partial systems*, (M') (N') (P'); then *each old triple plane* such as MNP, will furnish *three new triple planes*,

$$M'NP, \qquad MN'P, \qquad MNP';$$

while *each old line*, KL, will give *one new* triple plane, CKL: nor can any new triple plane be obtained in any other way. We have, therefore, this new *equation in differences:*

$$f(s+1) = 3f(s) + \phi(s).$$

But we have seen that $\quad \phi(s+1) = 2\phi(s) + 1;$

if then we write, for a moment,

$$f(s) + \phi(s) = \chi(s),$$

we have this other equation in finite differences,

$$\chi(s+1) = 3\chi(s) + 1.$$

Also, $\quad f(3) = 1, \quad \phi(3) = 3, \quad \chi(3) = 4:$

therefore, $\quad 2\chi(s) = 3^{s-1} - 1,$

and $\quad 2f(s) = 3^{s-1} - 2^s + 1.$

(5.) Finally, it is clear that we have the relation,

$$3f(s) + \psi(s) = \tfrac{1}{2}\phi(s) \,.\, (\phi(s) - 1) = (2^{s-1} - 1)(2^{s-2} - 1);$$

because the *triple* planes, each treated as *three*, and the *non*-triple planes, each treated as *one*, must jointly represent all the *binary combinations* of the *lines*, drawn through the mean point S of the whole system. Hence,

$$2\psi(s) = 2^{2s-2} + 3 \,.\, 2^{s-1} - 3^s - 1;$$

and $\quad \text{F}(s) = 2^{2s-3} + 2^{s-2} - 3^{s-1};$

so that $\quad \text{F}(s+1) - 4\text{F}(s) = 3^{s-1} - 2^{s-1},$

and $\quad \psi(s+1) - 4\psi(s) = 3f(s);$

which last equation in finite differences admits of an independent geometrical interpretation.

(6.) For instance, these general expressions give,

$$\phi(5) = 15; \quad f(5) = 25; \quad \psi(5) = 30; \quad \text{F}(5) = 55;$$

so that if we assume a *gauche pentagon*, or a system of *five points in space*, A . . E, and determine the *mean point* F of this system, there will in general be a set of *fifteen lines*, of the kind above considered, all passing through this sixth point F: and these will be arranged generally in *fifty-five distinct planes*, whereof *twenty-five* will be what we have called *triple*, the *thirty others* being of the *non-triple* kind.

97. More generally, if $a_1 \ldots a_m$ be, as before, a system of *m given* and *co-initial vectors*, and if $a_1, \ldots a_m$ be any system of *m given scalars* (17), then that

new co-initial vector β, or OB, which is deduced from these by the formula,

$$\beta = \frac{a_1\alpha + \ldots + a_m\alpha_m}{a_1 + \ldots + a_m} = \frac{\Sigma a\alpha}{\Sigma a}, \text{ or } \mathrm{OB} = \frac{\Sigma a \mathrm{OA}}{\Sigma a},$$

or by the equation

$$\Sigma a\,(\alpha - \beta) = 0, \quad \text{or} \quad \Sigma a \mathrm{BA} = 0,$$

may be said to be the *Complex Mean* of those m given *vectors* α, or OA, considered as *affected* (or combined) with that system of given *scalars*, a, as *coefficients*, or as *multipliers* (12, 14). It may also be said that the *derived point* B, of which (comp. 96) the *position* is independent of that of the *origin* O, is the *Barycentre* (or *centre of gravity*) of the given system of points $\mathrm{A}_1 \ldots$, considered as *loaded* with the given *weights* $a_1 \ldots$; and theorems of *intersections* of *lines* and *planes* arise, from the comparison of these *complex means*, or *barycentres*, of *partial* and *total systems*, which are entirely analogous to those lately considered (96), for *simple means* of *vectors* and of *points*.

(1.) As an example, in the case of Art. 24, the point C is the barycentre of the system of the *two* points, A and B, with the weights a and b; while, under the conditions of 27, the origin O is the barycentre of the *three* points A, B, C, with the three weights a, b, c; and if we use the formula for ρ, assigned in 34 or 36, the same three given points A, B, C, when loaded with xa, yb, zc as weights, have the point P in their plane for their barycentre. Again, with the equations of 65, E is the barycentre of the system of the *four* given points, A, B, C, D, with the weights a, b, c, d; and if the expression of 79 for the vector OP be adopted, then xa, yb, zc, wd are equal (or proportional) to the weights with which the same four points A . . D must be loaded, in order that the point P of space may be their barycentre. In all these cases, the *weights* are thus *proportional* (by 69) to certain *segments*, or *areas*, or *volumes*, of kinds which have been already considered; and what we have called the *anharmonic co-ordinates* of a *variable point* P, in a *plane* (36), or in *space* (79), may be said, on the same plan, to be *quotients of quotients of weights.*

(2.) The circumstance that the *position* of a *barycentre* (97), like that of a simple *mean point* (96), is *independent* of the position of the assumed *origin* of vectors, might induce us (comp. 69) to *suppress* the *symbol* O of that *arbitrary* and *foreign point;* and therefore to write* simply, under the lately supposed conditions,

$$\mathrm{B} = \frac{\Sigma a\,\mathrm{A}}{\Sigma a} \text{ or } b\mathrm{B} = \Sigma a\mathrm{A}, \text{ if } b = \Sigma a.$$

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* We should thus have some of the principal *notations* of the *Barycentric Calculus:* but used mainly with a reference to *vectors*. Compare the note to page 50.

It is easy to prove (comp. 96), by principles already established, that the *ordinate of the barycentre* of any given system of points is the *complex mean* (in the sense above defined, and with the same system of *weights*), *of the ordinates of the points* of that system, with reference to *any given plane*: and that the *projection of the barycentre*, on any such plane, is the *barycentre of the projected system*.

(3.) Without any reference to *ordinates*, or to any foreign *origin*, the *barycentric notation* $\mathrm{B} = \frac{\Sigma a \mathrm{A}}{\Sigma a}$ may be *interpreted*, by means of our *fundamental convention* (Art. 1) respecting the geometrical signification of the *symbol* $\mathrm{B} - \mathrm{A}$, considered as denoting the *vector* from A to B: together with the rules for *multiplying* such vectors by *scalars* (14, 17), and for taking the *sums* (6, 7, 8, 9) of those (generally new) vectors, which are (15) the *products* of such multiplications. For we have only to write the formula as follows,

$$\Sigma a\,(\mathrm{A} - \mathrm{B}) = 0,$$

in order to perceive that it may be considered as signifying, that the system of the *vectors from the barycentre* B, to the system of the *given points* $\mathrm{A}_1, \mathrm{A}_2, \ldots$ when multiplied respectively by the *scalars* (or coefficients) of the given system $a_1, a_2, \ldots$ becomes (generally) a new system of vectors with a *null sum*: in such a manner that these last vectors, $a_1 . \mathrm{BA}_1, a_2 . \mathrm{BA}_2, \ldots$ can be made (10) the *successive sides of a closed polygon*, by transports without rotation.

(4.) Thus if we meet the formula,

$$\mathrm{B} = \tfrac{1}{2}(\mathrm{A}_1 + \mathrm{A}_2),$$

we may indeed interpret it as an *abridged form* of the equation,

$$\mathrm{OB} = \tfrac{1}{2}(\mathrm{OA}_1 + \mathrm{OA}_2);$$

which implies that if O be any *arbitrary point*, and if O′ be the point which *completes* (comp. 6) the *parallelogram* $\mathrm{A}_1\mathrm{O}\mathrm{A}_2\mathrm{O}'$, then B is the point which *bisects the diagonal* OO′, and therefore also the *given line* $\mathrm{A}_1\mathrm{A}_2$, which is here the *other diagonal*. But we may *also* regard the formula as a mere *symbolical transformation* of the equation,

$$(\mathrm{A}_2 - \mathrm{B}) + (\mathrm{A}_1 - \mathrm{B}) = 0;$$

which (by the earliest principles of the present Book) expresses that the *two vectors*, from B to the two given points A_1 and A_2, have a *null sum*; or that they are *equal in length*, but *opposite in direction*: which can only be, by B bisecting $\mathrm{A}_1\mathrm{A}_2$, as before.

(5.) Again, the formula, $B_1 = \frac{1}{3}(A_1 + A_2 + A_3)$, *may* be interpreted as an *abridgment* of the equation,

$$OB_1 = \tfrac{1}{3}(OA_1 + OA_2 + OA_3),$$

which expresses that the point B *trisects the diagonal* OO′ of the *parallelepiped* (comp. 62), which has OA_1, OA_2, OA_3 for *three co-initial edges.* But the same formula may also be considered to express, in full consistency with the foregoing interpretation, that the *sum of the three vectors*, from B to the three points A_1, A_2, A_3, *vanishes:* which is the characteristic property (30) of the *mean point* of the *triangle* $A_1A_2A_3$. And similarly in more complex cases: the *legitimacy* of such *transformations* being here regarded as a consequence of the original *interpretation* (1) of the *symbol* B − A, and of the rules for *operations on vectors*, so far as they have been hitherto established.

SECTION 6.

On Anharmonic Equations, and Vector Expressions of Surfaces and Curves in Space.

98. When, in the expression 79 for the vector ρ of a *variable point* P of space, the four variable scalars, or anharmonic co-ordinates, $xyzw$, are *connected* (comp. 46) by a given algebraic equation,

$$f_p(x, y, z, w) = 0, \quad \text{or briefly } f = 0,$$

supposed to be rational and integral, and homogeneous of the p^{th} dimension, then the point P has for its *locus* a *surface of the p^{th} order*, whereof $f = 0$ may be said (comp. 56) to be the *local equation.* For if we substitute instead of the co-ordinates $x \,..\, w$, expressions of the forms,

$$x = tx_0 + ux_1, \,.. \qquad w = tw_0 + uw_1,$$

to indicate (82) that P is *collinear* with two given points P_0, P_1, the resulting algebraic *equation* in $t : u$ is of the p^{th} *degree;* so that (according to a received modern mode of speaking), the *surface* may be said to be *cut in p points* (distinct or coincident, and real or imaginary*), *by any arbitrary right line* P_0P_1.

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* It is to be observed, that *no interpretation is here proposed*, for *imaginary intersections* of this kind, such as those of a *sphere* with a *right line*, which is *wholly external* thereto. The *language* of modern *geometry* requires that *such* imaginary *intersections* should be *spoken of*, and even that they should be *enumerated:* exactly as the *language* of *algebra* requires that we should *count* what are called the *imaginary roots* of an equation. But it would be an error to confound *geometrical imaginaries*, of *this* sort, with those *square roots of negatives*, for which it will soon be seen that the *Calculus of Quaternions* supplies, from the outset, a *definite* and *real interpretation.*

And in like mamner, when the four anharmonic co-ordinates *lmnr* of a *variable plane* Π (80) are connected by an algebraical equation of the form,

$$F_q(l, m, n, r) = 0, \text{ or briefly } F = 0,$$

where F denotes a rational and integral function, supposed to be homogenous of the q^{th} dimension, then this plane Π has for its *envelope* (comp. 56) a *surface of the* q^{th} *class*, with $F = 0$ for its *tangential equation:* because if we make

$$l = tl_0 + ul_1, \ldots \qquad r = tr_0 + ur_1,$$

to express (comp. 82) that the variable plane Π passes *through a given right line* $\Pi_0 \cdot \Pi_1$, we are conducted to an algebraical *equation of the* q^{th} *degree*, which gives q (real or imaginary) values for the ratio $t : u$, and thereby assigns q (real or imaginary*) *tangent planes to the surface*, drawn through any such given but arbitrary right line. We may add (comp. 51, 56), that if the functions f and F be only *homogeneous* (without *necessarily* being *rational* and *integral*), then

$$[D_x f, \ D_y f, \ D_z f, \ D_w f]$$

is the *anharmonic symbol* (80) of the *tangent plane* to the surface $f = 0$, at the point $(xyzw)$; and that

$$(D_l F, \ D_m F, \ D_n F, \ D_r F)$$

is in like manner, a symbol for the *point of contact* of the plane $[lmnr]$, with its *enveloped surface*, $F = 0$; $D_x, \ldots D_l, \ldots$ being characteristics of *partial derivation*.

(1.) As an example, the *surface of the second order*, which passes through the *nine points* called lately

$$A, \ C', \ B, \ A', \ C, \ C_2, \ D, \ A_2, \ E,$$

has for its *local equation*,

$$0 = f = xz - yw;$$

which gives, by differentiation,

$$l = D_x f = z; \qquad m = D_y f = -w;$$

$$n = D_z f = x; \qquad r = D_w f = -y;$$

so that

$$[z, \ -w, \ x, \ -y]$$

is a symbol for the *tangent plane*, at the point (x, y, z, w).

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* As regards the *uninterpreted character* of such *imaginary contacts* in geometry, the preceding note to the present Article, respecting *imaginary intersections*, may be consulted.

(2.) In fact, the *surface* here considered is the *ruled* (or *hyperbolic*) *hyperboloid*, on which the *gauche quadrilateral* ABCD is *superscribed*, and which passes also through the point E. And if we write

$$\text{P} = (xyzw), \quad \text{Q} = (xy00), \quad \text{R} = (0yz0), \quad \text{S} = (00zw), \quad \text{T} = (x00w),$$

then QS and RT (see the annexed figure 31), namely, the lines drawn through P to intersect the two pairs, AB, CD, and BC, DA, of opposite sides of that quadrilateral ABCD, are the two generating lines, or *generatrices*, through that point; so that their plane, QRST, is the *tangent plane* to the surface, at the point P. If, then, we denote that tangent plane by the symbol $[lmnr]$, we have the equations of condition,

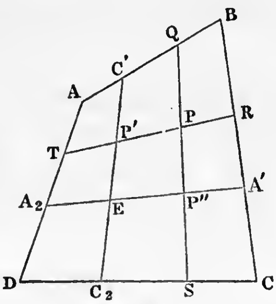

Fig. 31.

$$0 = lx + my = my + nz = nz + rw = rw + lx;$$

whence follows the proportion,

$$l : m : n : r = x^{-1} : -y^{-1} : z^{-1} : -w^{-1};$$

or, because $xz = yw$,

$$l : m : n : r = z : -w : x : -y,$$

as before.

(3.) At the same time we see that

$$(\text{AC}'\text{BQ}) = \frac{x}{y} = \frac{w}{z} = (\text{DC}_2\text{CS});$$

so that the *variable generatrix* QS *divides* (as is known) *the two fixed generatrices* AB and DC *homographically**; AD, BC, and $\text{C}'\text{C}_2$ being three of its positions. Conversely, if it were proposed *to find the locus of the right line* QS, which thus *divides* homographically (comp. 26) *two given right lines in space*, we might take AB and DC for those two given lines, and AD, BC, $\text{C}'\text{C}_2$ (with the recent meanings of the letters) for three given positions of the variable line; and then should have, for the two variable but *corresponding* (or *homologous*) *points* Q, S themselves, and for any arbitrary point P *collinear* with them, anharmonic symbols of the forms,

$$\text{Q} = (s, u, 0, 0), \quad \text{S} = (0, 0, u, s), \quad \text{P} = (st, tu, uv, vs);$$

because, by 82, we should have, between these three symbols, a relation of the form

$$(\text{P}) = t(\text{Q}) + v(\text{S}):$$

if then we write $\text{P} = (x, y, z, w)$, we have the anharmonic equation $xz = yw$, as

* Compare 298 of the *Géométrie Supérieure*.

before; so that the *locus*, whether of the *line* QS, or of the *point* P, is (as is known) a ruled *surface of the second order*.

(4.) As regards the known *double generation* of that surface, it may suffice to observe that if we write, in like manner,

$$\mathrm{R} = (0tv0), \qquad \mathrm{T} = (t00v), \qquad (\mathrm{P}) = u\,(\mathrm{R}) + s\,(\mathrm{T}),$$

we shall have *again* the expression,

$$\mathrm{P} = (st,\, tu,\, uv,\, vs), \quad \text{giving} \quad xz = yw,$$

as before: so that the *same* hyperboloid is *also* the *locus* of that *other line* RT, which divides the *other pair* of *opposite sides* BC, AD of the same gauche quadrilateral ABCD homographically; BA, CD, and $\mathrm{A}'\mathrm{A}_2$ being three of its positions; and the lines $\mathrm{A}'\mathrm{A}_2$, $\mathrm{C}'\mathrm{C}_2$ being still supposed to intersect each other in the given point E.

(5.) The symbol of an arbitrary point on the variable line RT is (by sub-art. 2) of the form, $t\,(0, y, z, 0) + u\,(x, 0, 0, w)$, or (ux, ty, tz, uw); while the symbol of an arbitrary point on the given line $\mathrm{C}'\mathrm{C}_2$ is (t', t', u', u'). And these two symbols represent one *common point* (comp. fig. 31),

$$\mathrm{P}' = \mathrm{RT} \cdot \mathrm{C}'\mathrm{C}_2 = (y, y, z, z),$$

when we suppose $\qquad t' = y,\ u' = z,\ t = 1,\ u = \dfrac{y}{x} = \dfrac{z}{w}.$

Hence the known theorem results, that a *variable generatrix*, RT, *of one system*, *intersects three fixed lines*, BC, AD, $\mathrm{C}'\mathrm{C}_2$, which are *generatrices of the other system*. Conversely, by the same comparison of symbols, for points on the two lines RT and $\mathrm{C}'\mathrm{C}_2$, we should be conducted to the equation $xz = yw$, as the *condition* for their *intersection;* and thus should obtain this other known theorem, that *the locus of a right line, which intersects three given right lines in space*, is generally an *hyperboloid* with those three lines for *generatrices*. A similar analysis shows that QS intersects $\mathrm{A}'\mathrm{A}_2$, in a point (comp. again fig. 31) which may be thus denoted:

$$\mathrm{P}'' = \mathrm{QS} \cdot \mathrm{A}'\mathrm{A}_2 = (xyyx).$$

(6.) As another example of the treatment of surfaces by their anharmonic and local equations, we may remark that the recent symbols for P′ and P″, combined with those of sub-art. (2.) for P, Q, R, S, T; with the symbols of 83, 86 for C′, A′, C_2, A_2, E; and with the equation $xz = yw$, give the expressions:

$$(\mathrm{P}) = (\mathrm{Q}) + (\mathrm{S}) = (\mathrm{R}) + (\mathrm{T}); \qquad (\mathrm{P}') = y\,(\mathrm{C}') + z\,(\mathrm{C}_2) = (\mathrm{R}) + \frac{y}{x}\,(\mathrm{T});$$

$$(\mathrm{E}) = (\mathrm{C}') + (\mathrm{C}_2) = (\mathrm{A}') + (\mathrm{A}_2); \qquad (\mathrm{P}'') = y\,(\mathrm{A}') + x\,(\mathrm{A}_2) = (\mathrm{Q}) + \frac{y}{z}(\mathrm{S});$$

whence it follows (84) that the two points P′, P″, and the sides of the quadrilateral ABCD, divide the four generating lines through P and E in the following anharmonic ratios:

$$(\mathrm{C'EC_2P'}) = (\mathrm{QP''SP}) = \frac{y}{z} = (\mathrm{BA'CR}) = (\mathrm{AA_2DT});$$

$$(\mathrm{A'EA_2P''}) = (\mathrm{RP'TP}) = \frac{y}{x} = (\mathrm{BC'AQ}) = (\mathrm{CC_2DS});$$

so that (as again is known) the *variable* generatrices, as well as the *fixed* ones, of the hyperboloid, are *all* divided *homographically*.

(7.) The *tangential equation* of the present surface is easily found, by the expressions in sub-art. (1.) for the co-ordinates $lmnr$ of the tangent plane, to be the following:

$$0 = \mathrm{F} = ln - mr;$$

which may be interpreted as expressing, that this hyperboloid is the *surface of the second class*, which *touches the nine planes*,

$$[1000], [0100], [0010], [0001], [1100], [0110], [0011], [1001], [1111];$$

or with the literal symbols lately employed (comp. 86, 87),

BCD, CDA, DAB, ABC, CDC″, DAA″, ABC′$_2$, BCA′$_2$, and [E].

Or we may interpret the same tangential equation $\mathrm{F} = 0$ as expressing (comp. again 86, 87, where Q, L, N are now replaced by T, R, Q), that the surface is the *envelope of a plane* QRST, which satisfies *either* of the *two* connected *conditions* of *homography:*

$$(\mathrm{BC'AQ}) = -\frac{l}{m} = -\frac{r}{n} = (\mathrm{CC_2DS});$$

$$(\mathrm{CA'BR}) = -\frac{m}{n} = -\frac{l}{r} = (\mathrm{DA_2AT});$$

a *double generation* of the hyperboloid thus showing itself in a new way. And as regards the *passage* (or *return*), from the *tangential* to the *local equation* (comp. 56), we have in the present example the formulæ:

$$x = \mathrm{D}_l\mathrm{F} = n; \quad y = \mathrm{D}_m\mathrm{F} = -r; \quad z = \mathrm{D}_n\mathrm{F} = l; \quad w = \mathrm{D}_r\mathrm{F} = -m;$$

whence $xz - yw = 0$, as before.

(8.) More generally, when the surface is of the *second order*, and therefore also of the *second class*, so that the two functions f and F, when presented under rational and integral forms, are both homogeneous of the *second dimension*, then whether we derive $l \,..\, r$ from $x \,..\, w$ by the formulæ,

$$l = \mathrm{D}_x f, \quad m = \mathrm{D}_y f, \quad n = \mathrm{D}_z f, \quad r = \mathrm{D}_w f,$$

or $x \,..\, w$ from $l \,..\, r$ by the *converse* formulæ,

$$x = \mathrm{D}_l \mathrm{F}, \quad y = \mathrm{D}_m \mathrm{F}, \quad z = \mathrm{D}_n \mathrm{F}, \quad w = \mathrm{D}_r \mathrm{F},$$

the *point* $\mathrm{P} = (xyzw)$ is, relatively to that surface, what is usually called (comp. 52) the *pole* of the *plane* $\Pi = [lmnr]$; and conversely, the plane Π is the *polar* of the point P; *wherever in space* the point P and plane Π, thus *related to each other*, may be situated. And because the *centre* of a surface of the second order is known to be (comp. again 52) the *pole* of (what is called) the *plane at infinity;* while (comp. 38) the *equation* and the *symbol* of this last *plane* are, respectively,

$$ax + by + cz + dw = 0, \quad \text{and} \quad [a, b, c, d],$$

if the four constants $abcd$ have still the same significations as in 65, 70, 79, &c., with reference to the system of the five given points ABCDE: it follows that we may denote this *centre* by the symbol,

$$\mathrm{K} = (\mathrm{D}_a \mathrm{F}_0, \ \mathrm{D}_b \mathrm{F}_0, \ \mathrm{D}_c \mathrm{F}_0, \ \mathrm{D}_d \mathrm{F}_0);$$

where F_0 denotes, for abridgment, the function $\mathrm{F}\ (abcd)$, and d is still a scalar constant.

(9.) In the recent example, we have $\mathrm{F}_0 = ac - bd$; and the anharmonic symbol for the centre of the hyperboloid becomes thus,

$$\mathrm{K} = (c, -d, a, -b).$$

Accordingly if we assume (comp. sub-arts. (3.), (4.),)

$$\mathrm{P} = (st, tu, uv, vs), \qquad \mathrm{P}' = (s't', -t'u', u'v', -v's'),$$

where s, t, u, v are any four scalars, and P' is a new point, while

$$s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bu;$$

if also we write, for abridgment,

$$e' = ac - bd, \qquad w' = ast + btu + cuv + dvs;$$

we shall then have the symbolic relations,

$$e'(\mathrm{P}) + (\mathrm{P}') = w'(\mathrm{K}), \qquad e'(\mathrm{P}) - (\mathrm{P}') = (\mathrm{P}''),$$

if $\mathrm{P}'' = (x''y''z''w'')$ be that new point, of which the co-ordinates are,

$$x'' = 2e'st - cw', \quad y'' = 2e'tu + dw', \quad z'' = 2e'uv - aw', \quad w'' = 2e'vs + bw',$$

and therefore, $$ax'' + by'' + cz'' + dw'' = 0.$$

That is to say, if PP′ be *any chord of the hyperboloid*, which passes *through the fixed point* K, and if P″ be the *harmonic conjugate* of that fixed point, with respect to that variable chord, so that (PKP′P″) = − 1, then this conjugate point P″ is *on the infinitely distant plane* [*abcd*] : or in other words, *the fixed point* K *bisects all the chords* PP′ *which pass through it*, and is therefore (as above asserted) the *centre* of the surface.

(10.) With the same meanings (65, 79) of the constants a, b, c, d, the *mean point* (96) of the quadrilateral ABCD, or of the system of its corners, may be denoted by the symbol,

$$\text{M} = (a^{-1},\ b^{-1},\ c^{-1},\ d^{-1})\ ;$$

if then this mean point be *on the surface*, so that

$$ac - bd = 0,$$

the *centre* K is *on the plane* $[a, b, c, d]$; or in other words, it is *infinitely distant:* so that the surface becomes, in this case, a *ruled* (or *hyperbolic*) *paraboloid.* In general (comp. sub-art. (8.)), if $\text{F}_0 = 0$, the surface of the second order is a *paraboloid* of *some* kind, because its *centre* is then *at infinity*, in virtue of the equation

$$(a\text{D}_a + b\text{D}_b + c\text{D}_c + d\text{D}_d)\ \text{F}_0 = 0\ ;$$

or because (comp. 50, 58) the *plane* [*abcd*] *at infinity* is then one of its *tangent planes*, as satisfying its *tangential equation*, $\text{F} = 0$.

(11.) It is evident that a *curve in space* may be represented by a system of *two* anharmonic and *local equations ;* because it may be regarded as the *intersection* of *two surfaces.* And then its *order*, or the *number of points* (real or imaginary*), in which it is *cut by an arbitrary plane*, is obviously the *product* of the *orders* of those two surfaces ; or the *product* of the *degrees* of their two local equations, supposed to be rational and integral.

(12.) A *curve of double curvature* may also be considered as the *edge of regression* (or *arête de rebroussement*) of a *developable surface*, namely of the *locus of the tangents to the curve ;* and this surface may be supposed to be *circumscribed* at once *to two given surfaces*, which are *envelopes of variable planes* (98), and are represented, as such, by their *tangential equations.* In *this* view, a *curve* of double curvature may *itself* be represented by a system of *two* anharmonic and *tangential* equations; and if the *class* of such a curve be defined to be the *number of its osculating planes*, which pass *through an arbitrary point of space*, then this class is the *product of the classes* of the *two curved sur-*

* Compare the notes to pages 87, 88.

faces just now mentioned: or (what comes to the same thing) it is the *product of the dimensions* of the two tangential *equations*, by which the curve is (on this plan) symbolized. But we cannot enter further into these details; the *mechanism* of calculation respecting which would indeed be found to be the same, as that employed in the known method (comp. 41) of *quadriplanar co-ordinates*.

99. Instead of anharmonic co-ordinates, we may consider *any other system* of *n variable scalars*, $x_1, \ldots x_n$, which enter into the expression of a *variable vector*, ρ; for example, into an expression of the form (comp. 96, 97),

$$\rho = x_1 a_1 + x_2 a_2 + \ldots = \Sigma x a.$$

And then, if those *n scalars x* be all *functions of one independent and variable scalar*, *t*, we may regard this *vector* ρ as being *itself a function* of that *single scalar;* and may write,

$$\text{I.} \ldots \rho = \phi\,(t).$$

But if the n scalars x .. be functions of *two* independent and scalar variables, t and u, then ρ becomes a function of those *two scalars*, and we may write accordingly,

$$\text{II.} \ldots \rho = \phi\,(t, u).$$

In the Ist case, the term P (comp. 1) of the variable vector ρ has *generally* for its *locus* a *curve in space*, which may be plane or of double curvature, or may even become a *right line*, according to the *form* of the *vector-function* ϕ; and ρ may be said to be *the vector of this line, or curve*. In the IInd case, ρ is *the vector of a surface*, plane or curved, according to the form of $\phi\,(t, u)$; or to the manner in which this *vector* ρ depends on the *two independent scalars* that enter into its expression.

(1.) As examples (comp. 25, 63), the expressions,

$$\text{I.} \ldots \rho = \frac{a + t\beta}{1 + t}; \qquad \text{II.} \ldots \rho = \frac{a + t\beta + u\gamma}{1 + t + u},$$

signify, Ist, that ρ is the vector of a variable point P *on the right line* AB; or that it is *the vector of that line itself*, considered as the *locus of a point;* and IInd, that ρ is the *vector of the plane* ABC, considered in like manner as the locus of an arbitrary point P thereon.

(2.) The equations,

$$\text{I.} \ldots \rho = xa + y\beta, \qquad \text{II.} \ldots \rho = xa + y\beta + z\gamma,$$

with $\quad x^2 + y^2 = 1$ for the Ist, and $x^2 + y^2 + z^2 = 1$ for the IInd,

signify Ist, that ρ is the *vector of an ellipse*, and IInd, that it is the *vector of* *an ellipsoid*, with the *origin* O for their common *centre*, and with OA, OB, or OA, OB, OC, for *conjugate semi-diameters*.

(3.) The equation (comp. 46),

$$\rho = t^2a + u^2\beta + (t + u)^2\gamma,$$

expresses that ρ is the *vector of a cone of the second order*, with O for its *vertex* (or *centre*), which is *touched by the three planes* OBC, OCA, OAB; the *section* of this *cone*, made by the plane ABC, being an *ellipse* (comp. fig. 25), which is *inscribed in the triangle* ABC; and the *middle points* A′, B′, C′, of the *sides* of that triangle, being the *points of contact* of those sides with that conic.

(4.) The equation (comp. 53),

$$\rho = t^{-1}a + u^{-1}\beta + v^{-1}\gamma, \quad \text{with} \quad t + u + v = 0,$$

expresses that ρ is the *vector of another cone of the second order*, with O still for *vertex*, but with OA, OB, OC for three of its *sides* (or *rays*). The *section* by the plane ABC is a new *ellipse*, *circumscribed* to the triangle ABC, and having its *tangents* at the *corners* of that triangle respectively *parallel* to the *opposite sides* thereof.

(5.) The equation (comp. 54),

$$\rho = t^3a + u^3\beta + v^3\gamma, \quad \text{with} \quad t + u + v = 0,$$

signifies that ρ is the vector of a *cone of the third order*, of which the vertex is still the origin; its section (comp. fig. 27) by the plane ABC being a *cubic curve*, whereof the sides of the triangle ABC are at once the *asymptotes*, and the three (real) *tangents of inflexion*; while the *mean point* (say O′) of that *triangle* is a *conjugate point* of the *curve*; and therefore the *right line* OO′, from the vertex O to that mean point, may be said to be a *conjugate ray* of the *cone*.

(6.) The equation (comp. 98, sub-art. (3.)),

$$\rho = \frac{sta\alpha + tub\beta + uvc\gamma + vsd\delta}{sta + tub + uvc + vsd},$$

in which $\frac{s}{u}$ and $\frac{t}{v}$ are two variable scalars, while a, b, c, d are still four constant scalars, and α, β, γ, δ are four constant vectors, but ρ is still a variable vector, expresses that ρ is the *vector of a ruled* (or *single-sheeted*) *hyperboloid*, on which the gauche quadrilateral ABCD is superscribed, and which passes through the given point E, whereof the vector ε is assigned in 65.

(7.) If we make (comp. 98, sub-art. (9.)),

$$\rho' = \frac{s't'a\alpha - t'u'b\beta + u'v'c\gamma - v's'd\delta}{s't'a - t'u'b + u'v'c - v's'd},$$

where $\quad s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bu,$

then $\rho' = \text{OP}'$ is the vector of *another point* P′ on the *same hyperboloid;* and because it is found that the *sum* of these two last vectors is *constant,*

$$\rho + \rho' = 2\kappa, \text{ if } \kappa = \frac{ac(\alpha + \gamma) - bd(\beta + \delta)}{2(ac - bd)},$$

it follows that κ is the vector of a *fixed point* K, which *bisects every chord* PP′ that passes through it: or in other words (comp. 52), that this point K is the *centre of the surface.*

(8.) The three vectors, $\quad \kappa, \quad \dfrac{\alpha + \gamma}{2}, \quad \dfrac{\beta + \delta}{2},$

are *termino-collinear* (24); if then a gauche quadrilateral ABCD be superscribed on a ruled hyperboloid, *the common bisector of the two diagonals,* AC, BD, *passes through the centre* K.

(9.) When $ac = bd$, or when we have the equation,

$$\rho = \frac{st\alpha + tu\beta + uv\gamma + vs\delta}{st + tu + uv + vs},$$

or simply, $\qquad \rho = st\alpha + tu\beta + uv\gamma + vs\delta$, with $s + u = t + v = 1$,

ρ is then the *vector of a ruled paraboloid,* of which the *centre* (comp. 52, and 98, sub-art. (10.)), is *infinitely distant,* but upon which the quadrilateral ABCD is still *superscribed.* And this surface passes *through the mean point* M of that quadrilateral, or of the system of the four given points A . . D; because, when $s = t = u = v = \frac{1}{2}$, the variable vector ρ takes the value (comp. 96, sub-art. (1.)),

$$\mu = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta).$$

(10.) In general, it is easy to prove, *from the last vector-expression* for ρ, that this paraboloid is the *locus of a right line,* which *divides similarly* the two *opposite sides* AB and DC of the same gauche quadrilateral ABCD; *or* the *other pair* of opposite sides, BC and AD.

SECTION 7.

On Differentials of Vectors.

100. The equation (99, I.),

$$\rho = \phi(t),$$

in which $\rho = \text{OP}$ is *generally* the vector of a point P of a *curve in space,* PQ . . ., gives evidently, for the vector OQ of *another point* Q of the same curve, an expression of the form

$$\rho + \Delta\rho = \phi(t + \Delta t);$$

so that the *chord* PQ, regarded as being itself a *vector*, comes thus to be represented (4) by the *finite difference*,

$$\text{PQ} = \Delta\rho = \Delta\phi\,(t) = \phi\,(t + \Delta t) - \phi\,(t).$$

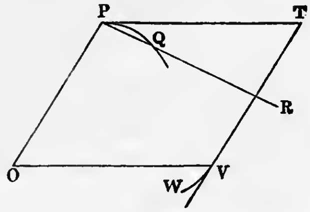

Fig. 32.

Suppose now that the *other* finite difference, Δt, is the n^{th} *part* of a *new scalar*, u; and that the chord $\Delta\rho$, or PQ, is in like manner (comp. fig. 32), the n^{th} part of a *new vector*, σ_n, or PR; so that we may write,

$$n\Delta t = u, \text{ and } n\Delta\rho = n\,.\,\text{PQ} = \sigma_n = \text{PR}.$$

Then, if we treat the *two scalars*, t and u, as *constant*, but the *number* n as *variable* (the *form* of the *vector-function* ϕ, and the *origin* O, being *given*), the *vector* ρ and the *point* P will be *fixed:* but the *two points* Q and R, the *two differences* Δt and $\Delta\rho$, and the *multiple vector* $n\Delta\rho$, or σ_n, will (in general) *vary together*. And if this number n be *indefinitely increased*, or made to *tend to infinity*, then *each* of the two differences Δt, $\Delta\rho$ will in general *tend to zero;* such being the *common limit*, of $n^{-1}u$, and of $\phi\,(t + n^{-1}u) - \phi\,(t)$: so that the *variable point* Q of the curve will *tend to coincide* with the *fixed point* P. But although the *chord* PQ will thus be *indefinitely shortened*, its n^{th} *multiple*, PR or σ_n, will *tend* (generally) to a *finite limit*,* depending on the supposed *continuity* of the *function* $\phi\,(t)$; namely, to a certain *definite vector*, PT, or σ_∞, or (say) τ, which vector PT will evidently be (in general) *tangential to the curve:* or, in other words, the *variable point* R will *tend to a fixed position* T, *on the tangent* to that curve at P. We shall thus have a *limiting equation*, of the form

$$\tau = \text{PT} = \lim.\ \text{PR} = \sigma_\infty = \lim_{n=\infty}.\ n\Delta\phi\,(t), \quad \text{if } n\Delta t = u;$$

t and u being, as above, *two given* and (generally) *finite scalars*. And if we then agree to *call* the *second* of these two given *scalars* the *differential* of the first, and to *denote* it by the *symbol* dt, we shall *define* that the *vector-limit*, τ or σ_∞, is the (corresponding) *differential of the vector* ρ, and shall *denote* it by the *corresponding symbol*, dρ; so as to have, under the supposed conditions,

$$u = \mathrm{d}t, \text{ and } \tau = \mathrm{d}\rho.$$

Or, eliminating the two symbols u and τ, and *not necessarily supposing* that P is a *point of a curve*, we may *express* our *Definition*† *of the Differential of a*

* Compare Newton's *Principia*.

† Compare the Note to page 35.

Vector ρ, considered as a *Function* ϕ *of a Scalar* t, by the following *General Formula:*

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \lim_{n=\infty} n\left\{\phi\left(t+\frac{\mathrm{d}t}{n}\right)-\phi(t)\right\},$$

in which t and $\mathrm{d}t$ are *two arbitrary and independent scalars, both generally finite;* and $\mathrm{d}\rho$ is, *in general*, a *new and finite vector, depending on those* TWO *scalars*, according to a *law* expressed by the *formula*, and *derived* from that *given law*, whereby the *old* or *former vector*, ρ or $\phi(t)$ *depends* upon the *single scalar*, t.

(1.) As an example, let the given *vector-function* have the form,

$$\rho = \phi(t) = \tfrac{1}{2}t^2\alpha, \text{ where } \alpha \text{ is a } \textit{given vector}.$$

Then, making $\Delta t = \frac{u}{n}$, where u is *any given scalar*, and n is a *variable whole number*, we have

$$\Delta\rho = \Delta\phi(t) = \frac{\alpha}{2}\left\{\left(t+\frac{u}{n}\right)^2 - t^2\right\} = \frac{\alpha u}{n}\left(t+\frac{u}{2n}\right);$$

$$\sigma_n = n\Delta\rho = \alpha u\left(t+\frac{u}{2n}\right); \quad \sigma_\infty = \alpha tu;$$

and finally, writing $\mathrm{d}t$ and $\mathrm{d}\rho$ for u and σ_∞,

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \mathrm{d}\left(\frac{t^2\alpha}{2}\right) = \alpha t\mathrm{d}t.$$

(2.) In general, let $\phi(t) = \alpha f(t)$, where α is still a given or *constant vector*, and $f(t)$ denotes a *scalar function* of the *scalar variable*, t. Then because α is a *common factor* within the brackets { } of the recent general formula (100) for $\mathrm{d}\rho$, we may write,

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \mathrm{d}\,.\,\alpha f(t) = \alpha\mathrm{d}f(t);$$

provided that we now *define* that the *differential of a scalar function*, $f(t)$, is a *new scalar function of two independent scalars*, t and $\mathrm{d}t$, determined by the precisely similar formula:

$$\mathrm{d}f(t) = \lim_{n=\infty} n\left\{f\left(t+\frac{\mathrm{d}t}{n}\right)-f(t)\right\};$$

which can easily be proved to *agree*, in all its *consequences*, with the *usual rules* for *differentiating functions of one variable.*

(3.) For example, if we write $\mathrm{d}t = nh$, where h is a *new variable scalar*, namely, the n^{th} *part* of the *given* and (generally) *finite differential*, $\mathrm{d}t$, we shall thus have the equation,

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{h=0}\frac{f(t+h)-f(t)}{h};$$

in which the first member is *here* considered as the actual *quotient of two finite* *scalars*, $\mathrm{d}f(t) : \mathrm{d}t$, and *not merely* as a *differential coefficient*. We *may*, however, as usual, consider this *quotient*, from the expression of which the *differential* $\mathrm{d}t$ *disappears*, as a *derived function* of the *former variable*, t; and may *denote* it, as such, by either of the *two usual symbols*,

$$f'(t) \text{ and } \mathrm{D}_t f(t).$$

(4.) In like manner we may write, for the *derivative of a vector-function*,* $\phi(t)$, the *formula*:

$$\rho' = \phi'(t) = \mathrm{D}_t\rho = \mathrm{D}_t\phi(t) = \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}\phi(t)}{\mathrm{d}t};$$

these two last forms denoting that *actual and finite vector*, ρ' or $\phi'(t)$, which is obtained, or *derived*, by *dividing* (comp. 16) the *not less actual* (or finite) *vector*, $\mathrm{d}\rho$ or $\mathrm{d}\phi(t)$, *by the finite scalar*, $\mathrm{d}t$. And if again we denote the n^{th} *part* of this last scalar by h, we shall thus have the *equally general formula*:

$$\mathrm{D}_t\rho = \mathrm{D}_t\phi(t) = \lim_{h=0} \frac{\phi(t+h) - \phi(t)}{h};$$

with the equations

$$\mathrm{d}\rho = \mathrm{D}_t\rho \,.\, \mathrm{d}t = \rho'\mathrm{d}t; \qquad \mathrm{d}\phi(t) = \mathrm{D}_t\phi(t) \,.\, \mathrm{d}t = \phi'(t) \,.\, \mathrm{d}t,$$

exactly as if the *vector-function*, ρ or ϕ, were a *scalar function*, f.

(5.) The particular value, $\mathrm{d}t = 1$, gives thus $\mathrm{d}\rho = \rho'$; so that the *derived vector* ρ' is (with our definitions) a particular but important *case* of the *differential of a vector*. In applications to *mechanics*, if t denote the *time*, and if the *term* P of the *variable vector* ρ be considered as a *moving point*, this *derived* vector ρ' may be called the *Vector of Velocity:* because its *length* represents the *amount*, and its *direction* is the *direction* of the velocity. And if, by setting off *vectors* OV $= \rho'$ (comp. again fig. 32) *from one origin*, to represent thus the *velocities* of a point moving in space according to any supposed law, expressed by the equation $\rho = \phi(t)$, we construct a *new curve* VW .. of which the corresponding equation may be written as $\rho' = \phi'(t)$, then this *new* curve has been defined to be the HODOGRAPH,† as the *old* PQ .. may be called the *orbit* of the motion, or of the moving point.

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* In the theory of *Differentials of Functions of Quaternions*, a *definition* of the *differential* $\mathrm{d}\phi(q)$ will be proposed, which is expressed by an equation of precisely the *same form* as those above assigned, for $\mathrm{d}f(t)$, and for $\mathrm{d}\phi(t)$; but it will be found that, *for quaternions*, the *quotient* $\mathrm{d}\phi(q) : \mathrm{d}q$ is *not generally independent of* $\mathrm{d}q$; and consequently that it cannot properly be called a *derived function*, such as $\phi'(q)$, *of the quaternion q alone*. (Compare again the Note to page 35.) [See 327.]

† The subject of the *Hodograph* will be resumed at a subsequent stage of this work. In fact, it almost requires the assistance of *Quaternions*, to connect it, in what appears to be the best mode, with Newton's Law of Gravitation. [Compare 419.]

(6.) We may *differentiate a vector-function twice* (or oftener), and so obtain its *successive differentials*. For example, if we *differentiate the derived vector* ρ', we obtain a result of the form,

$$d\rho' = \rho'' dt, \text{ where } \rho'' = D_t\rho' = D_t^2\rho,$$

by an obvious extension of notation; and *if we suppose that the second differential*, ddt or d^2t, *of the scalar* t *is zero*, then the *second differential of the vector* ρ is,

$$d^2\rho = dd\rho = d \cdot \rho' dt = d\rho' \cdot dt = \rho'' \cdot dt^2;$$

where dt^2, as usual, denotes $(dt)^2$; and where it is important to observe that, with the definitions adopted, $d^2\rho$ is *as finite a vector* as $d\rho$, or as ρ itself. In applications to *motion*, if t denote the *time*, ρ'' may be said to be the *Vector of Acceleration*.

(7.) We may also say that, in mechanics, the *finite differential* $d\rho$, *of the Vector of Position* ρ, represents, in length and in direction, the *right line* (suppose PT in fig. 32) *which would have been described, by a freely moving point* P, *in the finite interval of time* dt, immediately *following the time* t, *if* at the end of this time t *all foreign forces had ceased to act.**

(8.) In *geometry*, if $\rho = \phi(t)$ be the equation of a *curve of double curvature*, regarded as the *edge of regression* (comp. 98, (12.)) of a *developable surface*, then the equation of that *surface* itself, considered as the *locus of the tangents* to the curve, may be thus written (comp. 99, II.):

$$\rho = \phi(t) + u\phi'(t); \text{ or simply, } \rho = \phi(t) + d\phi(t),$$

if it be remembered that u, or dt, may be *any arbitrary scalar.*

(9.) If *any other curved surface* (comp. again 99, II.) be represented by an equation of the form, $\rho = \phi(x,y)$, where ϕ now denotes a *vector-function of two independent and scalar variables*, x and y, we may then *differentiate* this *equation*, or this expression for ρ, with respect to *either variable separately*, and so obtain what may be called *two partial* (but *finite*) *differentials*, $d_x\rho$, $d_y\rho$, and *two partial derivatives*, $D_x\rho$, $D_y\rho$, whereof the former are connected with the latter, and with the *two arbitrary* (but *finite*) *scalars*, dx, dy, by the relations,

$$d_x\rho = D_x\rho \cdot dx; \quad d_y\rho = D_y\rho \cdot dy.$$

And these two differentials (or derivatives) of the vector ρ of the surface denote *two tangential vectors*, or at least two vectors *parallel* to *two tangents* to

* As is well illustrated by Atwood's machine.

that surface at the point P: so that *their plane* is (or is parallel to) the *tangent plane* at that point.

(10.) The *mechanism* of all such *differentiations of vector-functions* is, at the present stage, precisely the *same* as in the *usual* processes of the *Differential Calculus;* because the *most general form* of such a *vector-function*, which has been considered in the present Book, is that of a *sum of products* (comp. 99) of the form $x\alpha$, where α is a *constant vector*, and x is a *variable scalar:* so that we have only to *operate on these scalar coefficients* x . ., by the *usual rules* of the calculus, the *vectors* α . . being treated as *constant factors* (comp. sub-art. 2). But when we shall come to consider *quotients* or *products* of *vectors*, or generally those *new functions of vectors* which can only be expressed (in our system) by *Quaternions*, then some few *new rules of differentiation* become necessary, although deduced from the *same* (or nearly the same) *definitions*, as those which have been established in the present section.

(11.) As an example of *partial differentiation* (comp. sub-art. 9) of a *vector function* (the word "vector" being *here* used as an *adjective*) of *two scalar variables*, let us take the equation

$$\rho = \phi(x,y) = \tfrac{1}{2}\{x^2\alpha + y^2\beta + (x+y)^2\gamma\};$$

in which ρ (comp. 99, (3.)) is the vector of a certain *cone of the second order;* or, more precisely, the vector of *one sheet* of such a cone, if x *and* y be supposed to be *real* scalars. Here, the two partial derivatives of ρ are the following:

$$\mathrm{D}_x\rho = x\alpha + (x+y)\gamma; \qquad \mathrm{D}_y\rho = y\beta + (x+y)\gamma;$$

and therefore,

$$2\rho = x\mathrm{D}_x\rho + y\mathrm{D}_y\rho;$$

so that the *three vectors*, ρ, $\mathrm{D}_x\rho$, $\mathrm{D}_y\rho$, if drawn (18) from one *common origin*, are contained (22) in one *common plane;* which implies that the *tangent plane* to the surface, at any point P, passes *through the origin* O: and thereby verifies the *conical* character of the *locus* of that *point* P, in which the *variable vector* ρ, or OP, *terminates*.

(12.) If, in the same example, we make $x=1$, $y=-1$, we have the values,

$$\rho = \tfrac{1}{2}(\alpha+\beta), \qquad \mathrm{D}_x\rho = \alpha, \qquad \mathrm{D}_y\rho = -\beta;$$

whence it follows that the middle point, say C′, of the right line AB, is one of the points of the conical locus; and that (comp. again the sub-art. 3 to Art. 99, and the recent sub-art. 9) the right lines OA and OB are parallel to two of the tangents to the surface at that point; so that the *cone* in question is *touched by the plane* AOB, *along the side* (or *ray*) OC′. And in like manner it may be proved, that the *same cone* is touched by the *two other planes* BOC and COA, at the middle points A′ and B′ of the *two other lines* BC and CA; and therefore along *the two other sides* (or rays), OA′ and OB′: which again agrees with former results.

(13.) It will be found that a *vector function* of the *sum of two scalar variables*, t and $\mathrm{d}t$, may generally be *developed*, by an extension of *Taylor's Series*, under the form,

$$\phi(t+\mathrm{d}t) = \phi(t) + \mathrm{d}\phi(t) + \tfrac{1}{2}\mathrm{d}^2\phi(t) + \frac{1}{2.3}\mathrm{d}^3\phi(t) + \ldots$$

$$= \left(1 + \mathrm{d} + \frac{\mathrm{d}^2}{2} + \frac{\mathrm{d}^3}{2.3} + \ldots\right)\phi(t) = \varepsilon^{\mathrm{d}}\phi(t);$$

it being supposed that $\mathrm{d}^2 t = 0$, $\mathrm{d}^3 t = 0$, &c. (comp. sub-art. 6). Thus, if $\phi t = \frac{1}{2}at^2$ (as in sub-art. 1), where a is a *constant vector*, we have $\mathrm{d}\phi t = at\mathrm{d}t$, $\mathrm{d}^2\phi t = a\mathrm{d}t^2$, $\mathrm{d}^3\phi t = 0$, &c.; and

$$\phi(t+\mathrm{d}t) = \tfrac{1}{2}a(t+\mathrm{d}t)^2 = \tfrac{1}{2}at^2 + at\mathrm{d}t + \tfrac{1}{2}\,a\mathrm{d}t^2,$$

rigorously, without any supposition that $\mathrm{d}t$ is *small*.

(14.) When we thus suppose $\Delta t = \mathrm{d}t$, and *develop the finite difference*, $\Delta\phi(t) = \phi(t+\mathrm{d}t) - \phi(t)$, the *first term* of the development so obtained, or the term of *first dimension* relatively to $\mathrm{d}t$, is hence (by a *theorem*, which holds good for *vector-functions*, as well as for *scalar* functions) the *first differential* $\mathrm{d}\phi t$ *of the function;* but we do not choose to *define* that this *Differential is* (or *means*) that *first term:* because the Formula (100), which we prefer, does not *postulate the possibility*, nor even suppose the *conception*, of any such *development.* Many recent remarks will perhaps appear more clear, when we shall come to *connect* them, at a later stage, with that theory of *Quaternions*, to which we next proceed.

[Compare generally III. ii. Two elementary illustrations of Hamilton's method are given in § 2 of the Chapter cited. It may be of interest to refer to Art. XXVIII. of J. Clerk Maxwell's "Matter and Motion." "Another mode of obtaining the diagram of velocities of a system at a given instant is to take a small interval of time, say the n^{th} part of the unit of time, so that the middle of this interval corresponds to the given instant. Take the diagram of displacements corresponding to this interval and magnify all its dimensions n times. The result will be a diagram of the *mean* velocities of the system during the interval. If we now suppose the number n to increase without limit the interval will diminish without limit, and the mean velocities will approximate to the actual velocities at the given instant. Finally, when n becomes infinite the diagram will represent accurately the velocities at the given instant." The unit of time is of course not necessarily small: compare sub-art. (5). In a letter to De Morgan, dated April 26th, 1852 (Graves's Life, vol. III., p. 629), Hamilton says:—"I lay no stress on the infinitely *great* value of n. It would suit me almost as well to define

$$dfq = \lim_{x=0} x^{-1}\{f(q + xdq) - f(q)\},$$

though I think the other form a little clearer. But the important thing is that I avoid—1st, commutation of factors; 2nd, development in series; 3rd, smallness of differentials."]

BOOK II.

ON QUATERNIONS, CONSIDERED AS QUOTIENTS OF VECTORS, AND AS INVOLVING ANGULAR RELATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

SECTION 1.

Introductory Remarks; First Principles adopted from Algebra.

Art. 101.—The only *angular relations*, considered in the foregoing Book, have been those of *parallelism* between *vectors* (Art. 2, &c.); and the only *quotients*, hitherto employed, have been of the three following kinds:

I. *Scalar quotients of scalars*, such as the *arithmetical fraction* $\frac{n}{m}$ in Art. 14;

II. *Vector quotients*, of *vectors divided by scalars*, as $\frac{\beta}{x} = \alpha$ in Art. 16;

III. *Scalar quotients of vectors*, with *directions* either *similar* or *opposite*, as $\frac{\beta}{\alpha} = x$ in the last cited Article. But we now propose to treat of *other geometric* Quotients (or *geometric Fractions*, as we shall also call them), such as

$$\frac{\text{OB}}{\text{OA}} = \frac{\beta}{\alpha} = q, \text{ with } \beta \textit{ not } \parallel \alpha \text{ (comp. 15)};$$

for each of which the *Divisor* (or *denominator*), α or OA, and the *Dividend* (or *numerator*), β or OB, shall not only *both* be *Vectors*, but shall also be *inclined* to each other at an Angle, *distinct* (in general) from *zero*, and from *two** *right angles*.

102. In introducing this *new conception*, of a *General Quotient of Vectors*, with *Angular Relations* in a given plane, or in space, it will obviously be necessary to employ some properties of *circles* and *spheres*, which were not wanted for

* More generally speaking, from *every even multiple* of a *right angle*.

the purpose of the former Book. But, on the other hand, it will be possible and useful to suppose a much less degree of acquaintance with many important theories* of *modern geometry*, than that of which the possession was assumed, in several of the foregoing sections. Indeed it is hoped that a very moderate amount of geometrical, algebraical, and trigonometrical preparation will be found sufficient to render the present Book, as well as the early parts of the preceding one, fully and easily intelligible to any attentive reader.

103. It may be proper to premise a few general principles respecting quotients of vectors, which are indeed *suggested* by *algebra*, but are here *adopted* by *definition*. And Ist, it is evident that the *supposed operation* of *division* (whatever its *full geometrical import* may afterwards be found to be), by which we here conceive ourselves to pass from a given *divisor-line* α, and from a given *dividend-line* β, to what we have called (provisionally) their *geometric quotient*, q, may (or rather must) be *conceived* to correspond to *some converse act* (as yet not *fully* known) of *geometrical multiplication*: in which new act the former *quotient*, q, becomes a FACTOR, and *operates on the line* α so as to *produce* (or *generate*) the *line* β. We shall therefore *write*, as in algebra,

$$\beta = q \,.\, \alpha, \text{ or simply, } \beta = q\alpha, \text{ when } \beta : \alpha = q\,;$$

even if the two lines α and β, or OA and OB, be supposed to be *inclined* to each other, as in fig. 33. And this very simple and natural *notation* (comp. 16) will then allow us to treat as *identities* the two following formulæ:

$$\left(\frac{\beta}{\alpha} \,.\, \alpha =\right) \frac{\beta}{\alpha}\alpha = \beta; \qquad \frac{q\alpha}{\alpha} = q\,;$$

although we shall, *for the present, abstain* from writing *also* such formulæ† as the following:

$$\frac{\beta\alpha}{\alpha} = \beta, \qquad \frac{q}{\alpha}\alpha = q,$$

where α, β still denote *two vectors*, and q denotes their geometrical *quotient*:

* Such as *homology*, *homography*, *involution*, and generally whatever depends on *anharmonic ratio*: although all that is needful to be known respecting such ratio, for the applications subsequently made, may be learned, without reference to any other treatise, from the *definitions* incidentally given, in Art. 25, &c. It was, perhaps, not strictly *necessary* to introduce any of these modern geometrical theories, in any part of the present work; but it was thought that it might interest one class, at least, of students, to see how they could be *combined* with that fundamental *conception* of the VECTOR, which the First Book was designed to develop.

† It will be seen, however, at a later stage, that these two formulæ are permitted, and even required, in the development of the Quarternion System.

because we have not *yet* even *begun* to consider the *multiplication of one vector by another*, or the *division of a quotient by a line*.

104. As a IInd general principle, suggested by algebra, we shall next lay it down, that if

$$\frac{\beta'}{\alpha'} = \frac{\beta}{\alpha}, \quad \text{and} \quad \alpha' = \alpha, \quad \text{then} \quad \beta' = \beta;$$

or in words, and under a slightly varied form, that *unequal vectors, divided by equal vectors, give unequal quotients*. The importance of this very natural and obvious assumption will soon be seen in its applications.

105. As a IIIrd principle, which indeed may be considered to pervade the whole of *mathematical language*, and without adopting which we could not usefully *speak*, in any case, of EQUALITY as existing between any two geometrical quotients, we shall next assume that two *such quotients can never be equal to the same third** *quotient*, *without being* at the same time *equal to each other*: or in symbols, that

$$\text{if } q' = q, \quad \text{and} \quad q'' = q, \quad \text{then} \quad q'' = q'.$$

106. In the IVth place, as a preparation for *operations on geometrical quotients*, we shall say that any two such quotients, or *fractions* (101), which have a *common divisor-line*, or (in more familiar words) a *common denominator*, are *added*, *subtracted*, or *divided*, among themselves, by adding, subtracting, or dividing their *numerators*: the common denominator being *retained*, in each of the two former of these three cases. In symbols, we thus define (comp. 14) that, *for any three* (actual) *vectors*, α, β, γ,

$$\frac{\gamma}{\alpha} + \frac{\beta}{\alpha} = \frac{\gamma + \beta}{\alpha}; \qquad \frac{\gamma}{\alpha} - \frac{\beta}{\alpha} = \frac{\gamma - \beta}{\alpha};$$

and

$$\frac{\gamma}{\alpha} : \frac{\beta}{\alpha} = \frac{\gamma}{\beta};$$

aiming still at agreement with algebra.

107. Finally, as a Vth principle, designed (like the foregoing) to assimilate, so far as can be done, the present Calculus to Algebra, in its *operations* on geometrical quotients, we shall define that the following formula holds good:

$$\left(\frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} = \right) \frac{\gamma}{\beta}\frac{\beta}{\alpha} = \frac{\gamma}{\alpha};$$

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* It is scarcely necessary to add, what is indeed *included* in this IIIrd principle, in virtue of the *identity* $q = q$, that if $q' = q$, then $q = q'$; or in words, that we shall never admit that any *two* geometrical quotients, q and q', are *equal* to each other *in one order*, without at the same time admitting that they are *equal*, in the *opposite order also*.

or that *if two geometrical fractions*, q and q', *be so related*, that the *denominator*, β, of the *multiplier* q' (here written towards the left-hand) is equal to the *numerator* of the *multiplicand* q, then the *product*, $q' . q$ or $q'q$, is that *third fraction*, whereof the numerator is the *numerator* γ *of the multiplier*, and the denominator is the *denominator* α *of the multiplicand*: all such denominators, or *divisor-lines*, being still supposed (16) to be *actual* (and not *null*) *vectors*.

SECTION 2.

First Motive for naming the Quotient of two Vectors a Quaternion.

108. Already we may see grounds for the application of the *name*, QUATERNION, to such a *Quotient of two Vectors* as has been spoken of in recent articles. In the first place, such a quotient cannot *generally* be what we have called (17) a SCALAR: or in other words, it cannot generally be equal to any of the (so-called) *reals of algebra*, whether of the *positive* or of the *negative* kind. For let x denote any such (actual*) *scalar*, and let α denote any (actual) *vector*; then we have seen (15) that the *product* $x\alpha$ denotes *another* (actual) *vector*, say β', which is either *similar* or *opposite* in direction to α, according as the scalar coefficient, or *factor*, x, is positive or negative; in *neither* case, then, can it represent any vector, such as β, which is *inclined* to α, at any actual *angle*, whether acute, or right, or obtuse: or in other words (comp. 2), the *equation* $\beta' = \beta$, or $x\alpha = \beta$, is impossible, under the conditions here supposed. But we have agreed (16, 103) to write, as in algebra, $\frac{x\alpha}{\alpha} = x$; we must, therefore (by the IInd principle of the foregoing section, stated in Art. 104), *abstain* from writing *also* $\frac{\beta}{\alpha} = x$, under the same conditions: x still denoting a *scalar*. Whatever *else* a *quotient of two inclined vectors* may be found to be, it is thus, at least, a NON-SCALAR.

109. Now, in forming the conception of the *scalar itself*, as the *quotient of two parallel*† *vectors* (17), we took into account not only *relative length*, or *ratio* of the usual kind, but also *relative direction*, under the form of *similarity* or *opposition*. In passing from α to $x\alpha$, we *altered* generally (15) the *length* of

* By an *actual scalar*, as by an *actual vector* (comp. 1), we mean here one that is *different from zero*. An *actual vector*, multiplied by a *null scalar*, has for product (15) a *null vector*; it is therefore unnecessary to prove that the *quotient* of *two actual vectors* cannot be a *null scalar*, or zero.

† It is to be remembered that we have proposed (15) to extend the use of this term *parallel*, to the case of two vectors which are (in the *usual* sense of the word) parallel to one *common line*, even when they happen to be *parts* of one and the *same* right line.

the line a, in the ratio of $\pm x$ to 1; and we *preserved* or *reversed* the *direction* of that line, according as the *scalar coefficient* x was *positive* or *negative*. And in like manner, in proceeding to form, more definitely than we have yet done, the conception of the *non-scalar quotient* (108), $q = \beta : a =$ OB : OA, of *two inclined vectors*, which for simplicity may be supposed (18) to be *co-initial*, we have *still* to take account both of the *relative length*, and of the *relative direction*, of the two lines compared. But while the *former element* of the *complex relation* here considered, between these two lines or vectors, is *still* represented by a simple RATIO (of the kind commonly considered in geometry), or by a *number** expressing that ratio; the *latter element* of the same complex relation is *now* represented by an ANGLE, AOB: and not simply (as it was before) by an *algebraical sign*, + or −.

110. Again, in estimating this *angle*, for the purpose of *distinguishing* one quotient of vectors from another, we must consider not only its *magnitude* (or *quantity*), but also its PLANE: since otherwise, in violation of the principle stated in Art. 104, we should have OB′ : OA = OB : OA, if OB and OB′ were *two distinct rays* or sides of a *cone* of revolution, with OA for its *axis*; in which case (by 2) they would necessarily be *unequal vectors*. For a similar reason, we must attend also to the *contrast* between two *opposite angles*, of equal magnitudes, and in one *common plane*. In short, for the purpose of knowing *fully* the *relative direction* of two co-initial lines OA, OB in *space*, we ought to know not only *how many degrees*, or other *parts* of some *angular unit*, the *angle* AOB contains; but also (comp. fig. 33) the *direction of the rotation* from OA to OB: including a knowledge of the *plane, in which* the rotation is performed; and of the *hand* (as *right* or *left*, when *viewed* from a known *side* of the plane), *towards which* the rotation is *directed*.

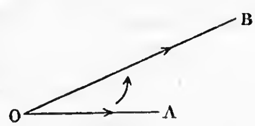

Fig. 33.

111. Or, if we agree to *select* some *one fixed hand* (suppose the *right*† hand), and to call all *rotations positive* when they are directed towards *this* selected

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* This *number*, which we shall presently call the *tensor* of the quotient, may be *whole* or *fractional*, or even *incommensurable* with unity; but it may always be *equated*, in calculation, to a *positive scalar*: although it might perhaps more *properly* be said to be a *signless number*, as being derived solely from comparison of *lengths*, without any reference to *directions*.

† If *right-handed rotation* be thus considered as *positive*, then the *positive axis* of the rotation AOB in fig. 33, must be conceived to be directed *downward*, or *below* the plane of the paper. [Compare the Note to 295 (2), and Art. 23 of Clerk Maxwell's *Electricity and Magnetism*. Hamilton compared the positive axis to a handle or turnscrew used in screwing a right-handed screw into a nut. It is now usual to regard the positive axis as drawn in the direction of the translation of a right-handed screw moving in a fixed nut, or Hamilton's *left-handed* rotation is now called *right-handed*.]

hand, but all rotations *negative* when they are directed towards the *other hand*, then, for *any given angle* AOB, supposed for simplicity to be less than two right angles, and considered as representing a *rotation in a given plane* from OA to OB, we may speak of *one perpendicular* OC *to that plane* AOB as being the *positive axis* of that rotation; and of the *opposite perpendicular* OC′ to the same plane as being the *negative axis* thereof: the rotation round the positive axis being *itself* positive, and *vice versâ*. And then the *rotation* AOB may be considered to be entirely *known*, if we know, Ist, its *quantity*, or the *ratio* which it bears to a *right rotation*; and IInd, the *direction* of its *positive axis*, OC: but not without a knowledge of these *two* things, or of some data equivalent to them. But whether we consider the *direction of an* AXIS, or the *aspect of a* PLANE, we find (as indeed is well known) that the *determination* of such a *direction*, or of such an *aspect*, depends on TWO *polar co-ordinates*,* or other *angular elements*.

112. It appears, then, from the foregoing discussion, that *for the complete determination*, of what we have called the *geometrical* QUOTIENT *of two co-initial Vectors*, a *System of Four Elements*, admitting each separately of numerical expression, *is generally required*. Of these four elements, *one* serves (109) to determine the *relative length* of the two lines compared; and the other *three* are in general necessary, in order to determine *fully* their *relative direction*. Again, of these three latter elements, *one* represents the mutual *inclination*, or *elongation*, of the two lines; or the *magnitude* (or quantity) of the *angle* between them; while the *two others* serve to determine the *direction* of the *axis*, perpendicular to their common *plane*, round which a *rotation* through that angle is to be performed, in a *sense* previously selected as the *positive* one (or towards a fixed and previously selected *hand*), for the purpose of *passing* (in the simplest way, and therefore in the plane of the two lines) *from* the *direction* of the *divisor-line*, *to* the direction of the *dividend-line*. And *no more than four* numerical *elements* are necessary, for our present purpose: because the *relative length* of two lines is not changed, when their two lengths are altered *proportionally*, nor is their *relative direction* changed, when the *angle* which they form is merely *turned* about, *in its own plane*. On account, then, of this *essential connexion* of that *complex relation* (109) between two lines, which is *compounded* of a *relation of lengths*, and of a *relation of directions*, and to which we have given (by an *extension* from the theory of *scalars*) the name of a

* The actual (or at least the frequent) *use* of *such* co-ordinates is foreign to the spirit of the present System: but the *mention* of them here seems likely to assist a student, by suggesting an appeal to results, with which his previous reading can scarcely fail to have rendered him familiar.

geometrical quotient, with a *System of* FOUR *numerical Elements*, we have already a *motive** for saying, that "*the Quotient of two Vectors is generally a Quaternion.*"

SECTION 3.

Additional Illustrations.

113. Some additional light may be thrown, on this first *conception* of a *Quaternion*, by the annexed figure 34. In that figure, the letters CDEFG are designed to indicate corners of a prismatic *desk*, resting upon a horizontal *table*. The *angle* HCD (supposed to be one of *thirty* degrees) represents a (left-handed) *rotation*, whereby the horizontal *ledge* CD of the desk is conceived to be *elongated* (or removed) from a *given horizontal line* CH, which may be imagined to be an *edge* of the table. The angle GCF (supposed here to contain *forty* degrees) represents the *slope*† of the desk, or the amount of its *inclination* to the table. On the *face* CDEF of the desk are drawn two *similar* and *similarly turned triangles*, AOB and A′O′B′, which are supposed to be halves of two equilateral triangles; in such a manner that *each rotation*, AOB or A′O′B′ is one of *sixty* degrees, and is directed towards *one common hand* (namely the *right* hand in the figure): while if *lengths* alone be attended to, the *side* OB is to the side OA, in *one triangle*, as the side O′B′ is to the side O′A′, in the *other*; or as the *number two* to *one*.

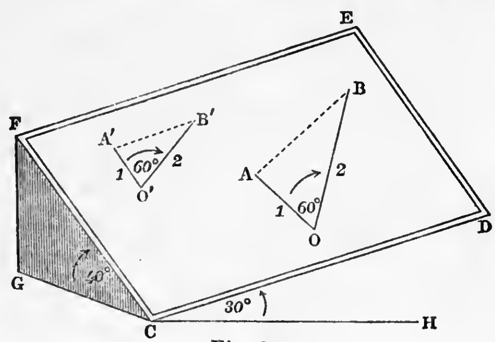

Fig. 34.

114. Under these conditions of construction, we consider the *two quotients*, or the *two geometric fractions*,

$$\mathrm{OB} : \mathrm{OA} \text{ and } \mathrm{OB}' : \mathrm{OA}', \text{ or } \frac{\mathrm{OB}}{\mathrm{OA}} \text{ and } \frac{\mathrm{O'B'}}{\mathrm{O'A'}},$$

as being *equal to each other*; because we regard the *two lines*, OA and OB, as having the *same relative length*, and the *same relative direction*, as the *two other*

---

* Several other reasons for thus speaking will offer themselves, in the course of the present work.

† *These two angles*, HCD and GCF, may thus be considered to correspond to *longitude of node*, and *inclination of orbit*, of a planet or comet in astronomy.

lines, O′A′ and O′B′. And we consider and speak of *each Quotient*, or *Fraction*, as a *Quaternion*: because its *complete construction* (or *determination*) depends, for all that is *essential* to its *conception*, and requisite to *distinguish* it from others, on a system of *four numerical elements* (comp. 112); which are, in this Example, the *four numbers*,

2, 60, 30, and 40.

115. Of these four *elements* (to recapitulate what has been above supposed), the Ist, namely the number 2, expresses that the *length* of the *dividend-line*, OB or O′B′, is *double* of the *length* of the *divisor-line*, OA or O′A′. The IInd numerical element, namely 60, expresses here that the *angle* AOB or A′O′B′, is one of *sixty degrees*; while the corresponding *rotation*, from OA to OB, or from O′A′ to O′B′, is *towards a known hand* (in this case the *right* hand, as seen by a person looking at the *face* CDEF of the desk), which *hand* is the *same for both* of these two *equal angles*. The IIIrd element, namely 30, expresses that the horizontal *ledge* CD of the desk makes an angle of *thirty degrees* with a known horizontal line CH, being removed from it, by that angular quantity, in a known direction (which in this case happens to be towards the *left* hand, as seen from above). Finally, the IVth element, namely 40, expresses here that the desk has an *elevation* of *forty degrees* as before.

116. Now an *alteration in any one of these Four Elements*, such as an alteration of the *slope* or *aspect* of the *desk* would make (in the view here taken) *an essential change in the Quaternion*, which is (in the same view) *the Quotient of the two lines* compared: although (as the figure is in part designed to suggest) *no such change* is *conceived to take place*, when the *triangle* AOB is merely *turned about, in its own plane*, without being *turned over* (comp. fig. 36); or when the *sides* of that triangle are *lengthened* or *shortened proportionally*, so as to *preserve the ratio* (in the old sense of that word), of any one to any other of those sides. We may then briefly say, in this mode of *illustrating* the *notion* of a QUATERNION* in *geometry*, by reference to an *angle on a desk*, that the *Four Elements* which it involves are the following:

Ratio, *Angle*, *Ledge*, and *Slope*;

although the *two latter elements* are in fact *themselves angles also*, but are not *immediately* obtained as such, from the simple comparison of the *two lines*, of which the *Quaternion* is the *Quotient*.

---

* As to the mere *word*, *Quaternion*, it signifies primarily (as is well known), like its Latin original, "Quaternio," or the Greek noun τετρακτύς, a *Set of Four*: but it is obviously used *here*, and elsewhere in the present work, *in a technical sense*.

SECTION 4.

On Equality of Quaternions; and on the Plane of a Quaternion.

117. It is an immediate consequence of the foregoing *conception* of a Quaternion, that *two quaternions*, or *two quotients of vectors*, supposed for simplicity to be all *co-initial* (18), are regarded as being EQUAL to each other, or that the EQUATION,

$$\frac{\delta}{\gamma} = \frac{\beta}{\alpha}, \quad \text{or} \quad \frac{\text{OD}}{\text{OC}} = \frac{\text{OB}}{\text{OA}},$$

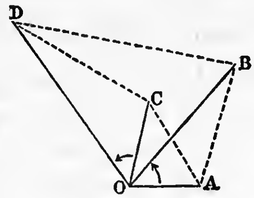

Fig. 35.

is by us considered and *defined* to hold good, *when the two triangles*, AOB and COD, *are similar and similarly turned*, and *in one common plane*, as represented in the annexed fig. 35: the RELATIVE LENGTH (109), and the RELATIVE DIRECTION (110), of the two lines, OA, OB, being then in all respects the *same* as the relative length and the relative direction of the *two other lines*, OC, OD.

118. Under the same conditions, we shall write the following *formula of direct similitude*,

$$\Delta\, \text{AOB} \propto \text{COD};$$

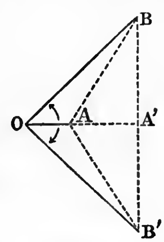

Fig. 36.

reserving this *other* formula,

$$\Delta\, \text{AOB} \propto' \text{AOB}', \quad \text{or} \quad \Delta\, \text{A}'\text{OB} \propto \text{A}'\text{OB}',$$

which we shall call a *formula of inverse similitude*, to denote that the two triangles, AOB and AOB′, or A′OB and A′OB′, although otherwise *similar* (and even, in this case, *equal*,* on account of their having a *common side*, OA or OA′), are *oppositely turned* (comp. fig. 36), as if one were the *reflexion* of the other in a mirror; or as if the one triangle were *derived* (or generated) from the other, by a *rotation of its plane* through *two right angles*. We may therefore write,

$$\frac{\text{OB}}{\text{OA}} = \frac{\text{OD}}{\text{OC}}, \text{ if } \Delta\, \text{AOB} \propto \text{COD}.$$

119. When the vectors are thus all drawn from one common origin O, the *plane* AOB of *any two* of them may be called the *Plane of the Quaternion*

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* That is to say, *equal* in *absolute amount* of *area*, but with *opposite algebraic signs* (28). The *two quotients* OB : OA, and OB′ : OA, although *not equal* (110), will soon be defined to be *conjugate quaternions*. Under the same conditions, we shall write also the formula,

$$\Delta\, \text{AOB}' \propto' \text{COD}.$$

(or of the Quotient), OB : OA; and of course also the plane of the *inverse* (or *reciprocal*) *quaternion* (or of the inverse quotient), OA : OB. And *any two quaternions*, which have a *common plane* (through O), may be said to be *Complanar** *Quaternions*, or complanar quotients, or fractions; but any two quaternions (or quotients), which have *different planes* (*intersecting* therefore in a right line through the origin), may be said, by contrast, to be *Diplanar*.

120. *Any two quaternions*, considered as *geometric fractions* (101), can be *reduced to a common denominator* without change of the *value*† of either of them, as follows. Let $\frac{OB}{OA}$ and $\frac{OD}{OC}$ be the two given fractions, or quaternions; and if they be *complanar* (119), let OE be *any line* in their *common plane*; but if they be *diplanar* (see again 119), then let OE be *any assumed part* of the *line of intersection* of the two planes: so that, in *each* case, the line OE is situated at once in the plane AOB, and *also* in the plane COD. We can then always conceive *two other lines*, OF, OG, to be determined so as to satisfy the *two conditions* of direct similitude (118),

$$\Delta\, EOF \propto AOB, \qquad \Delta\, EOG \propto COD;$$

and therefore also the *two equations* between quotients (117, 118),

$$\frac{OF}{OE} = \frac{OB}{OA}, \quad \frac{OG}{OE} = \frac{OD}{OC};$$

and thus the required *reduction* is effected, OE being the *common denominator* sought, while OF, OG are the new or *reduced numerators*. It may be added that if H be a new point in the plane AOB, such that Δ HOE $\propto$ AOB, we shall have also,

$$\frac{OE}{OH} = \frac{OB}{OA} = \frac{OF}{OE};$$

and therefore, by 106, 107,

$$\frac{OD}{OC} \pm \frac{OB}{OA} = \frac{OG \pm OF}{OE}; \qquad \frac{OD}{OC} : \frac{OB}{OA} = \frac{OG}{OF}; \qquad \frac{OD}{OC} \cdot \frac{OB}{OA} = \frac{OG}{OH};$$

whatever two geometric quotients (complanar or diplanar) may be represented by OB : OA and OD : OC.

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* It is, however, convenient to extend the use of this word, *complanar*, so as to include the case of quaternions represented by *angles in parallel planes*. Indeed, as all *vectors* which have *equal lengths*, and *similar directions*, are equal (2), so the *quaternion*, which is a *quotient* of *two* such vectors, ought not to be considered as undergoing *any change*, when either vector is merely changed in *position*, by a *transport without rotation*.

† That is to say, the *new* or *transformed* quaternions will be respectively *equal* to the old or *given* ones.

121. If now the two triangles AOB, COD are not only *complanar* but *directly similar* (118), so that Δ AOB ∝ COD, we shall evidently have Δ EOF ∝ EOG; so that we may write OF = OG (or F = G, by 20), the *two new lines* OF, OG (or the two new *points* F, G) in this case *coinciding*. The general construction (120), for the reduction to a common denominator, gives therefore here only *one new triangle*, EOF, and *one new quotient*, OF : OE, to which in this case *each* (comp. 105) of the *two* given *equal* and *complanar* quotients, OB : OA and OD : OC, is equal.

122. But if these two latter symbols (or the *fractional forms* corresponding) denote *two diplanar** *quotients*, then the *two new numerator-lines*, OF and OG, have *different directions*, as being situated *in two different planes*, drawn *through the new denominator-line* OE, without having either the direction of that *line itself*, or the direction *opposite* thereto; they are therefore (by 2) *unequal vectors*, even if they should happen to be *equally long*; whence it follows (by 104) that the *two new quotients*, and therefore also (by 105) that the two *old* or *given* quotients, are *unequal*, as a consequence of their *diplanarity*. It results, then, from this analysis, that *diplanar quotients of vectors*, and therefore that *Diplanar Quaternions* (119), *are always unequal*; a new and comparatively *technical* process thus *confirming* the conclusion, to which we had arrived by general considerations, and in (what might be called) a *popular* way before, and which we had sought to *illustrate* (comp. fig. 34) by the consideration of *angles on a desk*: namely, that a *Quaternion*, considered as the quotient of *two mutually inclined lines in space*, involves generally a *Plane*, as an *essential part* (comp. 110) of its constitution, and as necessary to the *completeness* of its *conception*.

123. We propose to use the mark

$$|||$$

as a *Sign of Complanarity*, whether of *lines* or of *quotients*; thus we shall write the formula,

$$\gamma \mathbin{|||} \alpha, \beta,$$

to express that the *three vectors*, α, β, γ, supposed to be (or to be made) *co-initial* (18), are situated *in one plane*; and the analogous formula,

$$q' \mathbin{|||} q, \quad \text{or } \frac{\delta}{\gamma} \mathbin{|||} \frac{\beta}{\alpha},$$

* And therefore *non-scalar* (108); for a *scalar*, considered as a *quotient* (17), has *no determined plane*, but must be considered as *complanar with every geometric quotient*; since it may be represented (or constructed) by the quotient of two similarly or oppositely directed lines, *in any proposed plane* whatever.

to express that the *two quaternions*, denoted here by q and q', and therefore that the *four vectors*, α, β, γ, δ, are *complanar* (119). And because we have just found (122) that *diplanar* quotients are *unequal*, we see that *one equation of quaternions includes two complanarities of vectors*; in such a manner that we may write,

$$\gamma \parallel\!| \alpha, \beta, \quad \text{and} \quad \delta \parallel\!| \alpha, \beta, \quad \text{if} \quad \frac{\delta}{\gamma} = \frac{\beta}{\alpha};$$

the *equation of quotients*, $\frac{\text{OD}}{\text{OC}} = \frac{\text{OB}}{\text{OA}}$, being impossible, *unless all the four lines* from O be *in one common plane*. We shall also employ the notation

$$\gamma \parallel\!| q,$$

to express that the *vector* γ is *in* (or *parallel to*) *the plane of the quaternion* q.

124. With the same notation for complanarity, we may write generally,

$$x\alpha \parallel\!| \alpha, \beta;$$

α and β being *any two vectors*, and x being *any scalar*; because, if $\alpha = \text{OA}$ and $\beta = \text{OB}$ as before, then (by 15, 17) $x\alpha = \text{OA}'$, where A′ is *some point on the indefinite right line* through the points O and A: so that the *plane* AOB contains the *line* OA′. For a similar reason, we have generally the following formula of *complanarity of quotients*,

$$\frac{y\beta}{x\alpha} \parallel\!| \frac{\beta}{\alpha},$$

whatever *two scalars* x and y may be; α and β still denoting any two *vectors*.

125. It is evident (comp. fig. 35) that

$$\text{if } \Delta\, \text{AOB} \propto \text{COD}, \quad \text{then } \Delta\, \text{BOA} \propto \text{DOC}, \quad \text{and } \Delta\, \text{AOC} \propto \text{BOD};$$

whence it is easy to infer that for quaternions, as well as for ordinary or algebraic quotients,

$$\text{if } \frac{\beta}{\alpha} = \frac{\delta}{\gamma}, \text{ then, } \textit{inversely}, \frac{\alpha}{\beta} = \frac{\gamma}{\delta}, \text{ and } \textit{alternately}, \frac{\gamma}{\alpha} = \frac{\delta}{\beta};$$

it being permitted *now* to establish the *converse* of the last formula of 118, or to say that

$$\text{if } \frac{\text{OB}}{\text{OA}} = \frac{\text{OD}}{\text{OC}}, \text{ then } \Delta\, \text{AOB} \propto \text{COD}.$$

Under the same condition, by *combining* inversion with alternation, we have also this other equation, $\frac{\alpha}{\gamma} = \frac{\beta}{\delta}$.

126. If the *sides*, OA, OB, of a triangle AOB, or those sides either way prolonged, be *cut* (as in fig. 37) by any *parallel*, A′B′ or A″B″, to the *base* AB, we have evidently the relations of *direct similarity* (118),

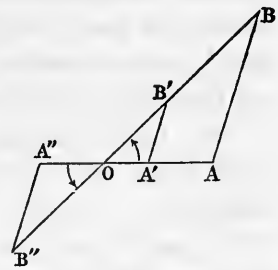

Fig. 37.

$$\Delta\, A'OB' \propto AOB, \quad \Delta\, A''OB'' \propto AOB;$$

whence (comp. Art. 13 and fig. 12) it follows that we may write, for quaternions as in algebra, the general equation, or identity,

$$\frac{x\beta}{x\alpha} = \frac{\beta}{\alpha};$$

where x is again *any scalar*, and α, β are *any two vectors*. It is easy also to see, that for any quaternion q, and any scalar x, we have the *product* (comp. 107),

$$xq = \frac{x\beta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{x\beta}{\alpha} = \frac{\beta}{x^{-1}\alpha} = \frac{\beta}{\alpha} \cdot \frac{\alpha}{x^{-1}\alpha} = qx;$$

so that, in the *multiplication of a quaternion by a scalar* (as in the multiplication of a *vector* by a *scalar*, 15), the *order of the factors* is *indifferent*.

SECTION 5.

On the Axis and Angle of a Quaternion; and on the Index of a Right Quotient, or Quaternion.

127. From what has been already said (111, 112), we are naturally led to define that the AXIS, or more fully that the *positive axis, of any quaternion* (or geometric *quotient*) OB : OA, is *a right line perpendicular to the plane* AOB of that quaternion; and is such that the *rotation round this axis, from the divisor-line* OA, *to the dividend-line* OB, is *positive*: or (as we shall henceforth assume) directed *towards the right-hand*,* like the motion of the hands of a watch.

128. To render still more *definite* this conception of the *axis of a quaternion*, we may add, Ist, that the *rotation*, here spoken of, is supposed (112) to be the *simplest possible*, and therefore to be *in the plane of the two lines* (or of the quaternion), being also *generally less than a semi-revolution* in that plane; IInd, that the axis shall be usually supposed to be a line OX drawn

* This is, of course, merely conventional, and the reader may (if he pleases) substitute the *left-hand* throughout. [The axis is supposed to be drawn outwards from the face of the watch. See Note, page 111.]

from the assumed origin O; and IIIrd, that the *length* of this line shall be supposed to be *given*, or *fixed*, and to be equal to some assumed *unit* of length: so that the *term* X, of this *axis* OX, is situated (by its construction) *on a given spheric surface* described about the *origin* O as *centre*, which surface we may call the surface of the UNIT-SPHERE.

129. In this manner, for every given *non-scalar quotient* (108), or for every given *quaternion* q which does not reduce itself (or degenerate) to a mere *positive or negative number*, the *axis* will be an entirely *definite vector*, which may be called an UNIT-VECTOR, on account of its assumed *length*, and which we shall *denote*,* for the present, by the *symbol* $\mathrm{Ax} . q$. Employing then the usual *sign of perpendicularity*, $\perp$, we may now write, for any two vectors α, β, the formula:

$$\mathrm{Ax} . \frac{\beta}{\alpha} \perp \alpha ; \quad \mathrm{Ax} . \frac{\beta}{\alpha} \perp \beta ; \text{ or briefly, } \mathrm{Ax} . \frac{\beta}{\alpha} \perp \begin{cases} \beta \\ \alpha \end{cases} .$$

130. The ANGLE *of a quaternion*, such as OB : OA, shall simply be, with us, the *angle* AOB *between the two lines*, of which the quaternion is the quotient; *this* angle being supposed here to be one of the *usual kind* (such as are considered by Euclid): and therefore being *acute*, or *right*, or *obtuse* (but not of any class *distinct* from these), when the quaternion is a *non-scalar* (108). We shall *denote* this *angle of a quaternion* q, by the *symbol*, $\angle q$; and thus shall have, *generally*, the two *inequalities*† following:

$$\angle q > 0 ; \qquad \angle q < \pi ;$$

where π is used as a symbol for *two right angles*.

131. When the *general quaternion*, q, *degenerates* into a *scalar*, x, then the *axis* (like the *plane*‡) becomes entirely *indeterminate* in its *direction*; and the *angle* takes, at the same time, either *zero* or *two right angles* for its value, according as the scalar is *positive* or *negative*. Denoting then, as above, any such *scalar* by x, we have:

$$\mathrm{Ax} . x = \text{an indeterminate unit-vector};$$
$$\angle x = 0, \text{ if } x > 0 ; \ \angle x = \pi, \text{ if } x < 0.$$

---

* At a later stage, reasons will be assigned for denoting this *axis*, $\mathrm{Ax} . q$, of a quaternion q, by the *less arbitrary* (or more systematic) *symbol*, $\mathrm{UV}q$; but for the present, the notation in the text may suffice. [See 291.]

† In some investigations respecting *complanar quaternions*, and *powers* or *roots* of quaternions, it is convenient to consider *negative angles*, and angles *greater than two* right angles: but these may then be called AMPLITUDES; and the word "Angle," like the word "Ratio," may thus be restricted, at least for the present, to its *ordinary geometrical sense*. [See 235.]

‡ Compare the Note to page 117. The *angle*, as well as the *axis*, becomes *indeterminate*, when the quaternion reduces itself to *zero*; unless we happen to know a *law*, according to which the dividend-line *tends* to become *null*, in the transition from $\frac{\beta}{\alpha}$ to $\frac{0}{\alpha}$.

132. Of *non-scalar quaternions*, the most important are those of which the *angle* is *right*, as in the annexed figure 38; and when we have thus,

$$q = \frac{\mathrm{OB}}{\mathrm{OA}}, \text{ and } \mathrm{OB} \perp \mathrm{OA}, \text{ or } \angle\, q = \frac{\pi}{2},$$

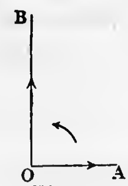
Fig. 38.

the quaternion q may then be said to be a RIGHT QUOTIENT;* or sometimes, a *Right Quaternion.*

(1.) If then $\alpha = $ OA and $\rho = $ OP, where O and A are *two given* (or *fixed*) *points*, but P is a *variable point*, the *equation*

$$\angle \frac{\rho}{\alpha} = \frac{\pi}{2}$$

expresses that the *locus* of this point P is the *plane through* O, *perpendicular to the line* OA; for it is equivalent to the *formula of perpendicularity* $\rho \perp \alpha$ (129).

(2.) More generally, if $\beta = $ OB, B being any *third given point*, the equation,

$$\angle \frac{\rho}{\alpha} = \angle \frac{\beta}{\alpha}$$

expresses that the *locus* of P is *one sheet of a cone of revolution*, with O for *vertex*, and OA for *axis*, and passing *through the point* B; because it implies that the *angles* AOB and AOP are *equal* in *amount*, but *not necessarily* in one *common plane.*

(3.) The equation (comp. 128, 129),

$$\mathrm{Ax} \,.\, \frac{\rho}{\alpha} = \mathrm{Ax} \,.\, \frac{\beta}{\alpha},$$

expresses that the *locus* of the variable point P is the *given plane* AOB; or rather the indefinite *half-plane*, which contains all the points P that are at once *complanar* with the *three given points* O, A, B, and are also *at the same side of the indefinite right line* OA, *as the point* B.

(4.) The system of the *two equations*,

$$\angle \frac{\rho}{\alpha} = \angle \frac{\beta}{\alpha}, \qquad \mathrm{Ax} \,.\, \frac{\rho}{\alpha} = \mathrm{Ax} \,.\, \frac{\beta}{\alpha},$$

expresses that the point P is situated, *either* on the *finite right line* OB, *or* on that line *prolonged through* B, but *not through* O; so that the *locus* of P may in this case be said to be the *indefinite half-line*, or *ray*, which sets out from O in the

* Reasons will afterwards be assigned, for *equating such a quotient*, or quaternion, *to a Vector*; namely to the *line* which will presently (133) be called the *Index of the Right Quotient.* [See 290.]

direction of the *vector* OB or β; and we may write $\rho = x\beta$, $x > 0$ (x being understood to be a *scalar*), instead of the equations assigned above.

(5.) This *other* system of *two* equations,

$$\angle\frac{\rho}{\alpha} = \pi - \angle\frac{\beta}{\alpha}, \quad \mathrm{Ax}.\frac{\rho}{\alpha} = -\mathrm{Ax}.\frac{\beta}{\alpha},$$

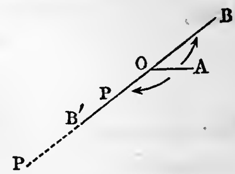

Fig. 33, *bis*.

expresses that the *locus* of P is the *opposite ray* from O; or that P is situated *on the prolongation of the revector* BO (1); or that $\rho = x\beta$, $x < 0$; or that

$$\rho = x\beta', \ x > 0, \text{ if } \beta' = \mathrm{OB}' = -\beta.$$

(Comp. fig. 33, *bis*.)

(6.) *Other notations*, for representing these and other geometric *loci*, will be found to be supplied, in great abundance, by the Calculus of Quaternions; but it seemed proper to point out these, at the present stage, as serving already to show that even the *two symbols* of the present section, Ax. and $\angle$, when considered as *Characteristics of Operation* on *quotients of vectors*, enable us to *express*, very simply and concisely, several useful geometrical *conceptions*.

133. If a *third line*, OI, be drawn in the *direction* of the *axis* OX of such a right quotient (and therefore *perpendicular*, by 127, 129, to *each* of the *two given rectangular lines*, OA, OB); and if the *length* of this new line OI bear to the *length* of that axis OX (and therefore also, by 128, to the assumed *unit* of length) the *same ratio*, which the *length of the dividend-line*, OB, bears *to the length of the divisor-line*, OA; then the line OI, thus determined, is said to be the INDEX *of the Right Quotient*. And it is evident, from this definition of such an *Index*, combined with our general definition (117, 118) of *Equality between Quaternions*, that *two right quotients are equal or unequal to each other, according as their two index-lines* (or *indices*) *are equal or unequal vectors*.

SECTION 6.

On the Reciprocal, Conjugate, Opposite, and Norm of a Quaternion; and on Null Quaternions.

134. The RECIPROCAL (or the *Inverse*, comp. 119) of a quaternion, such as $q = \frac{\beta}{\alpha}$, is that *other* quaternion,

$$q' = \frac{\alpha}{\beta},$$

which is formed by *interchanging the divisor-line and the dividend-line*; and in thus passing from any *non-scalar* quaternion to its *reciprocal*, it is evident that the *angle* (as lately defined in 130) remains *unchanged*, but that the *axis* (127, 128) is *reversed* in direction: so that we may write generally,

$$\angle \frac{\alpha}{\beta} = \angle \frac{\beta}{\alpha}; \qquad \mathrm{Ax} \, . \frac{\alpha}{\beta} = - \, \mathrm{Ax} \, . \, \frac{\beta}{\alpha}.$$

135. The *product* of *two reciprocal quaternions* is always equal to *positive unity*; and *each* is equal to the *quotient* of *unity divided by the other*; because we have, by 106, 107,

$$1 : \frac{\beta}{\alpha} = \frac{\alpha}{\alpha} : \frac{\beta}{\alpha} = \frac{\alpha}{\beta}, \quad \text{and} \quad \frac{\alpha}{\beta} . \frac{\beta}{\alpha} = \frac{\alpha}{\alpha} = 1.$$

It is therefore unnecessary to introduce any *new* or peculiar *notation*, to express the mutual relation existing between a quaternion and its *reciprocal*; since, if *one* be denoted by the symbol q, the *other* may (in the present System, as in Algebra) be denoted by the connected symbol,* $1 : q$, or $\frac{1}{q}$. We have thus the two general formulæ (comp. 134):

$$\angle \frac{1}{q} = \angle \, q \, ; \qquad \mathrm{Ax} \, . \, \frac{1}{q} = - \, \mathrm{Ax} \, . \, q.$$

136. Without *yet* entering on the *general* theory of multiplication and divisions of quaternions, beyond what has been done in Art. 120, it may be here remarked that if any two quaternions q and q' be (as in 134) *reciprocal* to each other, so that $q' \, . \, q = 1$ (by 135), and if q'' be *any third* quaternion, then (as in algebra), we have the general formula,

$$q'' : q = q'' \, . \, q' = q'' \, . \, \frac{1}{q};$$

because if (by 120) we *reduce* q and q'' to a *common denominator* α, and denote the *new numerators* by β and γ, we shall have (by the *definitions* in 106, 107),

$$q'' : q = \frac{\gamma}{\alpha} : \frac{\beta}{\alpha} = \frac{\gamma}{\beta} = \frac{\gamma}{\alpha} . \frac{\alpha}{\beta} = q'' \, . \, q'.$$

137. When *two complanar triangles* AOB, AOB′, with a *common side* OA, are (as in fig. 36) *inversely similar* (118), so that the formula Δ AOB′ $\propto'$ AOB holds good, then the *two unequal quotients*,† $\frac{\mathrm{OB}}{\mathrm{OA}}$ and $\frac{\mathrm{OB}'}{\mathrm{OA}}$ are said to be CONJUGATE

* The symbol q^{-1}, for the *reciprocal* of a quaternion q, is also permitted in the present Calculus; but we defer the use of it, until its legitimacy shall have been established, in connexion with a general theory of powers of Quaternions. [See 234.]

† Compare the Note to page 115.

QUATERNIONS; and if the *first* of them be still denoted by q, then the *second*, which is thus *the conjugate* of that *first*, or of any *other* quaternion which is *equal* thereto, is denoted by the *new symbol*, Kq: in which the letter K may be said to be the *Characteristic of Conjugation.* Thus, with the construction above supposed (comp. again fig. 36), we may write,

$$\frac{\text{OB}}{\text{OA}} = q; \qquad \frac{\text{OB}'}{\text{OA}} = \text{K}q = \text{K}\,\frac{\text{OB}}{\text{OA}}.$$

138. From this definition of conjugate quaternions, it follows, Ist, that if the equation $\frac{\text{OB}'}{\text{OA}} = \text{K}\,\frac{\text{OB}}{\text{OA}}$ hold good, then the *line* OB′ may be called (118) the *reflexion of the line* OB (and conversely, the *latter* line the reflexion of the *former*), *with respect to the line* OA; IInd, that, under the same condition, the line OA (prolonged if necessary) *bisects perpendicularly the line* BB′, in some point A′ (as represented in fig. 36); and IIIrd, that *any two conjugate quaternions* (like any two *reciprocal* quaternions, comp. 134, 135) have *equal angles*, but *opposite axes*: so that we may write, generally,

$$\angle\,\text{K}q = \angle\,q; \qquad \text{Ax}\,.\,\text{K}q = -\,\text{Ax}\,.\,q;$$

and therefore* (by 135),

$$\angle\,\text{K}q = \angle\,\frac{1}{q}; \qquad \text{Ax}\,.\,\text{K}q = \text{Ax}\,.\,\frac{1}{q}.$$

139. The *reciprocal* of a *scalar*, x, is simply *another scalar*, $\frac{1}{x}$, or x^{-1}, having the *same* algebraic *sign*, and in all other respects related to x as in algebra. But the *conjugate* Kx, of a *scalar* x, considered as a *limit* of a *quaternion*, is equal to that scalar x *itself*; as may be seen by supposing the *two equal* but opposite angles, AOB and AOB′, in fig. 36, to tend together to zero or to two right angles. We may therefore write, generally,

$$\text{K}x = x, \text{ if } x \text{ be any scalar};$$

and conversely,†

$$q = \text{a scalar, if } \text{K}q = q;$$

because then (by 104) we must have OB = OB′, BB′ = 0; and therefore each of the two (now coincident) points B, B′, must be situated somewhere on the indefinite right line OA.

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* It will soon be seen that these two last equations (138) express, that the *conjugate* and the *reciprocal*, of any proposed quaternion q, have always *equal versors*, although they have in general *unequal tensors*. [See 157.]

† Somewhat later it will be seen that the equation K$q = q$ may also be written as V$q = 0$; and that this last is another mode of expressing that *the quaternion*, q, *degenerates* (131) *into a scalar*. [See 204, XIV.]

140. In general, by the construction represented in the same figure, the *sum* (comp. 6) of the *two numerators* (or *dividend-lines*, OB and OB′), of the *two conjugate fractions* (or quotients, or quaternions), q and $\mathrm{K}q$ (137), is equal to the *double* of the line OA′; whence (by 106), the *sum of those two conjugate quaternions* themselves is,

$$\mathrm{K}q + q = q + \mathrm{K}q = \frac{2\mathrm{OA}'}{\mathrm{OA}};$$

this sum is therefore *always scalar*, being *positive* if the *angle* $\angle q$ be *acute*, but *negative* if that angle be *obtuse*.

141. In the *intermediate case*, when the angle AOB is *right*, the *interval* OA′ between the origin O and the line BB′ *vanishes*; and the two lately mentioned *numerators*, OB, OB′, become two *opposite vectors*, of which the *sum* is *null* (5). Now, in general, it is natural, and will be found useful, or rather *necessary* (for consistency with *former* definitions), to admit that a *null vector, divided by an actual vector*, gives always a NULL QUATERNION as the *quotient*; and to *denote* this *null quotient* by the usual symbol for *Zero*. In fact, we have (by 106) the equation,

$$\frac{0}{a} = \frac{a - a}{a} = \frac{a}{a} - \frac{a}{a} = 1 - 1 = 0;$$

the zero in the numerator of the *left*-hand fraction representing here a *null line* (or a *null vector*, 1, 2); but the zero on the *right*-hand side of the equation denoting a *null quotient* (or quaternion). And thus we are entitled to infer that the *sum*, $\mathrm{K}q + q$, or $q + \mathrm{K}q$, of a *right-angled quaternion*, or *right quotient* (132), and of its *conjugate*, is always equal to *zero*.

142. We have, therefore, the three following formulæ, whereof the *second* exhibits a *continuity* in the transition from the *first* to the *third*:

$$\text{I.} \ldots q + \mathrm{K}q > 0, \quad \text{if} \quad \angle q < \frac{\pi}{2};$$

$$\text{II.} \ldots q + \mathrm{K}q = 0, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

$$\text{III.} \ldots q + \mathrm{K}q < 0, \quad \text{if} \quad \angle q > \frac{\pi}{2}.$$

And because a quaternion, or *geometric quotient*, with an *actual* and *finite divisor-line* (as here OA), *cannot become equal to zero unless its dividend-line vanishes*, because by (104) the equation

$$\frac{\beta}{a} = 0 = \frac{0}{a} \text{ requires the equation } \beta = 0,$$

if α be any actual and finite vector, we may infer, conversely, that *the sum* $q + \mathrm{K}q$ *cannot vanish*, without the line OA′ *also* vanishing; that is, without the lines OB, OB′ becoming *opposite vectors*, and therefore the quaternion q becoming a *right quotient* (132). We are therefore entitled to establish the three following *converse* formulæ (which indeed result from the three former):

$$\text{I}'. \ldots \text{if } q + \mathrm{K}q > 0, \quad \text{then} \quad \angle q < \frac{\pi}{2};$$

$$\text{II}'. \ldots \text{if } q + \mathrm{K}q = 0, \quad \text{then} \quad \angle q = \frac{\pi}{2};$$

$$\text{III}'. \ldots \text{if } q + \mathrm{K}q < 0, \quad \text{then} \quad \angle q > \frac{\pi}{2}.$$

143. When *two opposite vectors* (1), as β and $-\beta$, are both divided by *one* common (and actual) vector, α, we shall say that the *two quotients*, thus obtained are OPPOSITE QUATERNIONS; so that the *opposite* of any quaternion q, or of any quotient $\beta : \alpha$, may be denoted as follows (comp. 4):

$$\frac{-\beta}{\alpha} = \frac{0-\beta}{\alpha} = \frac{0}{\alpha} - \frac{\beta}{\alpha} = 0 - q = -q;$$

while the quaternion q *itself* may, on the same plan, be denoted (comp. 7) by the symbol $0 + q$, or $+q$. The *sum* of any two opposite quaternions is *zero*, and their *quotient* is *negative unity*; so that we may write, as in algebra (comp. again 7),

$$(-q) + q = (+q) + (-q) = 0; \quad (-q) : q = -1; \quad -q = (-1)\, q;$$

because, by 106 and 141,

$$\frac{-\beta}{\alpha} + \frac{\beta}{\alpha} = \frac{\beta - \beta}{\alpha} = \frac{0}{\alpha} = 0, \quad \frac{-\beta}{\alpha} : \frac{\beta}{\alpha} = \frac{-\beta}{\beta} = -1, \text{ \&c.}$$

The *reciprocals* of *opposite* quaternions are themselves *opposite*; or in symbols (comp. 126),

$$\frac{1}{-q} = -\frac{1}{q}, \text{ because } \frac{\alpha}{-\beta} = \frac{-\alpha}{\beta} = -\frac{\alpha}{\beta}.$$

Opposite quaternions have *opposite axes*, and *supplementary angles* (comp. fig. 33, *bis*); so that we may establish (comp. 132, (5.)) the two following general formulæ,

$$\angle(-q) = \pi - \angle q; \quad \mathrm{Ax.}(-q) = -\,\mathrm{Ax}.\, q.$$

144. We may also now write, in full consistency with the recent formulæ II. and II′. of 142, the equation,

$$\text{II}''. \ldots \mathrm{K}q = -q, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

and conversely* (comp. 138),

$$\text{II}''' \ldots \text{ if } \mathrm{K}q = -q, \quad \text{then} \quad \angle \mathrm{K}q = \angle q = \frac{\pi}{2}.$$

In words, the *conjugate of a right quotient*, or of a *right-angled* (or *right*) *quaternion* (132), is the right quotient *opposite* thereto; and conversely, *if an actual quaternion* (that is, one which is *not null*) *be opposite to its own conjugate* it must be a *right quotient*.

(1) If then we meet the equation,

$$\mathrm{K}\frac{\rho}{\alpha} = -\frac{\rho}{\alpha}, \text{ or } \frac{\rho}{\alpha} + \mathrm{K}\frac{\rho}{\alpha} = 0,$$

we shall know that $\rho \perp \alpha$; and therefore (if $\alpha = \text{OA}$, and $\rho = \text{OP}$, as before), that the *locus* of the *point* P is the *plane through* O, *perpendicular to the line* OA (as in 132, (1.)).

(2.) On the other hand, the equation,

$$\mathrm{K}\frac{\rho}{\alpha} = +\frac{\rho}{\alpha}, \text{ or } \frac{\rho}{\alpha} - \mathrm{K}\frac{\rho}{\alpha} = 0,$$

expresses (by 139) that the quotient $\rho : \alpha$ is a *scalar*; and therefore (by 131) that its *angle* $\angle(\rho : \alpha)$ is either 0 or π; so that in *this* case, the *locus* of P is the *indefinite right line* through the two points O and A.

145. As the *opposite of the opposite*, or the *reciprocal of the reciprocal*, so also the *conjugate of the conjugate*, of any quaternion, is that quaternion *itself*; or in symbols,

$$-(-q) = +q; \qquad 1:(1:q) = q; \qquad \mathrm{KK}q = q = 1q;$$

so that, by *abstracting from the subject of the operation*, we may write briefly,

$$\mathrm{K}^2 = \mathrm{KK} = 1.$$

It is easy also to prove, that the *conjugates of opposite quaternions* are themselves *opposite* quaternions; and that the *conjugates of reciprocals* are *reciprocal*: or in symbols, that

$$\text{I.} \ldots \mathrm{K}(-q) = -\mathrm{K}q, \text{ or } \mathrm{K}q + \mathrm{K}(-q) = 0;$$

and

$$\text{II.} \ldots \mathrm{K}\frac{1}{q} = 1 : \mathrm{K}q, \quad \text{or} \quad \mathrm{K}q \, . \, \mathrm{K}\frac{1}{q} = 1.$$

* It will be seen at a later stage, that the equation $\mathrm{K}q = -q$, or $q + \mathrm{K}q = 0$, may be transformed to this other equation, $\mathrm{S}q = 0$; and that, under this last form, it expresses that the *scalar part* of the quaternion q *vanishes*: or that this quaternion is a *right quotient* (132). [See 196, II.]

(1.) The equation $K(-q) = -Kq$ is included (comp. 143) in this more general formula, $K(xq) = xKq$, where *x is any scalar*; and this last equation (comp. 126) may be proved, by simply conceiving that the two lines OB, OB′, in fig. 36, are multiplied by any common scalar; or that they are both cut by any parallel to the line BB′.

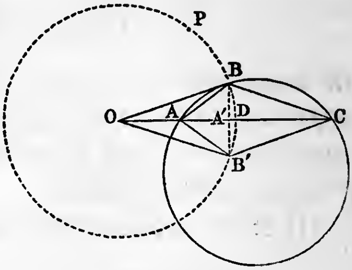

Fig. 36, *bis*.

(2.) To prove that conjugates of reciprocals are reciprocal, or that $Kq . K\frac{1}{q} = 1$, we may conceive that, as in the annexed figure 36, *bis*, while we have *still* the relation of *inverse* similitude,

$$\Delta\ \text{AOB}' \propto' \text{AOB}\ (118, 137),$$

as in the former figure 36, a new point C is determined, either on the line OA itself, or on that line prolonged through A, so as to satisfy either of the two following connected conditions of *direct* similitude:

$$\Delta\ \text{BOC} \propto \text{AOB}'; \qquad \Delta\ \text{B}'\text{OC} \propto \text{AOB};$$

or simply, as a relation between the *four points* O, A, B, C, the formula,

$$\Delta\ \text{BOC} \propto' \text{AOB}.$$

For then we shall have the transformations,

$$K\frac{1}{q} = K\frac{\text{OA}}{\text{OB}} = K\frac{\text{OB}'}{\text{OC}} = \frac{\text{OB}}{\text{OC}} = \frac{\text{OA}}{\text{OB}'} = \frac{1}{Kq}.$$

(3.) The two quotients OB : OA, and OB : OC, that is to say, the *quaternion q itself* and the *conjugate of its reciprocal*, or* the *reciprocal of its conjugate*, have the *same angle*, and the *same axis*; we may therefore write, generally,

$$\angle K\frac{1}{q} = \angle q; \qquad \text{Ax} . K\frac{1}{q} = \text{Ax} . q.$$

(4.) Since OA : OB and OA : OB′ have thus been *proved* (by sub-art. 2) to be a pair of *conjugate quotients*, we can now infer this *theorem*, that *any two geometric fractions*, $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta'}$, which have a *common numerator* α, are *conjugate*

* It will be seen afterwards, that the *common value* of these two equal quaternions, $K\frac{1}{q}$ and $\frac{1}{Kq}$, may be represented by either of the two new symbols, $Uq : Tq$, or $q : Nq$; or in words, that it is equal to the *versor divided by the tensor*; and also to the *quaternion* itself *divided by the norm*. [See 190, (3).]

quaternions, if the *denominator* β' of the *second* be the *reflexion* of the *denominator* β of the *first*, *with respect* to that common *numerator* (comp. 138, I.); whereas it had only been previously *assumed*, as a *definition* (137), that such *conjugation* exists, under the same geometrical condition, between the two *other* (or *inverse*) fractions, $\frac{\beta}{\alpha}$ and $\frac{\beta'}{\alpha}$; the three vectors α, β, β' being supposed to be all *co-initial* (18).

(5.) Conversely, if we meet, in any investigation, the formula

$$\text{OA} : \text{OB}' = \text{K}\,(\text{OA} : \text{OB}),$$

we shall know that the *point* B′ is the *reflexion* of the *point* B, with respect to the *line* OA; or that this line, OA, prolonged if necessary in either of two opposite directions, *bisects at right angles the line* BB′, in some point A′, as in either of the two figures 36 (comp. 138, II.).

(6.) Under the recent conditions of construction, it follows from the most elementary principles of geometry, that the *circle*, which passes *through the three points* A, B, C, is *touched at* B, *by the right line* OB; and that this line is, *in length*, a *mean proportional* between the lines OA, OC. Let then OD be such a *geometric mean*, and let it be set off from O in the common *direction* of the two last mentioned lines, so that the *point* D falls *between* A and C; also let the vectors OC, OD be denoted by the symbols γ, δ; we shall then have expressions of the forms,

$$\delta = a\alpha, \qquad \gamma = a^2\alpha,$$

where a is some positive scalar, $a > 0$; and the vector β of B will be connected (comp. sub-art. 2) with this scalar a, and with the vector α, by the formula

$$\frac{\text{OB}}{\text{OC}} = \text{K}\,\frac{\text{OA}}{\text{OB}}, \quad \text{or}\ \frac{\text{OC}}{\text{OB}} = \text{K}\,\frac{\text{OB}}{\text{OA}}, \quad \text{or}\ \frac{a^2\alpha}{\beta} = \text{K}\,\frac{\beta}{\alpha}.$$

(7.) Conversely, if we still suppose that $\gamma = a^2\alpha$, this last formula expresses the *inverse similitude of triangles*, Δ BOC $\propto'$ AOB; and it expresses *nothing more*: or in other words, it is satisfied by the vector β of *every point* B, which gives that inverse similitude. But for this purpose it is only requisite that the *length* of OB should be (as above) a *geometric mean* between the lengths of OA, OC; or that the *two lines*, OB, OD (sub-art. 6), should be *equally long*: or finally, that B should be situated *somewhere on the surface of a sphere*, which is described so as to pass *through the point* D (in fig. 36, *bis*), and to have the *origin* O for its *centre*.

(8.) If then we meet an equation of the form,

$$\frac{a^2\alpha}{\rho} = \mathrm{K}\,\frac{\rho}{\alpha}, \quad \text{or}\ \frac{\rho}{\alpha}\,\mathrm{K}\,\frac{\rho}{\alpha} = a^2,$$

in which α = OA, ρ = OP, and a is a scalar, as before, we shall know that the *locus of the point* P is a *spheric surface*, with its *centre* at the point O, and with the vector $a\alpha$ for a *radius*; and also that if we determine a point C by the equation OC = $a^2\alpha$, this *spheric locus* of P is a *common orthogonal* to *all the circles* APC, which can be described, so as to pass *through the two fixed points*, A and C: because *every radius* OP of the *sphere* is a *tangent*, at the variable point P, *to the circle* APC, exactly as OB is to ABC in the recent figure.

(9.) In the same fig. 36, *bis*, the similar triangles show (by elementary principles) that the *length* of BC is to that of AB in the *sub-duplicate ratio* of OC to OA; or in the *simple ratio* of OD to OA; or as the scalar a to 1. If then we meet, in any research, the recent equation in ρ (sub-art. 8), we shall know that

$$\textit{length of}\ (\rho - a^2\alpha) = a \times \textit{length of}\ (\rho - \alpha);$$

while the recent interpretation of the same equation gives this *other* relation of the same kind:

$$\textit{length of}\ \rho = a \times \textit{length of}\ \alpha.$$

(10.) At a subsequent stage [200 (3)], it will be shown that the *Calculus of Quaternions* supplies *Rules of Transformation*, by which we can pass from any one to any other of these last equations respecting ρ, *without* (at the time) *constructing any Figure*, or (*immediately*) appealing to *Geometry*: but it was thought useful to point out, already, *how much geometrical meaning** is contained in *so simple a formula*, as that of the last sub-art. 8.

(11.) The *product of two conjugate quaternions* is said to be their *common* NORM,† and is *denoted* thus:

$$q\mathrm{K}q = \mathrm{N}q.$$

* A student of ancient geometry may recognise, in the two equations of sub-art. 9, a sort of *translation*, into the *language of vectors*, of a celebrated *local theorem* of APOLLONIUS of Perga, which has been preserved through a citation made by his early commentator, Eutocius, and may be thus enunciated: Given any two points (as here A and C) in a plane, and any ratio of inequality (as here that of 1 to a), it is possible to construct a circle in the plane (as here the circle BDB'), such that the (lengths of the) two right lines (as here AB and CB, or AP and CP), which are inflected from the two given points to any common point (as B or P) of the circumference, shall be to each other in the given ratio. (*Δύο δοθέντων σημείων, κ. τ. λ.* Page 11 of Halley's Edition of Apollonius, Oxford, MDCCX.)

† This *name*, NORM, and the corresponding *characteristic*, N, are here adopted, as suggestions from the *Theory of Numbers*; but, in the present work, they will not be often *wanted*, although it may occasionally be convenient to employ them. For we shall soon introduce [in 187] the conception,

It follows that $NKq = Nq$; and that the *norm* of a *quaternion* is generally a *positive scalar*: namely, the *square of the quotient of the lengths* of the two lines of which (as *vectors*) the quaternion *itself* is the *quotient* (112). In fact we have, by sub-art. 6, and by the definition of a *norm*, the transformations:

$$N\frac{OB}{OA} = N\frac{OB'}{OA} = \frac{OC}{OB'}.\frac{OB'}{OA} = \frac{OC}{OB}.\frac{OB}{OA} = \frac{OC}{OA} = \left(\frac{OD}{OA}\right)^2;$$

$$Nq = N\frac{\beta}{\alpha} = \frac{\beta}{\alpha}K\frac{\beta}{\alpha} = \left(\frac{\text{length of }\beta}{\text{length of }\alpha}\right)^2.$$

As a *limit*, we may say that *the norm of a null quaternion is zero*; or in symbols, $N0 = 0$.

(12.) With this notation, the *equation of the spheric locus* (sub-art. 8), which has the point O for its centre, and the vector $a\alpha$ for one of its radii, assumes the shorter form:

$$N\frac{\rho}{\alpha} = a^2; \text{ or } N\frac{\rho}{a\alpha} = 1.$$

SECTION 7.

On Radial Quotients; and on the Square of a Quaternion.

146. It was early seen (comp. Art. 2, and fig. 4) that *any two radii*, AB, AC, of any *one circle*, or *sphere*, are necessarily *unequal vectors*; because their *directions differ*. On the other hand, when we are attending *only* to *relative direction* (110), we may suppose that *all* the vectors compared are not merely *co-initial* (18), but are also *equally long*; so that if their *common length* be taken for the *unit*, they are *all radii*, OA, OB, .. of what we have called the *Unit-Sphere* (128), described round the *origin* as *centre*; and may *all* be said to be *Unit-Vectors* (129). And then the quaternion, which is the quotient of any one such vector divided by any other, or generally the *quotient of any two equally long vectors*, may be called a *Radial Quotient*; or sometimes simply a RADIAL. (Compare the annexed figure 39.)

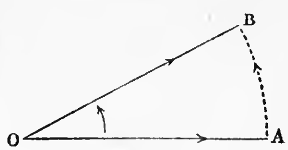

Fig. 39.

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and the characteristic, of the *Tensor*, Tq, of a quaternion, which is of greater *geometrical* utility than the *Norm*, but of which it will be proved that this norm is simply the *square*,

$$qKq = Nq = (Tq)^2.$$

Compare the Note to sub-art. 3.

147. The two *Unit-Scalars*, namely, *Positive and Negative Unity*, may be considered as *limiting cases* of *radial quotients*, corresponding to the *two extreme values*, 0 and π, of the *angle* AOB, or $\angle q$ (131). In the *intermediate case*, when AOB is a *right angle*, or $\angle q = \frac{\pi}{2}$, as in fig. 40, the resulting quotient, or quaternion, may be called (comp. 132) a *Right Radial Quotient*; or simply, a RIGHT RADIAL. The consideration of such *right radials* will be found to be of great importance, in the whole theory and practice of Quaternions.

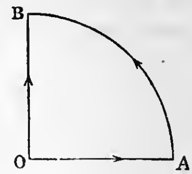

Fig. 40.

148. The most important *general property* of the quotients last mentioned is the following: that *the Square of every Right Radial is equal to Negative Unity*; it being understood that we write generally, as in algebra,

$$q \cdot q = qq = q^2,$$

and call this *product of two equal quaternions* the SQUARE of each of them.

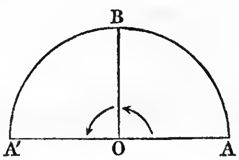

Fig. 41.

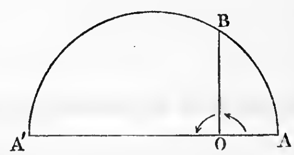

Fig. 41, *bis*.

For if, as in fig. 41, we describe a *semicircle* ABA′, with O for *centre*, and with OB for the *bisecting radius*, then the two right quotients, OB : OA, and OA′ : OB, are *equal* (comp. 117); and therefore their *common square* is (comp. 107) the *product*,

$$\left(\frac{\text{OB}}{\text{OA}}\right)^2 = \frac{\text{OA}'}{\text{OB}} \cdot \frac{\text{OB}}{\text{OA}} = \frac{\text{OA}'}{\text{OA}} = -1;$$

where OA and OB may represent *any two equally long, but mutually rectangular lines*. More generally, *the Square of every Right Quotient* (132) *is equal to a Negative Scalar*; namely, *to the negative of the square of the number*, which represents the *ratio of the lengths** of the two rectangular lines compared; or to *zero minus the square of the number* which denotes (comp. 133) the *length of the Index* of that Right Quotient: as appears from fig. 41, *bis*, in which OB is

* Hence, by 145 (11.), $q^2 = -\text{N}q$, if $\angle q = \frac{\pi}{2}$.

only an *ordinate*, and *not* (as before) a *radius*, of the semicircle ABA′; for we have thus,

$$\left(\frac{\text{OB}}{\text{OA}}\right)^2 = \frac{\text{OA}'}{\text{OA}} = -\left(\frac{\textit{length of } \text{OB}}{\textit{length of } \text{OA}}\right)^2, \text{ if } \text{OB} \perp \text{OA}.$$

149. Thus *every Right Radial* is, in the present System, *one of the Square Roots of Negative Unity;* and may therefore be said to be *one of the Values of the Symbol* $\sqrt{-1}$; which celebrated *symbol* has thus a certain degree of *vagueness*, or at least of *indetermination*, of *meaning* in this theory, on account of which we shall not often employ it. For although it thus admits of a *perfectly clear* and *geometrically real Interpretation*, as denoting what has been above called a *Right Radial Quotient*, yet the *Plane of that Quotient is arbitrary*; and therefore the *symbol itself* must be considered to have (in the present system) *indefinitely many values*; or in other words the *Equation*, $q^2 = -1$, has (in the Calculus of Quaternions) *indefinitely many Roots*,* which are *all Geometrical Reals*: *besides* any *other roots*, of a *purely symbolical character*, which the same equation may be conceived to possess, and which may be called *Geometrical Imaginaries*.† Conversely, if q be any *real quaternion*, which satisfies the equation $q^2 = -1$, *it must be a right radial*; for if, as in fig. 42, we suppose that Δ AOB $\propto$ BOC, we shall have

$$q^2 = \left(\frac{\text{OB}}{\text{OA}}\right)^2 = \frac{\text{OC}}{\text{OB}}.\frac{\text{OB}}{\text{OA}} = \frac{\text{OC}}{\text{OA}};$$

and this *square* of q cannot become equal to *negative unity*, except by OC being $= -$ OA, or $=$ OA′ in fig. 41; that is, by the line OB being *at right angles* to the line OA, and being at the same time *equally long*, as in fig. 40.

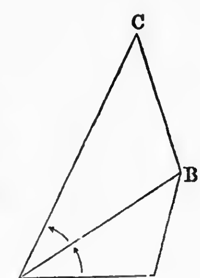

Fig. 42.

(1.) If then we meet the equation,

$$\left(\frac{\rho}{\alpha}\right)^2 = -1,$$

where $\alpha =$ OA, and $\rho =$ OP, as before, we shall know that the *locus* of the point

* It will be subsequently shown [in 222], that if x, y, z be *any three scalars, of which the sum of the squares is unity*, so that

$$x^2 + y^2 + z^2 = 1;$$

and if i, j, k be *any three right radials, in three mutually rectangular planes*; then the *expression*,

$$q = ix + jy + kz,$$

denotes *another right radial*, which satisfies (*as such*, and by symbolical *laws* to be assigned) the *equation* $q^2 = -1$; and is therefore *one of the geometrically real values of the symbol* $\sqrt{-1}$.

† *Such imaginaries* will be found to offer themselves, in the treatment by *Quaternions* (or rather by what will be called *Biquaternions*), of *ideal intersections*, and of *ideal contacts*, in *geometry* [see 214]; but we confine our attention, for the present, to *geometrical reals* alone. Compare the Notes to pages 87 and 88.

P is the *circumference of a circle*, with O for its *centre*, and with a *radius* which has the *same length* as the line OA; while the *plane* of the circle is *perpendicular* to that given line. In other words, the locus of P is a *great circle*, on a *sphere* of which the centre is the origin; and the given point A, on the same spheric surface, is *one of the poles* of that circle.

(2.) In general, the equation $q^2 = -a^2$, where a is *any* (real) *scalar*, requires that the quaternion q (if real) should be *some right quotient* (132); the number a denoting the *length of the index* (133), of that right quotient or quaternion (comp. Art. 148, and fig. 41, *bis*). But the *plane* of q is *still* entirely *arbitrary*; and therefore the equation

$$q^2 = -a^2,$$

like the equation $q^2 = -1$, which it includes, must be considered to have (in the present system) *indefinitely many geometrically real roots.*

(3.) Hence the equation,

$$\left(\frac{\rho}{a}\right)^2 = -a^2,$$

in which we may suppose that $a > 0$, expresses that the *locus* of the point P is a (new) *circular circumference*, with the line OA for its *axis*,* and with a *radius* of which the *length* $= a \times$ the length of OA.

150. It may be added that the *index* (133), and the *axis* (128), of a *right radial* (147), are the *same*; and that its *reciprocal* (134), its *conjugate* (137), and its *opposite* (143), are all *equal to each other*. Conversely, *if the reciprocal of a given quaternion q be equal to the opposite* of that quaternion, then *q is a right radial*; because its *square*, q^2, is then equal (comp. 136) to the quaternion itself, *divided by its opposite*; and therefore (by 143) to *negative unity*. But the *conjugate of every radial quotient* is equal to the *reciprocal* of *that* quotient; because if, in fig. 36 [p. 115], we conceive that the *three lines* OA, OB, OB′ are *equally long*, or if, in fig. 39, we *prolong* the *arc* BA, by an *equal arc* AB′, we have the equation,

$$Kq = \frac{OB'}{OA} = \frac{OA}{OB} = \frac{1}{q}.$$

And conversely,†

$$\text{if } Kq = \frac{1}{q}, \quad \text{or if } qKq = 1,$$

then the quaternion q is a radial quotient.

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* It being understood, that the *axis of a circle* is a right line perpendicular to the plane of that circle, and passing through its centre.

† Hence, in the *notation of norms* (145, (11.)), if $Nq = 1$, then q is a *radial*; and conversely, the *norm of a radial quotient*, is always equal to *positive unity*.

SECTION 8.

On the Versor of a Quaternion, or of a Vector; and on some General Formulæ of Transformation.

151. When a quaternion $q = \beta : \alpha$ is thus a *radial quotient* (146), or when the *lengths* of the two lines α and β are *equal*, the *effect* of this quaternion q, considered as a FACTOR (103), in the equation $q\alpha = \beta$, is simply the *turning* of the *multiplicand-line* α, *in the plane* of q (119), and *towards the hand* determined by the direction of the *positive axis* Ax . q (129), *through the angle* denoted by $\angle q$ (130); so as to bring that line α (or a revolving line which *had* coincided therewith) *into a new direction*: namely, into that of the *product-line* β. And with reference to this conceived *operation of turning*, we shall now say that *every Radial Quotient is a* VERSOR.

152. A *Versor* has thus, in general, a *plane*, an *axis*, and an *angle*; namely, those of the *Radial* (146) to which it *corresponds*, or is *equal*: the *only difference* between them being a difference in the *points of view** from which they are respectively regarded; namely, the *radial* as the *quotient*, q, in the formula, $q = \beta : \alpha$; and the *versor* as the (equal) *factor*, q, in the *converse formula*, $\beta = q \,.\, \alpha$; where it is still supposed that the two vectors, α and β, are *equally long*.

153. A *versor*, like a *radial* (147), cannot *degenerate* into a *scalar*, except by its *angle* acquiring one or other of the two *limit-values*, 0 and π. In the first case, it becomes *positive unity*; and in the second case, it becomes *negative unity*: each of these two *unit-scalars* (147) being here regarded as a *factor* (or *coefficient*, comp. 12), which *operates on a line*, to *preserve* or to *reverse* its *direction*. In this view, we may say that -1 is an *Inversor*; and that *every Right Versor* $\left(\text{or versor with an angle} = \dfrac{\pi}{2}\right)$ is a *Semi-inversor*:† because it *half-inverts* the *line* on which it *operates*, or *turns it through half of two right angles*

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* In a slightly *metaphysical* mode of expression it may be said, that the *radial quotient* is the *result* of an *analysis*, wherein *two radii* of one sphere (or circle) are *compared*, as regards their *relative direction*; and that the *equal versor* is the *instrument* of a corresponding *synthesis*, wherein *one radius* is conceived to be *generated*, by a certain *rotation*, from the *other*.

† This *word*, "semi-inversor," will not be often used; but the introduction of it here, in passing, seems adapted to throw light on the *view* taken, in the present work, of the *symbol* $\sqrt{-1}$, when regarded as denoting a certain important class (149) of *Reals in Geometry*. There *are* uses of that symbol, to denote *Geometrical Imaginaries* (comp. again Art. 149, and the Notes to pages 87 and 88), considered as connected with *ideal intersections*, and with *ideal contacts*; but *with such uses* of $\sqrt{-1}$ we have, *at present*, *nothing to do*.

(comp. fig. 41). For the same reason, we are led to consider *every right versor* (like *every right radial*, 149, from which indeed we have just seen, in 152, that it differs only as *factor* differs from *quotient*), as being *one of the square roots of negative unity*: or as *one of the values of the symbol* $\sqrt{-1}$.

154. In fact we may observe that the *effect* of a *right versor*, considered as *operating on a line* (in its own plane), is *to turn that line, towards a given hand, through a right angle.* If then q be *such a versor*, and if $q\alpha = \beta$, we shall have also (comp. fig. 41), $q\beta = -\alpha$; so that, if α be any line in the plane of a right versor q, we have the equation,

$$q \cdot q\alpha = -\alpha;$$

whence it is natural to write, under the same condition,

$$q^2 = -1,$$

as in 149. On the other hand, *no versor, which is not right-angled, can be a value of* $\sqrt{-1}$; or can satisfy the equation $q^2\alpha = -\alpha$, as fig. 42 may serve to illustrate. For it is included in the meaning of this last equation, as applied to the theory of *versors*, that a *rotation through* $2 \angle q$, or through the *double of the angle* of q itself, is equivalent to an *inversion* of *direction*; and therefore to a rotation through *two right angles*.

155. In general, if α be *any vector*, and if a be used as a temporary* symbol for the *number* expressing its *length*; so that a is here a *positive scalar*, which bears to *positive unity*, or to the scalar $+1$, the *same ratio* as that which the length of the line α bears to the assumed *unit of length* (comp. 128); then the *quotient* $\alpha : a$ denotes generally (comp. 16) a *new vector*, which has the *same direction* as the *proposed* vector α, but has its *length* equal to that assumed *unit*: so that it is (comp. 146) *the Unit-Vector in the direction of* α. We shall *denote this unit-vector* by the *symbol*, $U\alpha$; and so shall write, generally,

$$U\alpha = \frac{\alpha}{a}, \quad \text{if } a = \textit{length of } \alpha;$$

that is, more fully, if a be, as above supposed, the *number* (commensurable or incommensurable, but *positive*) which *represents* that length, with reference to some selected standard.

156. Suppose now that $q = \beta : \alpha$ is (as at first) a *general quaternion*, or the *quotient of any two vectors*, α and β, whether *equal* or *unequal* in *length*. *Such* a *Quaternion* will *not* (generally) be a *Versor* (or at least *not simply such*),

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* We shall soon propose [in 185] a general *notation* for representing the *lengths of vectors*, according to which the symbol $T\alpha$ will denote what has been above called a; but are unwilling to introduce more than *one new characteristic of operation*, such as K, or T, or U, &c., *at one time*.

according to the definition lately given; because its *effect*, when operating as a *factor* (103) on α, will *not* in general be *simply* to *turn* that line (151): but will (generally) *alter* the *length*,* as well as the *direction*. But if we *reduce* the two proposed vectors, α and β, *to the two unit-vectors* $U\alpha$ and $U\beta$ (155), and form the *quotient of these*, we shall *then* have taken account of *relative direction alone*: and the *result* will therefore be a *versor*, in the sense lately defined (151). We propose to call the quotient, or the versor, *thus* obtained, the *versor-element*, or briefly, *the* VERSOR, *of the Quaternion* q; and shall find it convenient to employ the *same*† *Characteristic*, U, to denote the *operation* of *taking the versor of a quaternion*, as that employed above to denote the operation (155) of *reducing a vector to the unit of length*, without any change of its *direction*. On this plan, the *symbol* Uq will denote the *versor of* q; and the foregoing *definitions* will enable us to establish the *General Formula*:

$$Uq = U\frac{\beta}{\alpha} = \frac{U\beta}{U\alpha};$$

in which the *two unit-vectors*, $U\alpha$ and $U\beta$, may be called, by analogy, and for other reasons which will afterwards appear, *the versors*‡ *of the vectors*, α and β.

157. In thus passing from a given *quaternion*, q, to its *versor*, Uq, we have only changed (in general) the *lengths* of the two lines compared, namely, by *reducing* each to the assumed *unit* of length (155, 156), without making any change in their *directions*. Hence the *plane* (119), the *axis* (127, 128), and the *angle* (130), of the quaternion, remain *unaltered* in this passage; so that we may establish the two following general formulæ:

$$\angle\, Uq = \angle\, q; \qquad \mathrm{Ax}\,.\,Uq = \mathrm{Ax}\,.\,q.$$

* By what we shall soon call an *act of tension*, which will lead us to the consideration of the *tensor* of a quaternion.

† For the moment, this double use of the characteristic U, to assist in denoting *both* the *unit-vector* $U\alpha$ derived from a given *line* α, and *also* the *versor* Uq derived from a *quaternion* q, may be regarded as established *here* by arbitrary *definition*; but as *permitted*, because the *difference* of the *symbols*, as here α and q, which serve for the present to denote *vectors* and *quaternions*, considered as the *subjects* of these *two operations* U, will prevent such *double use* of that *characteristic* from giving rise to any *confusion*. But we shall further find that several important *analogies* are by anticipation *expressed*, or at least *suggested*, when the proposed *notation* is employed. Thus it will be found (comp. the Note to page 121), that *every vector* α may usefully be *equated to that right quotient*, of which it is (133) the *index*; and that *then* the *unit-vector* $U\alpha$ may be, on the same plan, *equated to that right radial* (147), which is (in the sense lately defined) *the versor of that right quotient*. We shall also find ourselves led to regard *every unit-vector* as the *axis of a quadrantal* (or *right*) *rotation*, in a plane perpendicular to that axis; which will supply another inducement, to *speak* of every *such* vector as a *versor*. On the whole, it appears that there will be no inconvenience, but rather a prospective advantage, in our already *reading* the symbol $U\alpha$ as "*versor of* α"; just as we may *read* the analogous symbol Uq, as "*versor of* q." [Compare 286 and 290.]

‡ Compare the Note immediately preceding.

More generally we may write,

$$\angle q' = \angle q, \text{ and } \mathrm{Ax}\,.\,q' = \mathrm{Ax}\,.\,q, \text{ if } \mathrm{U}q' = \mathrm{U}q\,;$$

the *versor of a quaternion depending solely on*, but conversely being sufficient to *determine*, the *relative direction* (156) of the *two lines*, of which (as *vectors*) the quaternion itself is the *quotient* (112); or the axis and angle of the *rotation*, in the plane of those two lines, from the divisor to the dividend (128): so that any two quaternions, which have *equal versors*, must also have *equal angles*, and *equal* (or coincident) *axes*, as is expressed by the last written formula. Conversely, from this dependence of the *versor* Uq on *relative direction* alone*, it follows that any two quaternions, of which the angles and the axes are equal, have also *equal versors*; or in symbols, that

$$\mathrm{U}q' = \mathrm{U}q, \quad \text{if} \quad \angle q' = \angle q, \quad \text{and} \quad \mathrm{Ax}\,.\,q' = \mathrm{Ax}\,.\,q.$$

For example, we saw (in 138) that the *conjugate* and the *reciprocal* of any quaternion have thus their *angles* and their *axes* the *same*; it follows, therefore, that the *versor of the conjugate* is always *equal* to the *versor of the reciprocal*; so that we are permitted to establish the following general formula,†

$$\mathrm{UK}q = \mathrm{U}\frac{1}{q}.$$

158. Again, because

$$\mathrm{U}\left(1 : \frac{\beta}{\alpha}\right) = \mathrm{U}\frac{\alpha}{\beta} = \frac{\mathrm{U}\alpha}{\mathrm{U}\beta} = 1 : \frac{\mathrm{U}\beta}{\mathrm{U}\alpha} = 1 : \mathrm{U}\frac{\beta}{\alpha},$$

it follows that the *versor of the reciprocal* of any quaternion is, at the same time, the *reciprocal of the versor*; so that we may write,

$$\mathrm{U}\frac{1}{q} = \frac{1}{\mathrm{U}q}; \quad \text{or} \quad \mathrm{U}q\,.\,\mathrm{U}\frac{1}{q} = 1.$$

Hence, by the recent result (157), we have also, generally,

$$\mathrm{UK}q = \frac{1}{\mathrm{U}q}; \quad \text{or,} \quad \mathrm{U}q\,.\,\mathrm{UK}q = 1.$$

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* The *unit-vector* Uα, which we have recently proposed (156) to call *the versor of the vector* α, depends in like manner *on the direction of that vector alone*; which *exclusive reference*, in *each* of these two cases, to Direction, may serve as an additional *motive* for employing, as we have lately done, one *common name*, Versor, and one *common characteristic*, U, to assist in describing or denoting *both* the *Unit-Vector* Uα *itself, and* the *Quotient of two such Unit-Vectors*, $\mathrm{U}q = \mathrm{U}\beta : \mathrm{U}\alpha$; all danger of *confusion* being sufficiently guarded against (comp. the Note to Art. 156), by the *difference of the two symbols*, α and q, employed to denote the *vector* and the *quaternion*, which are respectively the *subjects of the two operations* U; while those two *operations agree* in this *essential* point, that *each* serves to *eliminate the quantitative element*, of absolute or relative *length*.

† Compare the Note to Art. 138.

Also, because the *versor* Uq is always a *radial* quotient (151, 152), it is (by 150) the *conjugate of its own reciprocal*; and therefore, at the same time (comp. 145), the *reciprocal of its own conjugate*; so that the *product of two conjugate versors*, or what we have called (145, (11.)) their *common* NORM, is always equal to *positive unity*; or in symbols (comp. 150),

$$\mathrm{NU}q = \mathrm{U}q \,.\, \mathrm{KU}q = 1.$$

For the same reason, the *conjugate of the versor* of any quaternion is equal to the *reciprocal of that versor*, or (by what has just been seen) to the *versor of the reciprocal* of that quaternion; and therefore also (by 157), to the *versor of the conjugate*; so that we may write generally, as a summary of recent results, the formula:

$$\mathrm{KU}q = \frac{1}{\mathrm{U}q} = \mathrm{U}\frac{1}{q} = \mathrm{UK}q;$$

each of these four symbols denoting a *new versor*, which has the *same plane*, and the *same angle*, as the *old* or *given versor* Uq, but has an *opposite axis*, or an *opposite direction of rotation*: so that, with respect to that given *Versor*, it may naturally be called a REVERSOR.

159. As regards the *versor itself*, whether of a vector or of a quaternion, the definition (155) of Uα gives,

$$\mathrm{U}x\alpha = +\,\mathrm{U}\alpha, \quad \text{or} \quad = -\,\mathrm{U}\alpha, \quad \text{according as} \quad x > \text{or} < 0;$$

because (by 15) the *scalar coefficient* x *preserves*, in the first case, but *reverses*, in the second case, the *direction* of the vector α; whence also, by the definition (156) of Uq, we have generally (comp. 126, 143),

$$\mathrm{U}xq = +\,\mathrm{U}q, \quad \text{or} \quad = -\,\mathrm{U}q, \quad \text{according as} \quad x > \text{or} < 0.$$

The *versor of a scalar*, regarded as the *limit of a quaternion* (131, 139), is equal to *positive* or *negative unity* (comp. 147, 153), according as the scalar itself is positive or negative; or in symbols,

$$\mathrm{U}x = +\,1, \quad \text{or} \quad = -\,1, \quad \text{according as} \quad x > \text{or} < 0;$$

the *plane* and *axis* of each of these two *unit scalars* (147), considered as *versors* (153), being (as we have already seen) *indeterminate*. The *versor of a null quaternion* (141) must be regarded as *wholly arbitrary*, unless we happen to know a *law*,* according to which the quaternion *tends to zero*, before actually *reaching* that limit; in which latter case, the *plane*, the *axis*, and the *angle* of

* Compare the Note to Art. 131.

the *versor** $U0$ may *all* become *determined*, as *limits* deduced from that law. The *versor of a right quotient* (132), or of a *right-angled quaternion* (141), is always a *right radial* (147), or a *right versor* (153); and therefore is, as such, *one of the square roots of negative unity* (149), or *one of the values of the symbol* $\sqrt{-1}$; while (by 150) the *axis* and the *index* of *such* a versor *coincide*; and in like manner its *reciprocal*, its *conjugate*, and its *opposite* are all *equal* to each other.

160. It is evident that if a proposed *quaternion* q be *already* a *versor* (151), in the sense of being a *radial* (146), the operation of *taking its versor* (156) produces *no change*; and in like manner that, if a given *vector* α be *already* an *unit-vector*, it remains the *same* vector, when it is *divided* (155) *by its own length*; that is, in this case, by the *number one*. For example, we have assumed (128, 129), that the *axis* of *every* quaternion is an *unit-vector*; we may therefore write, generally, in the notation of 155, the equation,

$$\mathrm{U}\,(\mathrm{Ax}\,.\,q) = \mathrm{Ax}\,.\,q.$$

A *second operation* U leaves thus the *result* of the *first* operation U *unchanged*, whether the *subject* of such successive operations be a *line*, or a *quaternion*; we have therefore the two following general formulæ, differing only in the symbols of that subject:

$$\mathrm{UU}\alpha = \mathrm{U}\alpha; \qquad \mathrm{UU}q = \mathrm{U}q;$$

whence, by *abstracting* (comp. 145) *from the subject* of the operation, we may write, briefly and symbolically,

$$\mathrm{U}^2 = \mathrm{UU} = \mathrm{U}.$$

161. Hence, with the help of 145, 158, 159, we easily deduce the following (among other) *transformations of the versor of a quaternion*:

$$\mathrm{U}q = \frac{1}{\mathrm{U}\frac{1}{q}} = \frac{1}{\mathrm{KU}q} = \frac{1}{\mathrm{UK}q} = \mathrm{KU}\frac{1}{q} = \mathrm{K}\frac{1}{\mathrm{U}q} = \mathrm{KUK}q$$

$$= \mathrm{U}\frac{1}{\mathrm{K}q} = \mathrm{UK}\frac{1}{q} = \mathrm{U}^2q = \mathrm{UKU}\frac{1}{q} = \mathrm{UK}\frac{1}{\mathrm{U}q} = (\mathrm{UK})^2q;$$

$$\mathrm{U}q = \mathrm{U}xq, \text{ if } x > 0; \qquad = -\,\mathrm{U}xq, \text{ if } x < 0.$$

We may also write, generally,

$$\frac{q}{\mathrm{K}q} = \frac{\mathrm{U}q}{\mathrm{KU}q} = (\mathrm{U}q)^2 = \mathrm{U}(q^2) = \mathrm{U}q^2;$$

* When the *zero* in this *symbol*, $U0$, is considered as denoting a *null vector* (2), the symbol itself denotes *generally*, by the foregoing principles, an *indeterminate unit-vector*; although the *direction* of this unit-vector *may*, in certain questions, *become* determined, as a *limit* resulting from a *law*.

the *parentheses* being here unnecessary, because (as will soon be more fully seen) the *symbol* Uq^2 denotes *one common versor*, whether we interpret it as denoting the *square of the versor*, or as the *versor of the square*, of q. The present Calculus will be found to abound in *General Transformations* of this sort; which all (or nearly all), like the foregoing, depend ultimately on very *simple geometrical conceptions*; but which, notwithstanding (or rather, perhaps, on account of) this extreme *simplicity* of their *origin*, are often *useful*, as *elements* of a new kind of *Symbolical Language* in *Geometry*: and generally, as *instruments of expression*, in all those mathematical or physical researches to which the *Calculus of Quaternions* can be applied. It is, however, by no means necessary that a student of the subject, at the present stage, should make himself *familiar* with *all* the recent transformations of Uq; although it may be well that he should satisfy himself of their correctness, in doing which the following remarks will perhaps be found to assist.

(1.) To give a *geometrical illustration*, which may also serve as a *proof*, of the recent equation,

$$q : Kq = (Uq)^2,$$

we may employ fig. 36, *bis* [p. 128]; in which, by 145, (2.), we have

$$q \cdot \frac{1}{Kq} = \frac{OB}{OA} \cdot \frac{OA}{OB'} = \frac{OB}{OB'} = \left(\frac{OB}{OD}\right)^2 = \left(U\frac{OB}{OA}\right)^2 = (Uq)^2.$$

(2.) As regards the equation, $U(q^2) = (Uq)^2$, we have only to conceive that the three lines OA, OB, OC, of fig. 42, are cut (as in fig. 42, *bis*) in three new points, A′, B′, C′, by an *unit-circle* (or by a circle with a radius equal to the unit of length), which is described about their common origin O as centre, and in their common plane; for then if these three lines be called α, β, γ, the three new lines OA′, OB′, OC′ are (by 155) the three unit-vectors denoted by the symbols, $U\alpha$, $U\beta$, $U\gamma$; and we have the transformations (comp. 148, 149),

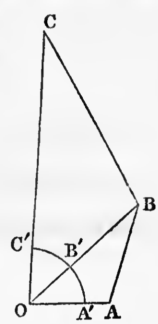

Fig. 42, *bis*.

$$U(q^2) = U \cdot \left(\frac{\beta}{\alpha}\right)^2 = U\frac{\gamma}{\alpha} = \frac{U\gamma}{U\alpha} = \frac{OC'}{OA'} = \left(\frac{OB'}{OA'}\right)^2 = (Uq)^2.$$

(3.) As regards *other* recent transformations (161), although we have seen (135) that it is *not necessary* to invent any *new* or peculiar *symbol*, to represent the *reciprocal* of a quaternion, yet if, for the sake of present convenience, and as a merely *temporary notation*, we write

$$Rq = \frac{1}{q},$$

employing thus, for a moment, the letter R as a *characteristic of reciprocation*, or of the operation of *taking the reciprocal*, we shall then have the *symbolical equations* (comp. 145, 158):

$$R^2 = K^2 = 1; \quad RK = KR; \quad RU = UR = KU = UK;$$

but we have also (by 160), $U^2 = U$; whence it easily follows that

$$U = RUR = RKU = RUK = KUR = KRU = KUK$$
$$= URK = UKR = UKUR = UKRU = (UK)^2 = \&c.$$

(4.) The equation

$$U\frac{\rho}{\alpha} = U\frac{\beta}{\alpha}, \quad \text{or simply,} \quad U\rho = U\beta,$$

expresses that the *locus* of the point P is the *indefinite right line*, or *ray* (comp. 132, (4.)), which is drawn *from* O *in the direction of* OB, but *not* in the *opposite* direction; because it is equivalent to

$$U\frac{\rho}{\beta} = 1; \quad \text{or} \quad \angle\frac{\rho}{\beta} = 0; \quad \text{or} \quad \rho = x\beta,\ x > 0.$$

(5.) On the other hand the equation,

$$U\frac{\rho}{\alpha} = -U\frac{\beta}{\alpha}, \quad \text{or} \quad U\rho = -U\beta,$$

expresses (comp. 132, (5.)) that the locus of P is the *opposite ray* from O; or that it is the *indefinite prolongation of the revector* BO; because it may be transformed to

$$U\frac{\rho}{\beta} = -1; \quad \text{or} \quad \angle\frac{\rho}{\beta} = \pi; \quad \text{or} \quad \rho = x\beta,\ x < 0.$$

(6.) If α, β, γ denote (as in sub-art. 2) the three lines OA, OB, OC of fig. 42 (or of fig. 42, *bis*), so that (by 149) we have the equation $\frac{\gamma}{\alpha} = \left(\frac{\beta}{\alpha}\right)^2$, then this other equation,

$$\left(U\frac{\rho}{\alpha}\right)^2 = U\frac{\gamma}{\alpha},$$

expresses *generally* that the *locus* of P is the *system* of the *two* last loci; or that it is the *whole indefinite right line*, both ways prolonged, through the two points O and B (comp. 144, (2.)).

(7.) But if it happen that the line γ, or OC, like OA′ in fig. 41 (or in fig. 41, *bis*), has the direction *opposite* to that of α, or of OA, so that the last equation takes the *particular form*,

$$\left(U\frac{\rho}{\alpha}\right)^2 = -1,$$

then $U\frac{\rho}{\alpha}$ must be (by 154) a *right versor*; and reciprocally, *every right versor*, with a plane containing α, will be (by 153) a value satisfying the equation. In *this* case, therefore, the *locus of the point* P is (as in 132, (1.), or in 144, (1.)) the *plane through* O, *perpendicular to the line* OA; and the recent *equation* itself, if supposed to be satisfied by a *real* vector* ρ, may be put under either of these two earlier but equivalent *forms*:

$$\angle\frac{\rho}{\alpha}=\frac{\pi}{2};\qquad \rho\perp\alpha.$$

SECTION 9.

On Vector-Arcs, and Vector-Angles, considered as Representatives of Versors of Quaternions; and on the Multiplication and Division of any one such Versor by another.

162. Since every *unit-vector* OA (129), drawn from the origin O, terminates in some point A on the surface of what we have called the *unit-sphere* (128), that *term* A (1) may be considered as a *Representative Point*, of which the *position* on that *surface* determines, and may be said to *represent*, the *direction of the line* OA *in space*; or of that line *multiplied* (12, 17) by *any positive scalar*. And then the *Quaternion* which is the *quotient* (112) of any *two* such unit-vectors, and which is in one view a *Radial* (146), and in another view a *Versor* (151), may be said to have the *arc of a great circle*, AB, upon the unit sphere, which *connects the terms* of the two vectors, for its *Representative Arc*. We may also call this arc a VECTOR ARC, on account of its having a *definite direction* (comp. Art. 1), such as is indicated (for example) by a *curved arrow* in fig. 39 [p. 131]; and as being thus *contrasted* with its own *opposite*, or with what may be called by analogy the *Revector Arc* BA (comp. again 1): this *latter* arc representing, on the present plan, at once the *reciprocal* (134), and the *conjugate* (137), of the former *versor*; because it represents the corresponding *Reversor* (158).

163. This mode of *representation*, of versors of quaternions by vector arcs, would obviously be very imperfect, unless *equals* were to be represented by *equals*. We shall therefore define, as it is otherwise natural to do, that a vector arc, AB, upon the unit sphere, is *equal* to every *other* vector arc CD which can be

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* Compare 149, (2.); also the second Note to the same Article; and the Notes to pages 87 and 88.

derived from it, by simply causing (or conceiving) it to *slide** *in its own great circle, without any change of length*, or *reversal of direction*. In fact, the two isosceles and plane triangles AOB, COD, which have the origin O for their common vector, and rest upon the chords of these two arcs as bases, are thus complanar, similar, and similarly turned; so that (by 117, 118) we may here write,

$$\Delta\,\mathrm{AOB} \propto \mathrm{COD}, \quad \frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\mathrm{OD}}{\mathrm{OC}};$$

the condition of the *equality of the quotients* (that is, here, of the *versors*), *represented* by the two *arcs*, being thus satisfied. We shall sometimes *denote* this sort of *equality of two vector arcs*, AB and CD, by the formula,

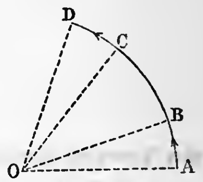

Fig. 35, *bis*.

$$\frown \mathrm{AB} = \frown \mathrm{CD};$$

and then it is clear (comp. 125, and the earlier Art. 3) that we shall also have, by what may be called *inversion* and *alternation*, these two *other* formulæ of *arcual equality*,

$$\frown \mathrm{BA} = \frown \mathrm{DC}; \qquad \frown \mathrm{AC} = \frown \mathrm{BD}.$$

(Compare the annexed figure 35, *bis*.)

164. Conversely, *unequal versors* ought to be *represented* (on the present plan) by *unequal vector arcs*; and accordingly, we purpose to regard any *two such arcs*, as being, for the present purpose, *unequal* (comp. 2), even when they *agree* in *quantity*, or contain the *same number of degrees*, provided that they *differ in direction*: which may happen in either of two principal ways, as follows. For, Ist, they may be *opposite arcs* of *one great circle*; as, for example, a *vector* arc AB, and the corresponding *revector* arc BA; and so may *represent* (162) a *versor*, OB : OA, and the corresponding *reversor*, OA : OB, respectively. Or, IInd, the *two arcs* may belong to *different great circles*, like AB and BC in fig. 43; in which latter case, they represent two *radial quotients* (146) *in different planes*; or (comp. 119) *two diplanar versors*, OB : OA, and OC : OB; but it has been shown generally (122), that *diplanar quaternions* are *always unequal*: we consider therefore, here again the *arcs*, AB and BC, *themselves*, to be (as has been said) *unequal vectors*.

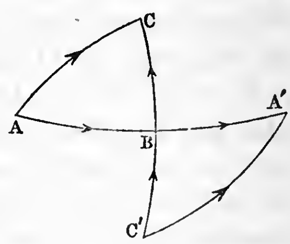

Fig. 43.

* Some aid to the conception may here be derived from the inspection of fig. 34 [p. 113]; in which *two equal angles* are supposed to be traced on the surface of *one common desk*. Or the four lines OA, OB, OC, OD, of fig. 35, may now be conceived to be *equally long*; or to be cut by a circle with O for centre as in the modification of that figure, which is given in Article 163, a little lower down.

165. In this manner, then, we may be led (comp. 122) to regard the *conception of a plane*, or of the *position of a great circle* on the unit sphere, as entering, *essentially*, in general,* into the *conception of a vector-arc*, considered as the *representative of a versor* (162). But even without expressly referring to *versors*, we may see that if, in fig. 43, we suppose that B is the middle point of an arc AA′ of a great circle, so that in a recent notation (163) we may establish the *arcual equation*,

$$\frown AB = \frown BA',$$

we ought then (comp. 105) not to write *also*,

$$\frown AB = \frown BC;$$

because the *two co-initial arcs*, BA′ and BC, which *terminate differently*, must be considered (comp. 2) to be, *as vector-arcs, unequal*. On the other hand, if we should refuse to admit (as in 163) that any *two complanar arcs*, if *equally long*, and *similarly* (not *oppositely*) *directed*, like AB and CD in the recent fig. 35, *bis*, are *equal vectors*, we could not usefully speak of *equality* between *vector-arcs* as existing under any circumstances. We are then thus led again to *include*, generally, the conception of a *plane*, or of one *great circle* as *distinguished* from another, as an *element* in the *conception* of a *Vector-Arc*. And hence an *equation* between *two* such arcs must in general be conceived to include *two relations of co-arcuality*. For example, the equation $\frown$ AB = $\frown$ CD, of Art. 163, includes generally, as a *part* of its signification, the assertion (comp. 123) that the *four points* A, B, C, D belong to one *common great circle* of the unit-sphere; or that *each* of the *two* points, C and D, is *co-arcual* with the *two other* points, A and B.

166. There is, however, a remarkable case of *exception*, in which two vector arcs may be said to be *equal*, although situated in *different planes*: namely, when they are both *great semicircles*. In fact, upon the present plan, *every* great semicircle, AA′, considered as a vector arc, *represents* an *inversor* (153) or it represents *negative unity* (OA′ : OA = $-\alpha : \alpha = -1$), considered as one *limit* of a *versor*; but we have seen (159) that *such* a versor has in general an *indeterminate plane*. Accordingly, whereas the *initial and final points*, or (comp. 1) the *origin* A and the *term* B, of a vector arc AB, are *in general* sufficient to determine the plane of that arc, considered as the *shortest* or the *most direct path* (comp. 112, 128) from the one point to the other on the sphere; in the particular *case* when one of the two given points is diametrically *opposite* to

* We say, *in general*; for it will soon be seen that there is a sense in which *all great semicircles*, considered as vector arcs, may be said to be *equal* to each other.

the other, as A′ to A, the *direction* of this *path* becomes, on the contrary, *indeterminate*. If then we only attend to the *effect produced*, in the way of *change of position of a point*, by a conceived *vection* (or *motion*) *upon the sphere*, we are permitted to say that *all great semicircles are equal vector arcs*; each serving simply, in the present view, to *transport a point* from *one* position to the *opposite*; and thereby to *reverse* (like the factor -1, of which it is here the *representative*) the *direction* of the *radius* which is drawn to that point of the unit sphere.

(1.) The equation, $$\frown \mathrm{AA}' = \frown \mathrm{BB}',$$

in which it is here supposed that A′ is opposite to A, and B′ to B, satisfies evidently the general *conditions of co-arcuality* (165); because the *four* points ABA′B′ are all on *one* great circle. It is evident that the same arcual equation admits (as in 163) of *inversion* and *alternation*; so that

$$\frown \mathrm{A'A} = \frown \mathrm{B'B}, \quad \text{and} \quad \frown \mathrm{AB} = \frown \mathrm{A'B'}.$$

(2.) We may also say (comp. 2) that *all null arcs are equal*, as producing *no effect* on the position of a point upon the sphere; and thus may write generally,

$$\frown \mathrm{AA} = \frown \mathrm{BB} = 0,$$

with the *alternate* equation, or identity, $\frown \mathrm{AB} = \frown \mathrm{AB}$.

(3.) Every such *null vector arc* AA is a *representative*, on the present plan, of the *other unit scalar*, namely *positive unity*, considered as another *limit* of a *versor* (153); and its *plane* is again *indeterminate* (159), unless some *law* be given, according to which the arcual *vection* may be conceived to *begin*, from a given point A, to an indefinitely *near* point B upon the sphere.

167. The principal *use* of *Vector Arcs*, in the present theory, is to assist in *representing*, and (so to speak) in *constructing*, by means of a *Spherical Triangle*, the *Multiplication* and *Division* of any two *Diplanar Versors* (comp. 119, 164). In fact, any two such versors of quaternions (156), considered as radial quotients (152), can easily be reduced (by the general process of Art. 120) to the forms,

$$q = \beta : \alpha = \mathrm{OB} : \mathrm{OA}, \qquad q' = \gamma : \beta = \mathrm{OC} : \mathrm{OB},$$

where A, B, C are corners of such a triangle on the unit sphere; and then (by 107), the former quotient multiplied by the latter will give for product:

$$q' \cdot q = \gamma : \alpha = \mathrm{OC} : \mathrm{OA}.$$

If then (on the plan of Art. 1) any *two successive arcs*, as AB and BC in fig. 43, be called (in relation to each other) *vector* and *provector*; while that *third arc*

AC, which is drawn from the initial point of the first to the final point of the second, shall be called (on the same plan) the *transvector*: we may now say that in the multiplication of any one versor (of a quaternion) by any other, if the *multiplicand** q be *represented* (162) by a *vector-arc* AB, and if the *multiplier* q' be in like manner represented by a *provector-arc* BC, which mode of representation is always possible, by what has been already shown, then the *product* $q' . q$, or $q'q$, *is represented*, at the same time, by the *transvector-arc* AC corresponding.

168. One of the most remarkable consequences of this *construction of the multiplication of versors* is the following: that *the value of the product of two diplanar versors* (164) *depends upon the order of the factors*; or that $q'q$ and qq' are *unequal*, unless q' be *complanar* (119) with q. For let AA′ and CC′ be any two arcs of great circles, *in different planes*, bisecting each other in the point B, as fig. 43 is designed to suggest; so that we have the *two arcual equations* (163),

$$\frown \text{AB} = \frown \text{BA}', \quad \text{and} \quad \frown \text{BC} = \frown \text{C}'\text{B};$$

then one or other of the two following alternatives will hold good. Either, Ist, the two mutually *bisecting arcs* will *both* be *semicircles*, in which case the *two new arcs*, AC and C′A′, will indeed both belong to *one great circle*, namely to that of which B is a *pole*, but will have *opposite directions* therein; because, in this case, A′ and C′ will be diametrically *opposite* to A and C, and therefore (by 166, (1.)) the equation

$$\frown \text{AC} = \frown \text{A}'\text{C}',$$

but *not* the equation

$$\frown \text{AC} = \frown \text{C}'\text{A}',$$

will be satisfied. Or, IInd, the arcs AA′ and CC′, which are supposed to bisect each other in B, will *not both* be semicircles, even if *one* of them happen to be such; and in this case, the arcs AC, C′A′ will belong to *two distinct great circles*, so that they will be *diplanar*, and therefore *unequal*, when considered as *vectors*. (Compare the Ist and IInd cases of Art. 164.) In *each* case, therefore, AC and C′A′ are *unequal vector arcs*; but the *former* has been seen (167) to *represent* the product $q'q$; and the *latter* represents, in like manner, the *other* product, qq', of the same two versors taken in the *opposite order*, because it is the *new transvector* arc, when C′B (= BC) is treated as the *new vector* arc, and BA′ (= AB) as the *new provector* arc, as is indicated by the curved arrows in

* Here, as in 107, and elsewhere, we write the symbol of the *multiplier* towards the *left-hand*, and that of the *multiplicand* towards the *right*.

fig. 43. The *two products*, $q'q$ and qq', are therefore *themselves unequal*, as above asserted, under the supposed condition of *diplanarity*.

169. On the other hand, when the two factors, q and q', are *complanar versors*, it is easy to prove, in several different ways, that their products, $q'q$ and qq', are *equal*, as in algebra. Thus we may conceive that the arc CC′, in fig. 43, is made to *turn* round its middle point B, until the spherical angle CBA′ *vanishes*; and then the two *new transvector-arcs*, AC and C′A′, will evidently become not only *complanar* but *equal*, in the sense of Art. 163, as being *still equally long*, and being *now similarly directed*. Or, in fig. 35, *bis*, of the last cited Article, we may conceive a point E, bisecting the arc BC, and therefore also the arc AD, which is *commedial* therewith (comp. Art. 2, and the second figure 3 of that Article); and then, if we represent the one versor q by either of the two equal arcs, AE, ED, we may at the same time represent the other versor q' by either of the two other equal arcs, EC, BE; so that the one product, $q'q$, will be represented by the arc AC, and the other product, qq', by the equal arc BD. Or, without reference to *vector arcs*, we may suppose that the two *factors* are,

$$q = \beta : \alpha = \text{OB} : \text{OA}, \qquad q' = \gamma : \alpha = \text{OC} : \text{OA},$$

OA, OB, OC being any three complanar and equally long right lines (see again fig. 35, *bis*); for thus we have only to determine a fourth line, δ or OD, of the same length, and in the same plane, which shall satisfy the equation $\delta : \gamma = \beta : \alpha$ (117), and therefore also (by 125) the alternate equation, $\delta : \beta = \gamma : \alpha$; and it will then immediately follow* (by 107) that

$$q' . q = \frac{\delta}{\beta} . \frac{\beta}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\gamma} . \frac{\gamma}{\alpha} = q . q'.$$

We may therefore infer, for *any two versors* of quaternions, q and q', the two following reciprocal relations:

$$\text{I.} \ldots q'q = qq', \quad \text{if} \quad q' \,|||\, q \,(123);$$

$$\text{II.} \ldots \text{if } q'q = qq', \quad \text{then} \quad q' \,|||\, q \,(168);$$

convertibility of factors (as regards their *places* in the *product*) being thus at once a *consequence* and a *proof* of *complanarity*.

170. In the Ist case of Art. 168, the *factors* q and q' are both *right versors* (153); and because we have seen that then their *two products*, $q'q$ and qq', are

* It is evident that, in this last process of reasoning, we make *no use* of the supposed *equality of lengths* of the four lines compared; so that we might prove, in exactly the same way, that $q'q = qq'$ if $q' \,|||\, q$ (123), *without* assuming that these two *complanar factors*, or *quaternions*, q and q', are *versors*.

versors *represented* by equally long but oppositely directed arcs of one great circle, as in the Ist case of 164, it follows (comp. 162) that these two products are at once *reciprocal* (134), and *conjugate* (137), to each other; or that they are related as *versor* and *reversor* (158). We may therefore write, generally,

$$\text{I.} \ldots qq' = \mathrm{K}q'q, \quad \text{and} \quad \text{II.} \ldots qq' = \frac{1}{q'q},$$

if q and q' be *any two right versors*; because the multiplication of any two *such* versors, in two opposite orders, may always be represented or constructed by a figure such as that lately numbered 43, in which the bisecting arcs AA′ and CC′ are *semicircles*. The IInd formula may also be thus written (comp. 135, 154):

$$\text{III.} \ldots \text{if } q^2 = -1, \text{ and } q'^2 = -1, \text{ then } q'q \,.\, qq' = +1;$$

and under this form it evidently agrees with ordinary algebra, because it expresses that, *under the supposed conditions*,

$$q'q \cdot qq' = q'^2 \cdot q^2;$$

but it will be found that this last equation is *not an identity* in the general theory of *quaternions*.

171. If the two bisecting semicircles cross each other *at right angles*, the conjugate products are represented by *two quadrants, oppositely turned*, of one great circle. It follows that *if two right versors, in two mutually rectangular planes, be multiplied together in two opposite orders, the two resulting products will be two opposite right versors, in a third plane, rectangular to the two formers*; or in symbols, that

$$\text{if } q^2 = -1, \; q'^2 = -1, \text{ and } \mathrm{Ax} \,.\, q' \perp \mathrm{Ax} \,.\, q,$$

then $$(q'q)^2 = (qq')^2 = -1, \quad q'q = -qq';$$

and $$\mathrm{Ax} \,.\, q'q \perp \mathrm{Ax} \,.\, q, \quad \mathrm{Ax} \,.\, q'q \perp \mathrm{Ax} \,.\, q'.$$

In *this* case, therefore, we have what would be in algebra a *paradox*, namely the equation,

$$(q'q)^2 = -q'^2 \cdot q^2,$$

if q and q' be *any two right versors, in two rectangular planes*; but we see that this result is *not more* paradoxical, in appearance, than the equation

$$q'q = -qq',$$

which exists, *under the same conditions*. And when we come to examine what, in the last analysis, may be said to be the *meaning* of this last equation, we find it to be simply this: *that any two quadrantal* or *right rotations, in planes* *perpendicular to each other, compound themselves into a third right rotation, as their resultant, in a plane perpendicular to each of them*: and that this *third* or *resultant* rotation has one or other of *two opposite directions*, according to the *order* in which the *two component rotations* are taken, so that one shall be *successive* to the other.

172. We propose to return, in the next section, to the consideration of such a *System of Right Versors* as that which we have here briefly touched upon: but desire at present to remark (comp. 167) that a *spherical triangle* ABC may serve to *construct*, by means of *representative arcs* (162), not only the *multiplication*, but also the *division*, of any one of *two diplanar versors* (or radial quotients) by the other. In fact, we have only to conceive (comp. fig. 43) that the *vector arc* AB represents a given *divisor*, say q, or $\beta : \alpha$, and that the *transvector arc* AC (167) represents a given *dividend*, suppose q'', or $\gamma : \alpha$; for then the *provector arc* BC (comp. again 167) will represent, on the same plan, the *quotient of these two versors*, namely $q'' : q$, or $\gamma : \beta$ (106), or the versor lately called q'; since we have generally, by 106, 107, 120, for quaternions, as in algebra, the two identities:

$$(q'' : q) \cdot q = q''; \quad q'q : q = q'.$$

173. It is however to be observed that, for reasons already assigned, we must *not* employ, for *diplanar versors*, such an equation as $q \cdot (q'' : q) = q''$; because we have found (168) that, for *such* versors, the ordinary *algebraic identity*, $qq' = q'q$, *ceases to be true.* In fact by 169, we may now establish the two converse formulæ:

$$\text{I.} \ldots \quad q\,(q'' : q) = q'', \quad \text{if} \quad q'' \parallel\!| \, q \; (123);$$

$$\text{II.} \ldots \text{if } q\,(q'' : q) = q'', \text{ then } q'' \parallel\!| \, q.$$

Accordingly, in fig. 43, if q, q', q'' be still represented by the arcs AB, BC, AC, the product $q\,(q'' : q)$, or qq', is *not* represented by AC, but by the *different arc* C′A′ (168), which *as a vector arc* has been seen to be *unequal* thereto: although it is true that these two last arcs, AC and C′A′, are always *equally long*, and therefore subtend *equal angles* at the *centre* O of the unit sphere; so that we may write, generally, for *any two versors* (or indeed for *any two quaternions*),* q and q'', the formula,

$$\angle\, q\,(q'' : q) = \angle\, q''.$$

* It will soon be seen [see 191] that several of the formulæ of the present section, respecting the *multiplication* and *division* of *versors*, considered as *radial quotients* (151), require little or no modification, in the passage to the corresponding *operations on quaternions*, considered as *general quotients of vectors* (112).

174. *Another mode of Representation of Versors*, or rather *two* such new modes, although intimately connected with each other, may be briefly noticed here.

Ist. We may consider the *angle* AOB, *at the centre* O *of the unit-sphere*, when conceived to have not only a definite *quantity*, but also a determined *plane* (110), and a given *direction* therein (as indicated by one of the curved arrows in fig. 39 [p. 131], or by the arrow in fig. 33 [p. 111]), as being what may be called by analogy a *Vector-Angle*; and may say that it *represents*, or that it is the *Representative Angle* of, the *Versor* OB : OA, where OA, OB are *radii* of the unit-sphere.

IInd. Or we may replace this *rectilinear* angle AOB at the *centre*, by the equal *Spherical Angle* AC′B, at what may be called the *Positive Pole* of the *representative arc* AB; so that C′A and C′B are *quadrants*; and the *rotation*, at this *pole* C′, from the first of these two quadrants to the second (as seen from a point outside the sphere), has the *direction* which has been selected (111, 127) for the *positive* one, as indicated in the annexed figure 44: and then we may consider this *spherical angle* as a *new Angular Representative* of the *same versor* q, or OB : OA, as before.

Fig. 44.

175. Conceive now that after employing a *first* spherical triangle ABC, to *construct* (as in 167) the *multiplication* of any one given versor q, by any other given versor q', we form a *second* or *polar* triangle, of which the corners A′, B′, C′ shall be respectively (in the sense just stated) the *positive poles* of the three *successive sides*, BC, CA, AB, of the former triangle; and that then we pass to a *third* triangle A′B″C′, as part of the *same lune* B′B″ with the second, by taking for B″ the point diametrically *opposite* to B′; so that B″ shall be the *negative* pole of the arc CA, or the *positive* pole of what was lately called (167) the *transvector-arc* AC: also let C″ be, in like manner, the point opposite to C′ on the unit sphere. Then we may not only write (comp. 129),

$$\mathrm{Ax}\,.\,q = \mathrm{OC}', \qquad \mathrm{Ax}\,.\,q' = \mathrm{OA}', \qquad \mathrm{Ax}\,.\,q'q = \mathrm{OB}'',$$

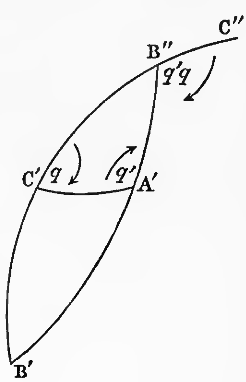

Fig. 45.

but shall also have the equations,

$$\angle q = \mathrm{B''C'A'}, \qquad \angle q' = \mathrm{C'A'B''}, \qquad \angle q'q = \mathrm{C''B''A'};$$

these three spherical angles, namely the *two base-angles* at C′ and A′, and the *external vertical angle* at B″, of the new or third triangle A′B″C′, will therefore *represent*, respectively, on the plan of 174, II., the *multiplicand*, q, the *multiplier*, q', and the *product*, $q'q$. (Compare the annexed figure 45.)

176. Without expressly referring to the former triangle ABC, we can *connect* this last *construction of multiplication of versors* (175) with the general formula (107), as follows.

Let α and β be now conceived to be two *unit-tangents** to the sphere at C′, perpendicular respectively to the two arcs C′B″ and C′A′, and drawn towards the same sides of those arcs as the points A′ and B′ respectively; and let two other unit-tangents, equal to these, and denoted by the same letters, be drawn (as in the annexed figure 45, *bis*) at the points B″ and A′, so as to be normal there to the same arcs C′B″ and C′A′, and to fall towards the same sides of them as before. Let also two other unit-tangents, equal to each other, and each denoted by γ, be drawn at the two last points B″ and A′, so as to be both perpendicular to the arc A′B″, and to fall towards the same side of it as the point C′. Then (comp. 174, II.) the *two quotients*, $\beta : \alpha$ and $\gamma : \beta$, will be equal to the *two versors*, q and q', which were lately *represented* (in fig. 45) by the *two base angles*, at C′ and A′, of the spherical triangle A′B″C′; the *product*, $q'q$, of these two versors, is therefore (by 107) equal to the *third quotient*, $\gamma : \alpha$; and consequently it is *represented*, as before, by the *external vertical angle* C″B″A′ of the same triangle, which is evidently *equal* in *quantity* to the *angle* of this third quotient, and has the *same axis* OB″, and the *same direction of rotation*, as the arrows in fig. 45, *bis*, may assist to show.

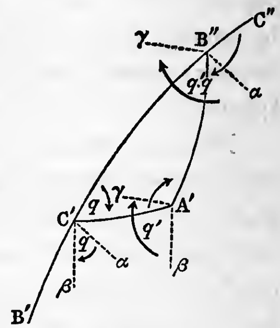

Fig. 45, *bis*.

177. In each of the two last figures, the *internal vertical angle* at B″ is thus equal to the *Supplement*, $\pi - \angle\, q'q$, of the *angle of the product*; and it is important to observe that the corresponding *rotation at the vertex* B″, *from the side* B″A′ *to the side* B″C′, or (as we may briefly express it) *from the point* A′ *to the point* C′, is *positive*; a result which is easily seen to be a *general* one, by the reasoning of the foregoing Article.† We may then infer, generally, that *when the multiplication of any two versors is constructed by a spherical triangle*, of which the two *base angles represent* (as in the two last Articles) the *factors*,

* By an *unit tangent* is here meant simply an *unit line* (or *unit vector*, 129) so drawn as to be *tangential to the unit-sphere*, and to have its *origin*, or its *initial point* (1), *on the surface* of that sphere, and *not* (as we have *usually* supposed) *at the centre* thereof.

† If a person be supposed to stand *on the sphere* at B″, and to *look* towards the arc A′C′, it would appear to him to have a *right-handed direction*, which is the one here adopted as positive (127).

while the *external vertical angle* represents the *product*, then the *rotation round the axis* (OB″) *of that product* $q'q$, *from the axis* (OA′) *of the multiplier* q', *to the axis* (OC′) *of the multiplicand* q, is *positive*: whence it follows that the rotation round the axis $\mathrm{Ax}\,.\,q'$ of the multiplier, from the axis $\mathrm{Ax}\,.\,q$ of the multiplicand, to the axis $\mathrm{Ax}\,.\,q'q$ of the product, is *also* positive. Or, to express the same thing more fully, since the only *rotations* hitherto considered have been *plane* ones (as in 128, &c.), we may say that if the two latter *axes* be *projected on a plane perpendicular to the former*, so as still to have a *common origin* O, then the rotation round $\mathrm{Ax}\,.\,q'$, *from the projection* of $\mathrm{Ax}\,.\,q$ to the projection of $\mathrm{Ax}\,.\,q'q$, will be directed (with our conventions) *towards the right hand.*

178. We have therefore thus a *new mode of geometrically exhibiting* the *inequality of the two products*, $q'q$ and qq', of *two diplanar versors* (168), when taken as factors in *two different orders.* For this purpose, let

$$\mathrm{Ax}\,.\,q = \mathrm{OP}, \quad \mathrm{Ax}\,.\,q' = \mathrm{OQ}, \quad \mathrm{Ax}\,.\,q'q = \mathrm{OR};$$

and prolong to some point S the arc PR of a great circle on the unit sphere. Then, for the spherical triangle PQR, by principles lately established, we shall have (comp. 175) the following values of the two internal base angles at P and Q, and of the external vertical angle at R:

$$\mathrm{RPQ} = \angle\, q; \qquad \mathrm{PQR} = \angle\, q'; \qquad \mathrm{SRQ} = \angle\, q'q;$$

and the rotation at Q, from the side QP to the side QR will be right-handed. Let fall an arcual perpendicular, RT, from the vertex R on the base PQ, and prolong this perpendicular to R′, in such a manner as to have

$$\frown \mathrm{RT} = \frown \mathrm{TR}';$$

also prolong PR′ to some point S′. We shall then have a new triangle PQR′, which will be a sort of *reflexion* (comp. 138) of the old one with respect to their common base PQ; and this *new triangle* will serve to *construct* the *new product*, qq'. For the rotation at P from PQ to PR′ will be right-handed, as it ought to be; and we shall have the equations,

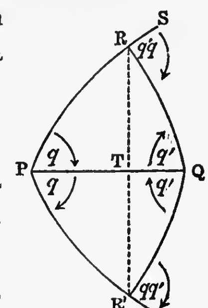

Fig. 46.

$$\mathrm{QPR}' = \angle\, q; \qquad \mathrm{R'QP} = \angle\, q'; \qquad \mathrm{QR'S'} = \angle qq'; \qquad \mathrm{OR}' = \mathrm{Ax}\,.\,qq';$$

so that the *new external and spherical angle*, QR′S′, will *represent the new versor*, qq', as the *old angle* SRQ represented the *old versor*, $q'q$, obtained from a different *order of the factors.* And although, no doubt, these *two angles*, at R and R′, are always *equal in quantity*, so that we may establish (comp. 173) the *general formula*,

$$\angle q'q = \angle qq',$$

yet *as vector angles* (174), and therefore as *representatives of versors*, they must be considered to be *unequal*: because they have *different planes*, namely, the *tangent planes* to the sphere at the *two vertices* R and R′; or the two planes respectively *parallel* to these, which are drawn *through the centre* O.

179. *Division of Versors* (comp. 172) can be *constructed* by means of *Representative Angles* (174), as well as by *representative arcs* (162). Thus to *divide* q'' by q, or rather to *represent* such division *geometrically*, on a plan entirely similar to that last employed for multiplication, we have only to determine the two points P and R, in fig. 46, by the two conditions,

$$\mathrm{OP} = \mathrm{Ax} \,.\, q, \qquad \mathrm{OR} = \mathrm{Ax} \,.\, q'',$$

and then to find a third point Q by the two angular equations,

$$\mathrm{RPQ} = \angle\, q, \qquad \mathrm{QRP} = \pi - \angle\, q'',$$

the rotation round P from PR towards PQ being positive; after which we shall have,

$$\mathrm{Ax} \,.\, (q'' : q) = \mathrm{OQ}; \qquad \angle\, (q'' : q) = \mathrm{PQR}.$$

(1.) Instead of conceiving, in fig. 46, that the dotted line RTR′, which connects the vertices of the two triangles, with PQ for their common base (178), is an *arc of a great circle*, perpendicularly bisected by that base, we may imagine it to be an *arc of a small circle*, described with the point P for its *positive pole* (comp. 174, II.). And then we may say that the *passage* (comp. 173) *from the versor* q'', or $q'q$, *to the unequal versor* $q\ (q'' : q)$, or qq', is *geometrically performed* by a *Conical Rotation of the Axis* $\mathrm{Ax} \,.\, q''$, *round the axis* $\mathrm{Ax} \,.\, q$, *through an angle* $= 2 \angle\, q$, *without any* (*quantitative*) *change of the angle* $\angle\, q''$; so that we have, as before, the general formula (comp. again 173),

$$\angle\, q\,(q'' : q) = \angle\, q''.$$

(2.) Or if we prefer to employ the construction of multiplication and division by representative *arcs*, which fig. 43 [p. 144] was designed to illustrate, and conceive that a new point C″ is determined in that figure by the condition $\frown \mathrm{A'C''} = \frown \mathrm{C'A'}$, we may then say that in the passage from the versor q'', which is represented by AC, to the versor $q\ (q'' : q)$, represented by C′A′ or by A′C″, the *representative arc of* q'' is made to *move, without change of length*, so as to preserve a *constant inclination* to the representative arc* AB *of* q, while *its initial point describes the double of that arc* AB, in passing from A to A′.

(3.) It may be seen, by these few examples, that if, even independently of some *new characteristics of operation*, such as K and U, *new combinations of old symbols*, such as $q\,(q'':q)$, occur in the present Calculus, which are not wanted in algebra, they admit for the most part of *geometrical interpretations*, of an easy and interesting kind; and in fact *represent conceptions*, which cannot well be dispensed with, and which it is useful to be able to *express*, with so much simplicity and conciseness. (Compare the remarks in Art. 161; and the sub-articles to 132, 145.)

180. In connexion with the construction indicated by the two figures 45, it may be here remarked, that if ABC be *any spherical triangle*, and if A′, B′, C′ be (as in 175) the *positive poles* of its three *successive sides*, BC, CA, AB, then the *rotation* (comp. 177, 179) *round* A′ *from* B′ *to* C′, or that round B′ from C′ to A′, &c., is *positive*. The easiest way, perhaps, of *seeing* the truth of this assertion is to conceive that if the rotation round A from B to C be not *already positive*, we *make* it such, by passing to the diametrically *opposite triangle* on the sphere, which will not change the *poles* A′, B′, C′. Assuming then that these poles are thus the *near* ones to the corresponding corners of the given triangle, we arrive without any difficulty at the conclusion stated above: which has been virtually employed in our *construction* of *multiplication* (and *division*) of *versors*, by means of *Representative Angles* (175, 176); and which may be otherwise justified (as before), by the consideration of the *unit-tangents* of fig. 45, *bis*.

(1.) Let then α, β, γ be *any three given unit vectors*, such that the rotation round the first, from the second to the third, is *positive* (in the sense of Art. 177); and let α', β', γ' be three other unit vectors, derived from these by the equations,

$$\alpha' = \mathrm{Ax}\,.\,(\gamma:\beta), \qquad \beta' = \mathrm{Ax}\,.\,(\alpha:\gamma), \qquad \gamma' = \mathrm{Ax}\,.\,(\beta:\alpha);$$

then the rotation round α', from β' to γ', will be positive also; and we shall have the converse formulæ,

$$\alpha = \mathrm{Ax}\,.\,(\gamma':\beta'), \qquad \beta = \mathrm{Ax}\,.\,(\alpha':\gamma'), \qquad \gamma = \mathrm{Ax}\,.\,(\beta':\alpha').$$

(2.) If the rotation round α from β to γ were given to be *negative*, α', β', γ' being still deduced from those three vectors by the same three equations as before, then the *signs* of α, β, γ would all require to be *changed*, in the three

* In a manner analogous to the motion of the *equator* on the *ecliptic*, by luni-solar *precession*, in astronomy.

last (or *reciprocal*) formulæ; but the rotation round α', from β' to γ', would *still* be *positive*.

(3.) Before closing this section, it may be briefly noticed, that it is sometimes convenient, from motives of analogy (comp. Art. 5), to speak of the *Transvector-Arc* (167), which has been seen to *represent* a *product* of two versors, as being the ARCUAL SUM of the two *successive vector-arcs*, which represent (on the same plan) the *factors*; *Provector* being still said to be *added* to *Vector*: but the *Order* of such *Addition of Diplanar Arcs* being *not now indifferent* (168), as the corresponding *order* had been early found (in 7) to be, when the *vectors* to be added were *right lines*. [Thus in fig. 43, $\frown \text{BC} + \frown \text{AB} = \frown \text{AC}$ and $\frown \text{BA}' + \frown \text{C}'\text{B} = \frown \text{C}'\text{A}'$. But $\frown \text{BA}' = \frown \text{AB}$ and $\frown \text{C}'\text{B} = \frown \text{BC}$, consequently $\frown \text{AB} + \frown \text{BC} = \frown \text{C}'\text{A}'$. If α and β are any two vector arcs, and if x is any scalar, $x(\alpha \pm \beta)$ is *not* equal to $x\alpha \pm x\beta$. Compare 14, and notice that the property there proved depends on the possibility of constructing similar plane triangles of different sizes.]

(4.) We may also speak occasionally, by an extension of the same analogy, of the *External Vertical Angle* of a spherical triangle, as being the SPHERICAL SUM *of the two Base Angles* of that triangle, taken in a suitable *order of summation* (comp. fig. 46); the *Angle* which represents (174) the *Multiplier* being then said to be *added* (as a sort of *Angular Provector*) to that *other Vector-Angle* which represents the *Multiplicand*; whilst what is here called the *sum* of these two angles (and is, with respect to them, a species of *Transvector-Angle*) represents, as has been proved, the *Product*.

(5.) This conception of *angular transvection* becomes perhaps a little more clear, when (on the plan of 174, I.) we assume the *centre* O as the *common vertex* of three angles AOB, BOC, AOC, situated generally in *three different planes*. For then we may conceive a *revolving radius* to be either carried by *two successive angular motions*, from OA to OB, and thence to OC; or to be transported immediately, by *one* such motion, from the *first* to the *third position*.

(6.) Finally, as regards the construction indicated by fig. 45, *bis*, in which *tangents* instead of *radii* were employed, it may be well to remark distinctly here, that $\text{A}'\text{B}''\text{C}'$, in that figure, may be *any given spherical triangle*, for which the rotation round B'' from A' to C' is *positive* (177); and that then, if the *two factors* q and q', be *defined* to be the *two versors*, of which the *internal angles* at C' and A' are (in the sense of 174, II.) the *representatives*, the reasonings of Art. 176 will prove, without necessarily referring, *even in thought*, to any *other* triangle (such as ABC), that the *external angle* at B'' is (in the same sense) the *representative of the product*, $q'q$, as before.

SECTION 10.

On a System of Three Right Versors, in Three Rectangular Planes; and on the Laws of the Symbols, i, j, k.

181. Suppose that OI, OJ, OK are any three given and co-initial but rectangular unit-lines, the rotation round the first from the second to the third being positive; and let OI′, OJ′, OK′ be the three unit-vectors respectively opposite to these, so that

$$\text{OI}' = -\text{OI}, \qquad \text{OJ}' = -\text{OJ}, \qquad \text{OK}' = -\text{OK}.$$

Let the three new symbols i, j, k denote a *system* (comp. 172) *of three right versors*, in *three mutually rectangular planes*, with the three given lines for their respective *axes*; so that

$$\text{Ax}\,.\,i = \text{OI}, \qquad \text{Ax}\,.\,j = \text{OJ}, \qquad \text{Ax}\,.\,k = \text{OK},$$

and

$$i = \text{OK} : \text{OJ}, \qquad j = \text{OI} : \text{OK}, \qquad k = \text{OJ} : \text{OI},$$

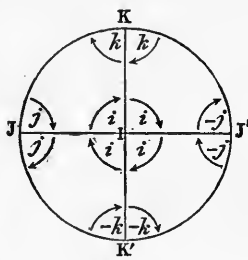

Fig. 47.

as figure 47 may serve to illustrate. We shall then have these other expressions for the same three versors:

$$i = \text{OJ}' : \text{OK} = \text{OK}' : \text{OJ}' = \text{OJ} : \text{OK}';$$

$$j = \text{OK}' : \text{OI} = \text{OI}' : \text{OK}' = \text{OK} : \text{OI}';$$

$$k = \text{OI}' : \text{OJ} = \text{OJ}' : \text{OI}' = \text{OI} : \text{OJ}';$$

while the three respectively *opposite* versors may be thus expressed:

$$-i = \text{OJ} : \text{OK} = \text{OK}' : \text{OJ} = \text{OJ}' : \text{OK}' = \text{OK} : \text{OJ}';$$

$$-j = \text{OK} : \text{OI} = \text{OI}' : \text{OK} = \text{OK}' : \text{OI}' = \text{OI} : \text{OK}';$$

$$-k = \text{OI} : \text{OJ} = \text{OJ}' : \text{OI} = \text{OI}' : \text{OJ}' = \text{OJ} : \text{OI}'.$$

And from the comparison of these different expressions several important symbolical consequences follow, which it will be worth while to enunciate separately here, although some of them are virtually included in the results of former sections.

182. In the *first* place, since

$$i^2 = (\text{OJ}' : \text{OK})\,.\,(\text{OK} : \text{OJ}) = \text{OJ}' : \text{OJ}, \ \&\text{c.},$$

we deduce (comp. 148) the following equal values for the *squares* of the new symbols:

$$\text{I.} \ldots i^2 = -1; \qquad j^2 = -1; \qquad k^2 = -1;$$

as might indeed have been at once inferred (154), from the circumstance that the three radial quotients, (146), denoted here by i, j, k, are all *right versors* (181).

In the *second* place, since

$$i \,.\, j = (\text{OJ} : \text{OK}') \,.\, (\text{OK}' : \text{OI}) = \text{OJ} : \text{OI}, \ \&\text{c.},$$

we have the following values for the *products* of the same three symbols, or versors, when taken *two by two*, and in a certain *order of succession* (comp. 168, 171):

$$\text{II.} \ldots ij = k; \qquad jk = i; \qquad ki = j.$$

But in the *third* place (comp. again 171), since

$$j \,.\, i = (\text{OI} : \text{OK}) \,.\, (\text{OK} : \text{OJ}) = \text{OI} : \text{OJ}, \ \&\text{c.},$$

we have these other and *contrasted* formulæ, for the *binary products* of the *same* three right versors, when taken as factors with an *opposite order*:

$$\text{III.} \ldots ji = -k; \qquad kj = -i; \qquad ik = -j.$$

Hence, while the *square of each* of the *three right versors*, denoted by these *three new symbols*, *ijk*, is equal (154) to *negative unity*, the *product of any two* of them is equal either to the *third itself*, or to the *opposite* (171) of that third versor, according as the *multiplier precedes* or *follows* the *multiplicand*, in the *cyclical succession*,

$$i, j, k, i, j, \ldots$$

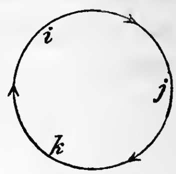

Fig. 47, *bis*.

which the annexed figure 47, *bis*, may give some help towards remembering.

(1.) To connect such multiplications of i, j, k with the theory of representative arcs (162), and of representative angles (174), we may regard any one of the four *quadrantal arcs*, JK, KJ′, J′K′, K′J, in fig. 47, or any one of the four *spherical right angles*, JIK, KIJ′, J′IK′, K′IJ, which those arcs subtend at their common pole I, as *representing* the versor i; and similarly for j and k, with the introduction of the point I′ opposite to I, which is to be conceived as being at the back of the figure.

(2.) The *squaring* of i, or the equation $i^2 = -1$, comes thus to be geometrically *constructed* by the *doubling* (comp. Arts. 148, 154, and figs. 41, 42) of an *arc*, or of an *angle*. Thus, we may conceive the *quadrant* KJ′ to be *added* to the *equal arc* JK, their sum being the *great semicircle* JJ′, which (by 166) *represents* an *inversor* (153), or *negative unity* considered as a *factor*. Or we may *add* the *right angle* KIJ′ to the *equal angle* JIK, and so obtain a *rotation* through *two* right angles at the *pole* I, or at the *centre* O; which rotation is equivalent (comp. 154, 174) to an *inversion of direction*, or to a passage from the radius OJ, to the opposite radius OJ′.

(3.) The *multiplication* of j by i, or the equation $ij = k$, may in like manner be *arcually constructed*, by the *addition* of K′J, as a *provector-arc* (167), to IK′ as a *vector-arc* (162), giving IJ, which is a *representative* of k, as the *transvector-arc*, or *arcual-sum* (180, (3.)). Or the same multiplication may be *angularly* constructed, with the help of the *spherical triangle* IJK; in which the *base-angles* at I and J represent respectively the *multiplier*, i, and the *multiplicand*, j, the rotation round I from J to K being *positive*: while their *spherical sum* (180, (4.)), or the *external vertical angle* at K (comp. 175, 176), represents the same *product*, k, as before.

(4.) The *contrasted* multiplication of i by j, or of j *into** i, may in like manner be *constructed*, or geometrically represented, either by the addition of the arc KI, as a *new provector*, to the arc JK as a new vector, which new process gives JI (instead of IJ) as the *new transvector*; or with the aid of the *new triangle* IJK′ (comp. figs. 46, 47), in which the rotation round I from J to the *new vertex* K′ is *negative*, so that the angle at I represents now the multiplicand, and the resulting angle at the *new pole* K′ represents the *new* and *opposite product*, $ji = -k$.

183. Since we have thus $ji = -ij$ (as we had $q'q = -qq'$ in 171), we see that the *laws of combination of the new symbols*, i, j, k, are *not in all respects the same* as the corresponding laws in *algebra*; since the *Commutative Property of Multiplication*, or the *convertibility* (169) of the places of the *factors* without change of value of the *product*, does *not here* hold good: which arises (168) from the circumstance, that the factors to be combined are here *diplanar versors* (181). It is therefore important to observe, that there *is* a respect in which the *laws of* i, j, k *agree* with usual and *algebraic laws*: namely, in the *Associative Property of Multiplication*; or in the property that the new symbols always obey the *associative formula* (comp. 9),

$$\iota \,.\, \kappa\lambda = \iota\kappa \,.\, \lambda,$$

whichever of them may be substituted for ι, for κ, and for λ; in virtue of which equality of values we may *omit the point*, in any such symbol of a

* A multiplicand is said to be multiplied *by* the multiplier; while, on the other hand, a multiplier is said to be multiplied *into* the multiplicand: a *distinction* of this sort between the *two factors* being necessary, as we have seen, for *quaternions*, although it is not needed for algebra.

ternary product (whether of equal or of unequal factors), and write it simply as $\iota\kappa\lambda$. In particular we have thus,

$$i \,.\, jk = i \,.\, i = i^2 = -1\,; \qquad ij \,.\, k = k \,.\, k = k^2 = -1\,;$$

or briefly,

$$ijk = -1.$$

We may, therefore, by 182, establish the following important *Formula*:

$$i^2 = j^2 = k^2 = ijk = -1\,; \tag{A}$$

to which we shall occasionally refer, as to "Formula A," and which we shall find to contain (virtually) *all the laws of the symbols ijk*, and therefore to be a *sufficient symbolical basis* for the whole *Calculus of Quaternions*:* because it will be shown that *every quaternion can be reduced to the Quadrinomial Form*,

$$q = w + ix + jy + kz,$$

where w, x, y, z compose a *system of four scalars*, while i, j, k are the same *three right versors* as above. [See 221.]

(1.) A direct proof of the equation, $ijk = -1$, may be derived from the definitions of the symbols in Art. 181. In fact, we have only to remember that those definitions were seen to give,

$$i = \mathrm{OJ}' : \mathrm{OK}, \qquad j = \mathrm{OK} : \mathrm{OI}', \qquad k = \mathrm{OI}' : \mathrm{OJ}\,;$$

and to observe that, by the general formula of multiplication (107), *whatever four lines* may be denoted by α, β, γ, δ, we have always,

$$\frac{\delta}{\gamma} \cdot \frac{\gamma}{\beta}\frac{\beta}{\alpha} = \frac{\delta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\delta}{\gamma}\frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha}\,;$$

or briefly, as in algebra,

$$\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha} = \frac{\delta}{\alpha},$$

the *point* being thus *omitted* without danger of confusion: so that

$$ijk = \mathrm{OJ}' : \mathrm{OJ} = -1, \text{ as before.}$$

* This formula (A) was accordingly made the *basis* of that Calculus in the first communication on the subject, by the present writer, to the Royal Irish Academy in 1843; and the letters *i*, *j*, *k*, continued to be, for some time, the *only peculiar symbols* of the Calculus in question. But it was gradually found to be useful to incorporate with these a few other *notations* (such as K and U, &c.), for representing *Operations on Quaternions*. It was also thought to be instructive to establish the *principles* of that Calculus, on a more *geometrical* (or less exclusively *symbolical*) *foundation* than at first; which was accordingly afterwards done, in the volume entitled: *Lectures on Quaternions* (Dublin, 1853); and is again attempted in the present work, although with many differences in the adopted *plan* of exposition, and in the *applications* brought forward, or suppressed.

Similarly, we have these two other ternary products:

$$jki = (\text{OK}' : \text{OI})\ (\text{OI} : \text{OJ}')\ (\text{OJ}' : \text{OK}) = \text{OK}' : \text{OK} = -1;$$

$$kij = (\text{OI}' : \text{OJ})\ (\text{OJ} : \text{OK}')\ (\text{OK}' : \text{OI}) = \text{OI}' : \text{OI} = -1.$$

(2.) On the other hand,

$$kji = (\text{OJ} : \text{OI})\ (\text{OI} : \text{OK})\ (\text{OK} : \text{OJ}) = \text{OJ} : \text{OJ} = +1;$$

and in like manner,

$$ikj = +1, \quad \text{and} \quad jik = +1.$$

(3.) The equations in 182 give also these other ternary products, in which the law of *association of factors* is still obeyed:

$$i\,.\,ij = ik = -j = i^2j = ii\,.\,j, \qquad iij = -j;$$

$$i\,.\,ji = i\,.-k = -ik = j = ki = ij\,.\,i, \qquad iji = +j;$$

$$i\,.\,jj = i\,.-1 = -i = kj = ij\,.\,j, \qquad ijj = -i;$$

with others deducible from these, by mere *cyclical permutation* of the letters, on the plan illustrated by fig. 47, *bis*.

(4.) In general, if the *Associative Law of Combination* exist for *any three symbols* whatever of a *given class*, and for a *given mode* of combination, as for *addition of lines* in Art. 9, or for *multiplication of ijk* in the present Article, the *same law* exists for any *four* (or more) symbols of the same class, and combinations of the same kind. For example, if *each* of the four letters ι, κ, λ, μ denote *some one* of the three symbols i, j, k (but not necessarily the *same* one), we have the formula,

$$\iota\,.\,\kappa\lambda\mu = \iota\,.\,\kappa\,.\,\lambda\mu = \iota\kappa\,.\,\lambda\mu = \iota\kappa\,.\,\lambda\,.\,\mu = \iota\kappa\lambda\,.\,\mu = \iota\kappa\lambda\mu.$$

(5.) Hence, *any multiple* (or *complex*) *product* of the symbols *ijk*, in any manner *repeated*, but taken in one given *order*, may be interpreted, with one definite *result*, by *any mode of association*, or of *reduction to partial factors*, which can be performed *without commutation*, or *change of place* of the given factors. For example, the symbol *ijkkji* may be interpreted in either of the *two* following (among other) ways:

$$ij\,.\,kk\,.\,ji = ij\,.-ji = i\,.-j^2\,.\,i = ii = -1; \qquad ijk\,.\,kji = -1\,.\,1 = -1.$$

184. The formula (A) of 183 *includes* obviously the three equations (I.) of 182. To show that it includes *also* the six other equations, (II.), (III.), of the last cited Article, we may observe that it gives, with the help of the

associative principle of multiplication (which may be suggested to the memory by the absence of the *point* in the symbol ijk),

$$ij = -ij\,.\,kk = -ijk\,.\,k = +k\,; \qquad jk = -i\,.\,ijk = +i\,;$$
$$ji = j\,.\,jk = j^2k = -k\,; \qquad ik = i\,.\,ij = i^2j = -j\,;$$
$$kj = ij\,.\,j = ij^2 = -i\,; \qquad ki = -k^2j = -ji^2 = +j.$$

And then it is easy to prove, *without any reference to geometry*, if the foregoing *laws of the symbols* be admitted, that we have also,

$$jki = kij = -1, \qquad kji = jik = ikj = +1,$$

as otherwise and *geometrically* shown in recent sub-articles. It may be added that the mere *inspection* of the formula (A) is sufficient to show that the *three** *square roots of negative unity*, denoted in it by i, j, k, *cannot be subject to all the ordinary rules of algebra*: because that formula gives, at sight,

$$i^2j^2k^2 = (-1)^3 = -1 = -(ijk)^2\,;$$

the *non-commutative character* (183), of the *multiplication of such roots* among themselves, being thus put in evidence.

[Conversely if three symbols i, j, and k satisfy the equations

$$jk + kj = 0, \qquad ki + ik = 0, \qquad ij + ji = 0\,;$$

and if the associative property hold good,

$$i^2\,.\,j = i\,.\,ij = -i\,.\,ji = -ij\,.\,i = j\,.\,i^2.$$

i^2 is therefore commutative in multiplication with i, j, and k and with products formed from them, and cannot be distinguished from a scalar. Assuming therefore that the squares of the symbols are scalars, and that the symbols have been multiplied by suitable numerical coefficients so that their squares are equal,

$$i^2 = j^2 = k^2 = P.$$

Again $\qquad i\,.\,jk = -i\,.\,kj = kij = -kji = jki = -jik = Q$ suppose,

and $\qquad iQ = i\,.\,ijk = P\,.\,jk = jki\,.\,i = Qi.$

The product Q is likewise commutative with i, j, and k, and is indistinguishable from a scalar. Also $Q^2 = -ijk\,.\,kji = -P^3$, so if $P = -1$, $Q = \pm 1$.

* It is evident that $-i$, $-j$, $-k$ are *also*, on the same principles, values of the symbol $\sqrt{-1}$; because they also are *right versors* (153); or because $(-q)^2 = q^2$. More generally (comp. a Note to page 133), if x, y, z be *any three scalars* which satisfy the condition $x^2 + y^2 + z^2 = 1$, it will be proved, at a later stage, that

$$(ix + jy + kz)^2 = -1.$$

Mr. Oliver Heaviside takes $P = +1$ *but* $i = jk, j = ki$ and $k = ij$, and consequently $i^2 . j = j$ but $i . ij = ik = -j$, and in his system the associative property does not hold, or the product of three symbols has no definite meaning (see Art. 25 of a paper "On the Forces in the Electro-magnetic Field," Trans. Roy. Soc. A., 1892).

Grassmann supposes $P = 0$, and his progressive multiplication is associative, but his regressive multiplication is not. Q is taken as a scalar differing from zero. Hamilton (p. 61 of the preface to the "Lectures on Quaternions") refers to the octaves of Messrs. J. T. Graves and Arthur Cayley as not obeying the associative principle. See Prof. Cayley's paper "On the 8-square Imaginaries," Am. Jour. of Math., 1881. When the associative principle does not hold, a distinct operation of grouping must be combined with multiplication to render a product definite.]

SECTION 11.

On the Tensor of a Vector, or of a Quaternion; and on the Product or Quotient of any two Quaternions.

185. Having now sufficiently availed ourselves, in the two last sections, of the conceptions (alluded to, so early as in the First Article of these Elements) of a *vector-arc* (162), and of a *vector-angle* (174) in *illustration** of the laws of *multiplication* and *division* of *versors* of quaternions; we propose to *return* to that use of the word, VECTOR, with which alone the First Book, and the first eight sections of this First Chapter of the Second Book, have been concerned: and shall therefore henceforth mean again, *exclusively*, by that word "vector," a *Directed Right Line* (as in 1). And because we have already considered and expressed the *Direction* of any such line, by introducing the conception and notation (155) of the *Unit-Vector*, $\mathrm{U}\alpha$, which has the *same direction* with the line α, and which we have proposed (156) to call *the Versor of that Vector*, α; we now propose to consider and express the *Length* of the same line α, by introducing the *new name* TENSOR, and the *new symbol*,† $\mathrm{T}\alpha$;

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* One of the chief uses of *such* vectors, in connexion with those laws, has been to illustrate the *non-commutative property* (168) of *multiplication of versors*, by exhibiting a corresponding property of what has been called, by analogy to the earlier operation of the same kind on *linear* vectors (5), the *addition* of *arcs* and *angles* on a *sphere*. Compare 180, (3.), (4.).

† Compare the Note to Art. 155.

which latter symbol we shall *read*, as *the Tensor of the Vector* α : and shall *define* it to be, or to denote, *the Number* (comp. again 155) *which represents the Length of that line* α, by expressing the *Ratio* which that length bears to some assumed standard, or *Unit* (128).

186. To connect more closely these two conceptions, of the *versor* and the *tensor* of a *vector*, we may remember that when we employed (in 155) the letter a as a temporary symbol for the number which thus expresses the length of the line α, we had the equation, $\mathrm{U}\alpha = \alpha : a$, as one form of the *definition* of the *unit-vector* denoted by $\mathrm{U}\alpha$. We might therefore have written also these two other forms of equation (comp. 15, 16),

$$\alpha = a \, . \, \mathrm{U}\alpha, \qquad a = \alpha : \mathrm{U}\alpha,$$

to express the dependence of the *vector*, α, and of the *scalar*, a, on each other, and on what has been called (156) the *versor*, $\mathrm{U}\alpha$. For example, with the construction of fig. 42, *bis* (comp. 161, (2.)), we may write the three equations,

$$a = \mathrm{OA} : \mathrm{OA}', \qquad b = \mathrm{OB} : \mathrm{OB}', \qquad c = \mathrm{OC} : \mathrm{OC}',$$

if a, b, c be thus the *three positive scalars*, which denote the *lengths* of the three lines, OA, OB, OC; and these three scalars may then be considered as *factors*, or as *coefficients* (12), by which the *three unit-vectors* $\mathrm{U}\alpha$, $\mathrm{U}\beta$, $\mathrm{U}\gamma$, or OA′, OB′, OC′ (in the cited figure), are to be respectively *multiplied* (15), in order to change them into the three other vectors α, β, γ, or OA, OB, OC, by *altering their lengths*, without any change in their *directions*. But such an exclusive *Operation*, on the *Length* (or on the *extension*) of a line, may be said to be an *Act of Tension*;* as an operation *on direction alone* may be called (comp. 151) an *act of version*. We have then thus a *motive* for the introduction of the *name*, *Tensor*, as applied to the *positive number* which (as above) represents the *length* of a line. And when the *notation* $\mathrm{T}\alpha$ (instead of a) is employed for such a *tensor*, we see that we may write generally, *for any vector* α, the equations (compare again 15, 16):

$$\mathrm{U}\alpha = \alpha : \mathrm{T}\alpha; \qquad \mathrm{T}\alpha = \alpha : \mathrm{U}\alpha; \qquad \alpha = \mathrm{T}\alpha \, . \, \mathrm{U}\alpha = \mathrm{U}\alpha \, . \, \mathrm{T}\alpha.$$

For example, if α be an unit-vector, so that $\mathrm{U}\alpha = \alpha$ (160), then $\mathrm{T}\alpha = 1$; and therefore, generally, *whatever vector* may be denoted by α, we have always,

$$\mathrm{TU}\alpha = 1.$$

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* Compare the first Note in page 137.

For the same reason, *whatever quaternion* may be denoted by q, we have always (comp. again 160) the equation,

$$\mathrm{T}\,(\mathrm{Ax}\,.\,q) = 1.$$

(1.) Hence the equation

$$\mathrm{T}\rho = 1,$$

where $\rho = \mathrm{OP}$, expresses that the *locus* of the variable point P is the surface of the *unit sphere* (128).

(2.) The equation $\mathrm{T}\rho = \mathrm{T}\alpha$ expresses that the locus of P is the spheric surface with O for centre, which passes through the point A.

(3.) On the other hand, for the sphere through O, which has its *centre* at A, we have the equation,

$$\mathrm{T}\,(\rho - \alpha) = \mathrm{T}\alpha\,;$$

which expresses that the lengths of the two lines, AP, AO, are equal.

(4.) More generally, the equation,

$$\mathrm{T}\,(\rho - \alpha) = \mathrm{T}\,(\beta - \alpha),$$

expresses that the locus of P is the spheric surface through B, which has its centre at A.

(5.) The equation of the Apollonian* Locus, 145, (8.), (9.), may be written under either of the two following forms:

$$\mathrm{T}\,(\rho - a^2\alpha) = a\mathrm{T}\,(\rho - \alpha)\,; \qquad \mathrm{T}\rho = a\mathrm{T}\alpha\,;$$

from each of which we shall find ourselves able to pass to the other, at a later stage, by general *Rules of Transformation*, without appealing to *geometry* (comp. 145, (10.) [and 200 (3.), (4.)]).

(6.) The equation, $\quad \mathrm{T}\,(\rho + \alpha) = \mathrm{T}\,(\rho - \alpha),$

expresses that the locus of P is the plane through O, perpendicular to the line OA; because it expresses that if $\mathrm{OA}' = -\,\mathrm{OA}$, then the point P is equally distant from the two points A and A′. It represents therefore the *same locus* as the equation,

$$\angle\frac{\rho}{\alpha} = \frac{\pi}{2}, \text{ of 132, (1.)};$$

or as the equation,

$$\frac{\rho}{\alpha} + \mathrm{K}\frac{\rho}{\alpha} = 0, \text{ of 144, (1.)};$$

or as

$$\left(\mathrm{U}\frac{\rho}{\alpha}\right)^2 = -\,1, \text{ of 161, (7.)};$$

* Compare the first Note to page 130.

or as the simple geometrical formula, $\rho \perp \alpha$ (129). And in fact it will be found possible, by *General Rules* of this Calculus, to *transform any one* of these *five* formulæ into *any other* of them; or into this *sixth form*,

$$\mathrm{S}\frac{\rho}{\alpha} = 0,$$

which expresses that the *scalar part** of the *quaternion* $\frac{\rho}{\alpha}$ is *zero*, and therefore that this quaternion is a *right quotient* (132).

(7.) In like manner, the equation

$$\mathrm{T}(\rho - \beta) = \mathrm{T}(\rho - \alpha)$$

expresses that the locus of P is the plane which perpendicularly bisects the line AB; because it expresses that P is equally distant from the two points A and B.

(8.) The *tensor*, $\mathrm{T}\alpha$, being generally a *positive scalar*, but *vanishing* (as a *limit*) *with* α, we have,

$$\mathrm{T}x\alpha = \pm x\mathrm{T}\alpha, \quad \text{according as} \quad x > \text{ or } < 0;$$

thus, in particular,

$$\mathrm{T}(-\alpha) = \mathrm{T}\alpha; \quad \text{and} \quad \mathrm{T}0\alpha = \mathrm{T}0 = 0.$$

(9.) That

$$\mathrm{T}(\beta + \alpha) = \mathrm{T}\beta + \mathrm{T}\alpha, \quad \text{if} \quad \mathrm{U}\beta = \mathrm{U}\alpha,$$

but not otherwise (α and β being any two actual vectors), will be seen, at a later stage, to be a symbolical consequence from the *rules* of the present *Calculus*; but in the mean time it may be *geometrically* proved, by conceiving that while α = OA, as usual, we make $\beta + \alpha$ = OC, and therefore β = OC − OA = AC (4); for thus we shall see that while, *in general*, the three points O, A, C are corners of a *triangle*, and therefore the *length* of the *side* OC is *less* than the *sum* of the lengths of the two other sides OA and AC, the former length becomes, on the contrary, *equal* to the latter sum, in the particular *case* when the triangle vanishes, by the point A falling *on the finite line* OC; in which case, OA and AC, or α and β, have one *common direction*, as the equation $\mathrm{U}\alpha = \mathrm{U}\beta$ implies.

(10.) If α and β be any actual vectors, and if their *versors be unequal* ($\mathrm{U}\alpha$ *not* = $\mathrm{U}\beta$), then

$$\mathrm{T}(\beta + \alpha) < \mathrm{T}\beta + \mathrm{T}\alpha;$$

an inequality which results at once from the consideration of the recent

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* Compare the Note to page 127; and the following Section of the present Chapter.

triangle OAC; but which (as it will be found) may also be *symbolically* proved, by *rules* of the calculus of quaternions. [See 210 (15.)]

(11.) If $U\beta = -U\alpha$, then $T(\beta + \alpha) = \pm(T\beta - T\alpha)$, according as $T\beta >$ or $< T\alpha$; but

$$T(\beta + \alpha) > \pm (T\beta - T\alpha), \quad \text{if} \quad U\beta \; \textit{not} = -U\alpha.$$

187. The *quotient*, $U\beta : U\alpha$, *of the versors* of the two vectors, α and β, has been called (156) the *Versor of the Quotient*, or quaternion, $q = \beta : \alpha$; and has been denoted, as such, by the symbol, Uq. On the same plan, we propose now to call the quotient, $T\beta : T\alpha$, of the *tensors* of the same two vectors, the *Tensor* of the Quaternion q*, or $\beta : \alpha$, and to denote it by the corresponding *symbol*, Tq. And then, as we have called the letter U (in 156) the characteristic of the operation of *taking the versor*, so we may now speak of T as the *Characteristic of the* (corresponding) *Operation of taking the Tensor*, whether of a *Vector*, α, or of a *Quaternion*, q. We shall thus have, generally,

$$T(\beta : \alpha) = T\beta : T\alpha, \text{ as we had } U(\beta : \alpha) = U\beta : U\alpha \text{ (156)};$$

and may say that as the *versor* Uq depended solely on, but conversely was sufficient to determine, the *relative direction* (157), so the *tensor* Tq depends on and determines the *relative length*† (109), of the two vectors, α and β, of which the *quaternion* q is the *quotient* (112).

(1.) Hence the equation $T\frac{\rho}{\alpha} = 1$, like $T\rho = T\alpha$, to which it is equivalent, expresses that the locus of P is the sphere with O for centre, which passes through the point A.

(2.) The equation (comp. 186, (6.)),

$$T\frac{\rho + \alpha}{\rho - \alpha} = 1,$$

expresses that the locus of P is the plane through O, perpendicular to the line OA.

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* Compare the Note to Art. 109, in page 111; and the first Note in page 137.

† It has been shown, in Art. 112, and in the *Additional Illustrations* of the third section of the present Chapter (113–116), that *Relative Length*, as well as *relative direction*, enters as an *essential element* into the very *Conception* of a *Quaternion*. Accordingly, in Art. 117, an *agreement* of relative *lengths* (as well as an agreement of relative directions) was made one of the *conditions of equality*, between any *two* quaternions, considered as quotients of vectors: so that we may now say, that *the tensors* (as well as the *versors*) *of equal quaternions are equal*. Compare the first Note to page 138, as regards what was there called the *quantitative element*, of absolute or relative *length*, which was *eliminated* from α, or from q, by means of the *characteristic* U; whereas, the *new* characteristic, T, of the present section, serves on the contrary to *retain that element alone*, and to eliminate what may be called by contrast the *qualitative* element, of absolute or relative *direction*.

(3.) Other examples of the same sort may easily be derived from the sub-articles to 186, by introducing the notation (187) for the *tensor of a quotient*, or quaternion, as additional to that for the *tensor of a vector* (185).

(4.) $\mathrm{T}(\beta : \alpha) >, =,$ or < 1, according as $\mathrm{T}\beta >, =,$ or $< \mathrm{T}\alpha$.

(5.) The *tensor of a right quotient* (132) is always equal to the tensor of its *index* (133).

(6.) The *tensor of a radial* (146) is always *positive unity*; thus we have, generally, by 156,

$$\mathrm{TU}q = 1;$$

and in particular, by 181,

$$\mathrm{T}i = \mathrm{T}j = \mathrm{T}k = 1.$$

(7.) $$\mathrm{T}xq = \pm x\mathrm{T}q, \text{ according as } x > \text{ or } < 0;$$

thus, in particular, $\mathrm{T}(-q) = \mathrm{T}q$, or the tensors of *opposite* quaternions are *equal*.

(8.) $$\mathrm{T}x = \pm x, \text{ according as } x > \text{ or } < 0;$$

thus, the tensor of a *scalar* is that scalar *taken positively*.

(9.) Hence, $$\mathrm{TT}\alpha = \mathrm{T}\alpha, \qquad \mathrm{TT}q = \mathrm{T}q;$$

so that, by abstracting from the *subject* of the operation T (comp. 145, 160), we may establish the symbolical equation,

$$\mathrm{T}^2 = \mathrm{TT} = \mathrm{T}.$$

(10.) Because the tensor of a quaternion is generally a positive scalar, such a tensor is *its own conjugate* (139); its *angle* is *zero* (131); and its *versor* (159) is positive *unity*: or in symbols,

$$\mathrm{KT}q = \mathrm{T}q; \qquad \angle \mathrm{T}q = 0; \qquad \mathrm{UT}q = 1.$$

(11.) $$\mathrm{T}(1 : q) = \mathrm{T}(\alpha : \beta) = \mathrm{T}\alpha : \mathrm{T}\beta = 1 : \mathrm{T}q;$$

or in words, the *tensor of the reciprocal* of a quaternion is equal to the *reciprocal of the tensor*.

(12.) Again, since the two lines, OB and OB′, in fig. 36 [p. 115], are *equally long*, the definition (137) of a conjugate gives

$$\mathrm{TK}q = \mathrm{T}q;$$

or in words, the tensors of *conjugate* quaternions are *equal*.

(13.) It is scarcely necessary to remark, that any two quaternions which have *equal tensors*, and *equal versors*, are themselves *equal*: or in symbols, that

$$q' = q, \quad \text{if} \quad \mathrm{T}q' = \mathrm{T}q, \quad \text{and} \quad \mathrm{U}q' = \mathrm{U}q.$$

188. Since we have, generally,

$$\frac{\beta}{\alpha} = \frac{\mathrm{T}\beta \,.\, \mathrm{U}\beta}{\mathrm{T}\alpha \,.\, \mathrm{U}\alpha} = \frac{\mathrm{T}\beta}{\mathrm{T}\alpha} \cdot \frac{\mathrm{U}\beta}{\mathrm{U}\alpha} = \frac{\mathrm{U}\beta}{\mathrm{U}\alpha} \cdot \frac{\mathrm{T}\beta}{\mathrm{T}\alpha} \text{ (comp. 126, 186)},$$

we may establish the two following general formulæ of *decomposition of a quaternion into two factors*, of the *tensor* and *versor* kinds:

$$\text{I.} \ldots q = \mathrm{T}q \,.\, \mathrm{U}q; \qquad \text{II.} \ldots q = \mathrm{U}q \,.\, \mathrm{T}q;$$

which are exactly analogous to the formulæ (186) for the corresponding decomposition of a *vector*, into *factors* of the same two kinds: namely,

$$\text{I}' \ldots \alpha = \mathrm{T}\alpha \,.\, \mathrm{U}\alpha; \qquad \text{II}' \ldots \alpha = \mathrm{U}\alpha \,.\, \mathrm{T}\alpha.$$

To illustrate this last decomposition of a quaternion, q, or OB : OA, into factors, we may conceive that AA′ and BB′ are two concentric and circular, but oppositely directed arcs, which terminate respectively on the two lines OB and OA, or rather on the longer of those two lines itself, and on the shorter of them prolonged, as in the annexed figure 48; so that OA′ has the *length* of OA, but the *direction* of OB, while OB′, on the contrary, has the length of OB, but the direction of OA; and that therefore we may write,

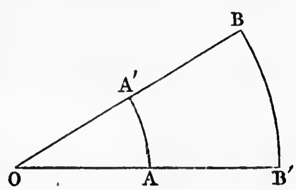

Fig. 48.

by what has been defined respecting *versors* and *tensors* of *vectors* (155, 156, 185, 186),

$$\text{OA}' = \mathrm{T}\alpha \,.\, \mathrm{U}\beta; \qquad \text{OB}' = \mathrm{T}\beta \,.\, \mathrm{U}\alpha.$$

Then, by the definitions in 156, 187, of the versor and tensor of a *quaternion*,

$$\mathrm{U}q = \mathrm{U}\,(\text{OB} : \text{OA}) = \text{OA}' : \text{OA} = \text{OB} : \text{OB}';$$

$$\mathrm{T}q = \mathrm{T}\,(\text{OB} : \text{OA}) = \text{OB}' : \text{OA} = \text{OB} : \text{OA}';$$

whence, by the general formula of multiplication of quotients (107),

$$\text{I.} \,.\, q = \text{OB} : \text{OA} = (\text{OB} : \text{OA}') \,.\, (\text{OA}' : \text{OA}) = \mathrm{T}q \,.\, \mathrm{U}q;$$

and

$$\text{II.} \,.\, q = \text{OB} : \text{OA} = (\text{OB} : \text{OB}') \,.\, (\text{OB}' : \text{OA}) = \mathrm{U}q \,.\, \mathrm{T}q,$$

as above.

189. In words, if we wish to pass from the vector α to the vector β, or from the line OA to the line OB, we are at liberty either, Ist, to *begin by turning*, from OA to OA′, and then to *end by stretching*, from OA′ to OB, as fig. 48 may serve to illustrate; or, IInd, to *begin* by stretching, from OA to OB′, and *end*

by turning, from OB′ to OB. The *act of multiplication* of a *line* α by a *quaternion* q, considered as a *factor* (103), which affects *both* length and direction (109), may thus be *decomposed* into *two* distinct and *partial acts*, of the kinds which we have called *Version* and *Tension*; and these two acts may be performed, at pleasure, in *either* of two *orders* of succession. And although, if we attended *merely to lengths*, we might be led to say that the *tensor* of a quaternion was a *signless number*,* expressive of a geometrical *ratio* of magnitudes, yet when the recent *construction* (fig. 48) is adopted, we see, by either of the two resulting expressions (188) for Tq, that there is a *propriety* in treating this tensor as a *positive scalar*, as we have lately done, and propose systematically to do.

190. Since $\mathrm{TK}q = \mathrm{T}q$, by 187, (12.), and $\mathrm{UK}q = 1 : \mathrm{U}q$, by 158, we may write, generally, for any quaternion and its conjugate, the two connected expressions:

$$\text{I.} \ldots q = \mathrm{T}q \,.\, \mathrm{U}q\,; \qquad \text{II.} \ldots \mathrm{K}q = \mathrm{T}q : \mathrm{U}q\,;$$

whence, by multiplication and division,

$$\text{III.} \ldots q \,.\, \mathrm{K}q = (\mathrm{T}q)^2\,; \qquad \text{IV.} \ldots q : \mathrm{K}q = (\mathrm{U}q)^2.$$

This last formula had occurred before; and we saw (161) that in it the *parentheses* might be omitted, because $(\mathrm{U}q)^2 = \mathrm{U}(q^2)$. In like manner (comp. 161, (2.)), we have also

$$(\mathrm{T}q)^2 = \mathrm{T}(q^2) = \mathrm{T}q^2,$$

parentheses being again omitted; or in words, the *tensor of the square* of a quaternion is always equal to the *square of the tensor*: as appears (among other ways) from inspection of fig. 42, *bis* [p. 141], in which the *lengths* of OA, OB, OC form a *geometrical progression*; whence

$$\mathrm{T}\,.\left(\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2 = \mathrm{T}\frac{\mathrm{OC}}{\mathrm{OA}} = \frac{\mathrm{T}\,.\,\mathrm{OC}}{\mathrm{T}\,.\,\mathrm{OA}} = \left(\frac{\mathrm{T}\,.\,\mathrm{OB}}{\mathrm{T}\,.\,\mathrm{OA}}\right)^2 = \left(\mathrm{T}\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2.$$

At the same time, we see again that the *product* $q\mathrm{K}q$ *of two conjugate quaternions*, which has been called (145, (11.)) their common *Norm*, and denoted by the symbol Nq, represents geometrically the *square of the quotient of the lengths* of the two lines, of which (when considered as *vectors*) the quaternion q is itself the quotient (112). We may therefore write generally,†

$$\text{V.} \ldots q\mathrm{K}q = \mathrm{T}q^2 = \mathrm{N}q\,; \qquad \text{VI.} \ldots \mathrm{T}q = \sqrt{}\,\mathrm{N}q = \sqrt{}\,(q\mathrm{K}q).$$

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* Compare the Note, in page 111, to Art. 109.

† Compare the second Note in page 130.

(1.) We have also, by II., the following other general transformations for the tensor of a quaternion:

$$\text{VII.} \ldots \mathrm{T}q = \mathrm{K}q \,.\, \mathrm{U}q; \qquad \text{VIII.} \ldots \mathrm{T}q = \mathrm{U}q \,.\, \mathrm{K}q;$$

of which the geometrical significations might easily be exhibited by a diagram, but of which the validity is sufficiently proved by what precedes.

(2.) Also (comp. 158),

$$\frac{1}{\mathrm{U}q} = \frac{\mathrm{K}q}{\mathrm{T}q} = \mathrm{K}\,\frac{q}{\mathrm{T}q} = \mathrm{KU}q; \qquad \mathrm{K}\frac{1}{\mathrm{U}q} = \frac{q}{\mathrm{T}q} = \mathrm{U}q.$$

(3.) The reciprocal of a quaternion, and the conjugate* of that reciprocal, may now be thus expressed:

$$\frac{1}{q} = \frac{\mathrm{K}q}{\mathrm{T}q^2} = \frac{\mathrm{K}q}{\mathrm{N}q} = \frac{\mathrm{KU}q}{\mathrm{T}q} = \frac{1}{\mathrm{U}q} \cdot \frac{1}{\mathrm{T}q} = \frac{1}{\mathrm{T}q} \cdot \frac{1}{\mathrm{U}q};$$

$$\mathrm{K}\frac{1}{q} = \frac{q}{\mathrm{N}q} = \frac{q}{\mathrm{T}q^2} = \frac{\mathrm{U}q}{\mathrm{T}q} = \frac{1}{\mathrm{K}q}.$$

(4.) We may also write, generally,

$$\text{IX.} \ldots \mathrm{K}q = \mathrm{T}q \,.\, \mathrm{KU}q = \mathrm{N}q : q.$$

191. In general, let *any two* quaternions, q and q', be considered as multiplicand and multiplier, and let them be reduced (by 120) to the forms $\beta : \alpha$ and $\gamma : \beta$; then the tensor and versor of that *third* quaternion, $\gamma : \alpha$, which is (by 107) their *product* $q'q$, may be thus expressed:

$$\text{I.} \ldots \mathrm{T}q'q = \mathrm{T}(\gamma : \alpha) = \mathrm{T}\gamma : \mathrm{T}\alpha = (\mathrm{T}\gamma : \mathrm{T}\beta) \,.\, (\mathrm{T}\beta : \mathrm{T}\alpha) = \mathrm{T}q' \,.\, \mathrm{T}q;$$

$$\text{II.} \ldots \mathrm{U}q'q = \mathrm{U}(\gamma : \alpha) = \mathrm{U}\gamma : \mathrm{U}\alpha = (\mathrm{U}\gamma : \mathrm{U}\beta) \,.\, (\mathrm{U}\beta : \mathrm{U}\alpha) = \mathrm{U}q' \,.\, \mathrm{U}q;$$

where $\mathrm{T}q'q$ and $\mathrm{U}q'q$ are written, for simplicity, instead of $\mathrm{T}(q'.\,q)$ and $\mathrm{U}(q'.\,q)$. Hence, in any such multiplication, the *tensor of the product* is the *product of the tensor*; and the *versor of the product* is the *product of the versors*; the *order* of the factors being generally *retained* for the *latter* (comp. 168, &c.), although it may be *varied* for the *former*, on account of the *scalar* character of a *tensor*. In like manner, for the *division* of any one quaternion q', by any other q, we have the analogous formulæ:

$$\text{III.} \ldots \mathrm{T}(q' : q) = \mathrm{T}q' : \mathrm{T}q; \qquad \text{IV.} \ldots \mathrm{U}(q' : q) = \mathrm{U}q' : \mathrm{U}q;$$

or in words, the *tensor of the quotient* of any two quaternions is equal to the

* Compare Art. 145, and the Note to page 128.

quotient of the tensors; and similarly, the *versor of the quotient* is equal to the *quotient of the versors.* And because multiplication and division of *tensors* are performed according to the rules of *algebra*, or rather of *arithmetic* (a tensor being always, by what precedes, a *positive number*), we see that the difficulty (whatever it may be) of the general *multiplication and division of quaternions* is thus reduced to that of the corresponding *operations on versors*: for which *latter* operations geometrical *constructions* have been assigned, in the ninth section of the present Chapter.

(1.) The two products, $q'q$ and qq', of any two quaternions taken as factors in two different orders, are *equal* or *unequal*, according as those two factors are *complanar* or *diplanar*; because such equality (169), or inequality (168), has been already proved to exist, for the case* when each tensor is unity: but we have always (comp. 178),

$$\mathrm{T}q'q = \mathrm{T}qq', \quad \text{and} \quad \angle\, q'q = \angle\, qq'.$$

(2.) If $\angle\, q = \angle\, q' = \frac{\pi}{2}$, then $qq' = \mathrm{K}q'q$ (170); so that the products of two *right* quotients, or right quaternions (132), taken in *opposite orders*, are always *conjugate* quaternions.

(3.) If $\angle\, q = \angle\, q' = \frac{\pi}{2}$, and $\mathrm{Ax}\,.\,q' \perp \mathrm{Ax}\,.\,q$, then $qq' = -\,q'q$,

$$\angle\, qq' = \angle\, q'q = \frac{\pi}{2}, \qquad \mathrm{Ax}\,.\,q'q \perp \mathrm{Ax}\,.\,q, \qquad \mathrm{Ax}\,.\,q'q \perp \mathrm{Ax}\,.\,q' \ (171);$$

so that *the product of two right quaternions, in two rectangular planes, is a third right quaternion, in a plane rectangular to both*; and is *changed to its own opposite*, when the *order* of the factors is *reversed*: as we had $ij = k = -ji$ (182).

(4.) In general, if q and q' be any two *diplanar* quaternions, the *rotation* round $\mathrm{Ax}\,.\,q'$, from $\mathrm{Ax}\,.\,q$ to $\mathrm{Ax}\,.\,q'q$, is *positive* (177).

(5.) Under the same condition, $q\,(q' : q)$ is a quaternion with the *same tensor*, and *same angle*, as q', but with a *different axis*; and this new axis, $\mathrm{Ax}\,.\,q(q' : q)$, may be derived (179, (1.)) from the old axis, $\mathrm{Ax}\,.\,q'$, by a *conical rotation* (in the positive direction) round $\mathrm{Ax}\,.\,q$, through an angle $= 2\,\angle\, q$.

(6.) The product or quotient of two *complanar* quaternions is, in general, a *third* quaternion complanar with both; but if they be both scalar, or both right, then this product or quotient *degenerates* (131) into a scalar.

* Compare the Notes to pages 148, 150.

(7.) Whether q and q' be complanar or diplanar, we have always as in algebra (comp. 106, 107, 136) the two identical equations:

$$\text{V.} \ldots (q':q) \,.\, q = q'; \qquad \text{VI.} \ldots (q'.q) : q = q'.$$

(8.) Also, by 190, V., and 191, I., we have this other general formula:

$$\text{VII.} \ldots \mathrm{N}q'q = \mathrm{N}q'.\mathrm{N}q;$$

or in words, the *norm of the product* is equal to the *product of the norms*.

192. Let $q = \beta : \alpha$, and $q' = \gamma : \beta$, as before; then

$$1 : q'q = 1 : (\gamma : \alpha) = \alpha : \gamma = (\alpha : \beta) \,.\, (\beta : \gamma) = (1 : q) \,.\, (1 : q');$$

so that the *reciprocal of the product* of any two quaternions is equal to the *product of the reciprocals*, taken in an *inverted order*: or briefly,

$$\text{I.} \ldots \mathrm{R}q'q = \mathrm{R}q \,.\, \mathrm{R}q',$$

if R be again used (as in 161, (3.)) as a (temporary) *characteristic of reciprocation.* And because we have then (by the same sub-article) the symbolical equation, $\mathrm{KU} = \mathrm{UR}$, or in words, the *conjugate of the versor* of any quaternion q is equal (158) to the *versor of the reciprocal* of that quaternion; while the *versor of a product* is equal (191) to the product of the versors: we see that

$$\mathrm{KU}q'q = \mathrm{UR}q'q = \mathrm{UR}q \,.\, \mathrm{UR}q' = \mathrm{KU}q \,.\, \mathrm{KU}q'.$$

But

$$\mathrm{K}q = \mathrm{T}q \,.\, \mathrm{KU}q, \text{ by } 190, \text{IX.}; \text{ and } \mathrm{T}q'q = \mathrm{T}q'.\mathrm{T}q = \mathrm{T}q \,.\, \mathrm{T}q',$$

by 191; we arrive then thus at the following other important and general formula:

$$\text{II.} \ldots \mathrm{K}q'q = \mathrm{K}q \,.\, \mathrm{K}q';$$

or in words, the *conjugate of the product* of any two quaternions is equal to the *product of the conjugates*, taken (still) in an *inverted order*.

(1.) These two results, I., II., may be illustrated, for *versors* ($\mathrm{T}q = \mathrm{T}q' = 1$), by the consideration of a *spherical triangle* ABC (comp. fig. 43 [p. 144]); in which the sides AB and BC (comp. 167) may *represent* q and q', the arc AC then representing $q'q$. For then the new multiplier $\mathrm{R}q = \mathrm{K}q$ (158) is represented (162) by BA, and the new multiplicand $\mathrm{R}q' = \mathrm{K}q'$ by CB; whence the new product, $\mathrm{R}q \,.\, \mathrm{R}q' = \mathrm{K}q \,.\, \mathrm{K}q'$, is represented by the *inverse arc* CA, and is therefore at once the *reciprocal* $\mathrm{R}q'q$, and the *conjugate* $\mathrm{K}q'q$, of the old product $q'q$.

(2.) If q and q' be *right* quaternions, then $\mathrm{K}q = -q$, $\mathrm{K}q' = -q'$ (by 144); and the recent formula II. becomes, $\mathrm{K}q'q = qq'$, as in 170.

(3.) In general, that formula II. (of 192) may be thus written:

$$\text{III.} \ldots \mathrm{K}\frac{\gamma}{\alpha} = \mathrm{K}\frac{\beta}{\alpha} \cdot \mathrm{K}\frac{\gamma}{\beta};$$

where α, β, γ may denote *any three vectors*.

(4.) Suppose then that, as in the annexed fig. 49, we have the two following relations of *inverse similitude* of triangles (118),

$$\Delta\, \text{AOB} \propto' \text{BOC}, \qquad \Delta\, \text{BOE} \propto' \text{DOB};$$

and therefore (by 137) the two equations,

$$\frac{\gamma}{\beta} = \mathrm{K}\frac{\beta}{\alpha}, \qquad \frac{\beta}{\delta} = \mathrm{K}\frac{\varepsilon}{\beta};$$

we shall have, by III.,

$$\frac{\gamma}{\delta} = \mathrm{K}\frac{\varepsilon}{\alpha}, \quad \text{or} \quad \Delta\, \text{DOC} \propto' \text{AOE};$$

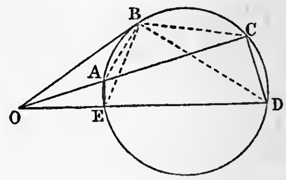

Fig. 49.

so that this *third* formula of inverse similitude is a *consequence* from the other *two*.

(5.) If then (comp. 145, (6.)) *any two circles*, whether in one plane or in space, *touch* one another at a point B: and if from any point O, on the *common tangent* BO, *two secants* OAC, OED be drawn, to these two circles; the *four points of section*, A, C, D, E, will be *on one common circle*: for such *concircularity* is an easy consequence (through *equal angles*, &c.), from the last *inverse similitude*.

(6.) The same conclusion (respecting concircularity, &c.) may be otherwise and *geometrically* drawn, from the equality of the *two rectangles*, AOC and DOE, each being equal to the *square* of the *tangent* OB; which may serve as an instructive *verification* of the recent formula III., and as an example of the *consistency* of the results, to which calculations with quaternions conduct.

(7.) It may be noticed that the construction would *in general* give *three circles*, although only *one* is drawn in the figure; but that if the two triangles ABC and DBE be situated *in different planes*, then these three circles, and of course the *five points* ABCDE, are situated *on one common sphere*.

193. An important application of the foregoing general theory of Multiplication and Division, is the case of *Right Quaternions* (132), taken in connexion with their *Index-Vectors*, or *Indices* (133).

Considering *division* first, and employing the general formula of 106, let β and γ be each $\perp \alpha$; and let β' and γ' be the respective indices of the two right quotients, $q = \beta : \alpha$, and $q' = \gamma : \alpha$. We shall thus have the two complanarities, $\beta' \parallel\!\mid \beta, \gamma$, and $\gamma' \parallel\!\mid \beta, \gamma$ (comp. 123), because the four lines β, γ,

β', γ' are all perpendicular to α; and within their common plane it is easy to see, from definitions already given, that these four lines form a *proportion of vectors*, in the same sense in which α, β, γ, δ did so, in the fourth section of the present Chapter: so that we may write the *equation of quotients*,

$$\gamma' : \beta' = \gamma : \beta.$$

In fact, we have (by 133, 185, 187) the following relations of *length*,

$$\mathrm{T}\beta' = \mathrm{T}\beta : \mathrm{T}\alpha, \quad \mathrm{T}\gamma' = \mathrm{T}\gamma : \mathrm{T}\alpha, \text{ and } \therefore \mathrm{T}(\gamma' : \beta') = \mathrm{T}(\gamma : \beta);$$

while the relation of *directions*, expressed by the formula,

$$\mathrm{U}(\gamma' : \beta') = \mathrm{U}(\gamma : \beta), \quad \text{or} \quad \mathrm{U}\gamma' : \mathrm{U}\beta' = \mathrm{U}\gamma : \mathrm{U}\beta,$$

is easily established by means of the equations,

$$\angle(\gamma' : \gamma) = \angle(\beta' : \beta) = \frac{\pi}{2}; \qquad \mathrm{Ax}.(\gamma' : \gamma) = \mathrm{Ax}.(\beta' : \beta) = \mathrm{U}\alpha.$$

We arrive, then, at this general Theorem (comp. again 133): that "*the Quotient of any two Right Quaternions is equal to the Quotient of their Indices.*"*

(1.) For example (comp. 150, 159, 181), the indices of the right versors i, j, k are the *axes* of those three versors, namely, the lines OI, OJ, OK; and we have the equal quotients,

$$j : i = \text{OI} : \text{OJ}' = k = \text{OJ} : \text{OI}, \&c.$$

(2.) In like manner, the indices of $-i$, $-j$, $-k$ are OI′, OJ′, OK′; and

$$i : -j = \text{OJ}' : \text{OI}' = k = \text{OI} : \text{OJ}', \&c.$$

(3.) In general the *quotient of any two right versors* is equal to the *quotient of their axes*; as the theory of *representative arcs*, and of their *poles*, may easily serve to illustrate.

194. As regards the *multiplication* of two right quaternions, in connexion with their indices, it may here suffice to observe that, by 106 and 107, the *product* $\gamma : \alpha = (\gamma : \beta) . (\beta : \alpha)$ is equal (comp. 136) to the *quotient*, $(\gamma : \beta) : (\alpha : \beta)$;

---

* We have thus a new point of *agreement*, or of *connexion*, between *right quaternions*, and their *index-vectors*, tending to justify the ultimate assumption (not yet made), of *equality* between the former and the latter [see 290]. In fact, we shall soon *prove* that the *index of the sum* (or difference), of any two right quotients (132), is equal to the *sum* (or difference) *of their indices* [see 206]; and shall find it convenient subsequently to *interpret* the *product* $\beta\alpha$ *of any two vectors*, as being the *quaternion-product* (194) *of the two right quaternions*, of which those *two lines* are the *indices* (133): after which, the above-mentioned assumption of *equality* will appear natural, and be found to be useful. (Compare the Notes to pages 121, 137). [In 198 the notation Iq is proposed as an abridgment of "Index of q."]

whence it is easy to infer that "*the product, $q'q$, of any two Right Quaternions, is equal to the Quotient of the Index of the Multiplier, q', divided by the Index of the Reciprocal of the Multiplicand, q.*"

It follows that the *plane*, whether of the product or of the quotient of two right quaternions, coincides with the *plane of their indices*; and therefore also with the plane of their *axes*; because we have, generally, by principles already established, the transformation,

$$\text{if } \angle q = \frac{\pi}{2}, \text{ then } \textit{Index of } q = \mathrm{T}q \,.\, \mathrm{Ax} \,.\, q.$$

SECTION 12.

On the Sum or Difference of any two Quaternions; and on the Scalar (or Scalar Part) of a Quaternion.

195. The *Addition* of any given quaternion q', considered as a geometrical quotient or *fraction* (101), to any other given quaternion q, considered also as a fraction, can always be accomplished by the first general formula of Art. 106,* when these two fractions have a *common denominator*; and if they be not already *given* as having such, they can always be *reduced* so as to have one, by the process of Art. 120. And because the *addition of any two lines* was early seen to be a *commutative operation* (7, 9), so that we have always $\gamma + \beta = \beta + \gamma$, it follows (by 106) that the addition of any two quaternions is *likewise* a commutative operation, or in symbols, that

$$\text{I.} \ldots q + q' = q' + q;$$

so that the Sum *of any two*† *Quaternions* has a *Value*, which is independent of their *Order*: and which (by what precedes) must be considered to be *given*, or at least *known*, or *definite*, when the two *summand* quaternions are given. It is easy also to see that the *conjugate* of any such *sum* is equal to the *sum of the conjugates*, or in symbols, that

$$\text{II.} \ldots \mathrm{K}(q' + q) = \mathrm{K}q' + \mathrm{K}q.$$

(1.) The important formula last written becomes geometrically evident, when it is presented under the following form. Let OBDC be any parallelogram, and let OA be any right line, drawn from one corner of it, but not

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* [This formula is a definition.]

† It will be found [in 207] that this result admits of being extended to the case of *three* (or *more*) quaternions; but, for the moment, we content ourselves with *two*. [As an example of non-commutative addition contrast Art. 180 (3.)]

generally in its plane. Let the three other corners, B, C, D, be *reflected* (in the sense of 145, (5.)) with respect to that line OA, into three new points, B′, C′, D′; or let the three lines OB, OC, OD be reflected (in the sense of 138) with respect to the same line OA; which thus bisects at right angles the three joining lines, BB′, CC′, DD′, as it does BB′ in fig. 36 [p. 115]. Then *each* of the *lines* OB, OC, OD, and therefore also the whole *plane figure* OBDC, may be considered to have simply *revolved* round the line OA as an *axis*, by a *conical rotation* through *two right angles*; and consequently the *new figure* OB′D′C′, like that *old* one OBDC, must be a *parallelogram*. Thus (comp. 106, 137), we have

$$\text{OD}' = \text{OC}' + \text{OB}', \qquad \delta' = \gamma' + \beta', \qquad \delta' : \alpha = (\gamma' : \alpha) + (\beta' : \alpha);$$

and the recent formula II. is justified.

(2.) Simple as this last reasoning is, and unnecessary as it appears to be to draw any new diagram to illustrate it, the reader's attention may be once more invited to the great *simplicity of expression*, with which many important *geometrical conceptions*, respecting *space of three dimensions*, are *stated* in the present Calculus: and are thereby kept *ready* for future application, and for easy combination with *other* results of the same kind. Compare the remarks already made in 132, (6.); 145, (10.); 161; 179, (3.); 192, (6.); and some of the shortly following sub-articles to 196, respecting properties of an *oblique cone* with circular base.

196. One of the most important *cases* of *addition*, is that of *two conjugate summands*, q and $\mathrm{K}q$; of which it has been seen (in 140) that the *sum* is always a *scalar*. We propose now to denote the *half* of this sum by the *symbol*, $\mathrm{S}q$; thus writing generally,

$$\text{I.} \ldots q + \mathrm{K}q = \mathrm{K}q + q = 2\mathrm{S}q;$$

or *defining* the new symbol $\mathrm{S}q$ by the formula,

$$\text{II.} \ldots \mathrm{S}q = \tfrac{1}{2}(q + \mathrm{K}q); \text{ or briefly, } \text{II}'. \ldots \mathrm{S} = \tfrac{1}{2}(1 + \mathrm{K}).$$

For reasons which will soon more fully appear, we shall also call this new quantity, $\mathrm{S}q$, the *scalar part*, or simply *the* SCALAR, *of the Quaternion*, q; and shall therefore call the letter S, thus used, the *Characteristic of the Operation of taking the Scalar* of a quaternion. (Comp. 132, (6.); 137; 156; 187.) It follows that not only *equal quaternions*, but also *conjugate quaternions*, have *equal scalars*; or in symbols,

$$\text{III.} \ldots \mathrm{S}q' = \mathrm{S}q, \quad \text{if} \quad q' = q; \quad \text{and} \quad \text{IV.} \ldots \mathrm{SK}q = \mathrm{S}q;$$

or briefly,

$$\text{IV}'. \ldots \mathrm{SK} = \mathrm{S}.$$

And because we have seen that $Kq = +q$, if q be a *scalar* (139), but that $Kq = -q$, if q be a *right quotient* (144), we find that the *scalar of a scalar* (considered as a *degenerate quaternion*, 131) is equal to that scalar *itself*, but that the *scalar of a right quaternion* is *zero*. We may therefore now write (comp. 160):

$$\text{V.}\,.\; Sx = x,\ \text{if } x \text{ be a scalar;} \qquad \text{VI.}\,.\,.\; SSq = Sq, \quad S^2 = SS = S;$$

and
$$\text{VII.}\,.\,.\; Sq = 0, \quad \text{if} \quad \angle q = \frac{\pi}{2}.$$

Again, because OA′ in fig. 36 [p. 115] is multiplied by x, when OB is multiplied thereby, we may write, generally,

$$\text{VIII.}\,.\,.\; Sxq = xSq, \ \text{if } x \text{ be any scalar;}$$

and therefore in particular (by 188),

$$\text{IX.}\,.\,.\; Sq = S\,(Tq\,.\,Uq) = Tq\,.\,SUq.$$

Also because $SKq = Sq$, by IV., while $KUq = U\frac{1}{q}$, by 158, we have the general equation,

$$\text{X.}\,.\,.\; SUq = SU\frac{1}{q}; \quad \text{or} \quad \text{X}'.\,.\,.\; SU\frac{\beta}{\alpha} = SU\frac{\alpha}{\beta};$$

whence, by IX.,

$$\text{XI.}\,.\,.\; Sq = Tq\,.\,SU\frac{1}{q}; \quad \text{or} \quad \text{XI}'.\,.\,.\; S\frac{\beta}{\alpha} = T\frac{\beta}{\alpha}\,.\,SU\frac{\alpha}{\beta};$$

and therefore also, by 190, (V.), since $Tq\,.\,T\frac{1}{q} = 1$,

$$\text{XII.}\,.\,.\; Sq = Tq^2\,.\,S\frac{1}{q} = Nq\,.\,S\frac{1}{q}; \quad \text{XII}'.\,.\,.\; S\frac{\beta}{\alpha} = N\frac{\beta}{\alpha}\,.\,S\frac{\alpha}{\beta}.$$

The results of 142, combined with the recent definition I. or II., enable us to extend the recent formula VII., by writing,

$$\text{XIII.}\,.\,.\; Sq >, =, \text{ or } < 0, \ \text{according as } \angle q <, =, \text{ or } > \frac{\pi}{2};$$

and conversely,

$$\text{XIV.}\,.\,.\; \angle q <, =, \text{ or } > \frac{\pi}{2}, \ \text{according as } Sq >, =, \text{ or } < 0.$$

In fact, if we compare that *definition* I. with the formula of 140, and with fig. 36, we see at once that because, in that figure,

$$S\,(\text{OB} : \text{OA}) = \text{OA}' : \text{OA},$$

we may write, generally,

$$\text{XV.}\,.\,.\; Sq = Tq\,.\cos\angle q; \quad \text{or} \quad \text{XVI.}\,.\,.\; SUq = \cos\angle q;$$

equations which will be found of great importance, as serving to *connect quaternions with Trigonometry*; and which show that

$$\text{XVII.} \ldots \angle q' = \angle q, \quad \text{if} \quad \mathrm{SU}q' = \mathrm{SU}q,$$

the angle $\angle q$ being still taken (as in 130), so as not to fall outside the limits 0 and π; whence also,

$$\text{XVIII.} \ldots \angle q' = \angle q, \quad \text{if} \quad \mathrm{S}q' = \mathrm{S}q, \quad \text{and} \quad \mathrm{T}q' = \mathrm{T}q,$$

the *angle of a quaternion* being thus *given*, when the *scalar* and the *tensor* of that quaternion are given, or known. Finally because, in the same figure 36 (comp. 15, 103), the *line*,

$$\text{OA}' = (\text{OA}' : \text{OA}) \,.\, \text{OA} = \text{OA} \,.\, \mathrm{S}\,(\text{OB} : \text{OA}),$$

may be said to be the *projection* of OB on OA, since A′ is the *foot of the perpendicular* let fall from the *point* B upon this latter line OA, we may establish this other general formula:

$$\text{XIX.} \ldots \alpha \mathrm{S}\frac{\beta}{\alpha} = \mathrm{S}\frac{\beta}{\alpha} \,.\, \alpha = \textit{projection of } \beta \textit{ on } \alpha;$$

a result which will be found to be of great utility, in investigations respecting *geometrical loci*, and which may be also written thus:

$$\text{XX.} \ldots \textit{Projection of } \beta \textit{ on } \alpha = \mathrm{U}\alpha \,.\, \mathrm{T}\beta \,.\, \mathrm{SU}\frac{\beta}{\alpha};$$

with other transformations deducible from principles stated above. It is scarcely necessary to remark that, on account of the *scalar* character of Sq, we have, generally, by 159, and 187, (8.), the expressions,

$$\text{XXI.} \ldots \mathrm{US}q = \pm 1; \qquad \text{XXII.} \ldots \mathrm{TS}q = \pm \mathrm{S}q;$$

while, for the same reason, we have always, by 139, the equation (comp. IV.),

$$\text{XXIII.} \ldots \mathrm{KS}q = \mathrm{S}q; \quad \text{or} \quad \text{XXIII}'. \ldots \mathrm{KS} = \mathrm{S};$$

and, by 131,

$$\text{XXIV.} \ldots \angle\, \mathrm{S}q = 0, \quad \text{or} \quad = \pi, \quad \text{unless } \angle\, q = \frac{\pi}{2};$$

in which last case Sq = 0, by VII., and therefore $\angle$ Sq is indeterminate:* USq becoming at the same time indeterminate, by 159, but TSq vanishing, by 186, 187.

* Compare the Note in page 120, to Art. 131.

(1.) The equation, $$S\frac{\rho}{\alpha} = 0,$$

is now seen to be equivalent to the formula, $\rho \perp \alpha$; and therefore to denote the *same plane locus* for P, as that which is represented by any one of the four other equations of 186, (6.); or by the equation,

$$T\frac{\rho + \alpha}{\rho - \alpha} = 1, \text{ of 187, (2.)}.$$

(2.) The equation,

$$S\frac{\rho - \beta}{\alpha} = 0, \quad \text{or} \quad S\frac{\rho}{\alpha} = S\frac{\beta}{\alpha},$$

expresses that BP $\perp$ OA; or that the *points* B and P have the *same projection* on OA; or that the *locus* of P is the *plane through* B, *perpendicular to the line* OA.

(3.) The equation,

$$SU\frac{\rho}{\alpha} = SU\frac{\beta}{\alpha},$$

expresses (comp. 132, (2.)) that P is on *one sheet* of a *cone of revolution*, with O for *vertex*, and OA for *axis*, and passing *through the point* B.

(4.) The *other sheet* of the *same cone* is represented by this other equation,

$$SU\frac{\rho}{\alpha} = -SU\frac{\beta}{\alpha};$$

and *both sheets* jointly by the equation,

$$\left(SU\frac{\rho}{\alpha}\right)^2 = \left(SU\frac{\beta}{\alpha}\right)^2.$$

(5.) The equation,

$$S\frac{\rho}{\alpha} = 1, \quad \text{or} \quad SU\frac{\rho}{\alpha} = T\frac{\alpha}{\rho},$$

expresses that the *locus* of P is the *plane through* A, *perpendicular to the line* OA; because it expresses (comp. XIX.) that the *projection* of OP on OA is the line OA *itself*; or that the *angle* OAP is *right*; or that $S\frac{\rho - \alpha}{\alpha} = 0$.

(6.) On the other hand the equation,

$$S\frac{\beta}{\rho} = 1, \quad \text{or} \quad SU\frac{\beta}{\rho} = T\frac{\rho}{\beta},$$

expresses that the projection of OB on OP is OP itself; or that the angle OPB is right; or that the locus of P is that *spheric surface* which has the line OB for a *diameter*.

(7.) Hence the *system* of the *two equations*,

$$\mathrm{S}\frac{\rho}{\alpha} = 1, \quad \mathrm{S}\frac{\beta}{\rho} = 1,$$

represents the *circle*, in which the *sphere* (6.), with OB for a *diameter*, is *cut* by the *plane* (5.), with OA for the *perpendicular* let fall on it from O.

(8.) And therefore this new equation,

$$\mathrm{S}\frac{\rho}{\alpha} \,.\, \mathrm{S}\frac{\beta}{\rho} = 1,$$

obtained by multiplying the two last, represents the *Cyclic** *Cone* (or *cone of the second order*, but *not generally* of *revolution*), which *rests* on this last *circle* (7.) as its *base*, and has the point O for its *vertex*. In fact, the equation (8.) is evidently satisfied, when the two equations (7.) are so; and therefore *every point* of the *circular circumference*, denoted by those two equations, must be *a point of the locus*, represented by the equation (8.). But the latter equation remains unchanged, at least essentially, when ρ is changed to $x\rho$, x being *any scalar*; the locus (8.) is, therefore, *some conical surface*, with its *vertex* at the *origin*, O; and consequently it can be none other than *that particular cone* (both ways prolonged), which *rests* (as above) on the given *circular base* (7.).

(9.) The system of the *two* equations,

$$\mathrm{S}\frac{\rho}{\alpha} \,.\, \mathrm{S}\frac{\beta}{\rho} = 1, \qquad \mathrm{S}\frac{\rho}{\gamma} = 1,$$

(in writing the first of which the *point* may be omitted), represents a *conic section*; namely *that* section, in which the *cone* (8.) is *cut* by the new *plane*, which has OC for the *perpendicular* let fall upon it, from the origin of vectors O.

(10.) Conversely, *every plane ellipse* (or other *conic section*) *in space*, of which the plane does not pass through the origin, may be represented by a *system* of two equations, of this last *form* (9.); because the *cone* which rests on any such *conic* as its *base*, and has its *vertex* at *any* given point O, is known to be a *cyclic* cone.

(11.) The *curve* (or rather the *pair of curves*), in which an *oblique* but *cyclic cone* (8.) is cut by a *concentric sphere* (that is to say, a cone resting on a circular

* Historically speaking, the *oblique cone with circular base* may deserve to be named the *Apollonian Cone*, from Apollonius of Perga, in whose great work on *Conics* (κωνικῶν), already referred to in a Note to page 130, the properties of *such* a cone appear to have been first treated systematically; although the *cone of revolution* had been studied by Euclid. But the designation "*cyclic cone*" is shorter; and it seems more natural, *in geometry*, to speak of the above-mentioned oblique cone *thus*, for the purpose of marking its connexion with the *circle*, than to call it, as is now usually done, a *cone of the second order*, or of the *second degree*: although these phrases also have their advantages.

base by a sphere which has its centre at the vertex of that cone), has come, in modern times, to be called a *Spherical Conic.* And any *such* conic may, on the foregoing plan, be represented by the system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\beta}{\rho} = 1, \qquad \mathrm{T}\rho = 1;$$

the length of the *radius* of the sphere being here, for simplicity, supposed to be the *unit* of length. But, by writing $\mathrm{T}\rho = a$, where a may denote any constant and positive scalar, we can at once remove this last restriction, if it be thought useful or convenient to do so.

(12.) The equation (8.) may be written, by XII. or XII'., under the form (comp. 191, VII.):

$$\mathrm{S}\frac{\alpha}{\rho}\,\mathrm{S}\frac{\rho}{\beta} = \mathrm{N}\frac{\alpha}{\beta} = \left(\mathrm{T}\frac{\alpha}{\beta}\right)^2;$$

or briefly,

$$\mathrm{S}\frac{\beta'}{\rho}\,\mathrm{S}\frac{\rho}{\alpha'} = 1,$$

$$\text{if } \alpha' = \beta\mathrm{T}\frac{\alpha}{\beta} = \mathrm{T}\alpha\,.\,\mathrm{U}\beta, \quad \text{and} \quad \beta' = \alpha\mathrm{T}\frac{\beta}{\alpha} = \mathrm{T}\beta\,.\,\mathrm{U}\alpha;$$

so that α' and β' are here the lines OA′ and OB′, of Art. 188, and fig. 48.

(13.) Hence the cone (8.) is cut, not only by the plane (5.) in the circle (7.), which is on the sphere (6.), but also by the (generally) *new plane*, $\mathrm{S}\frac{\rho}{\alpha'} = 1$, in the (generally) *new circle*, in which this new plane cuts the (generally) *new sphere*, $\mathrm{S}\frac{\beta'}{\rho} = 1$; or in the circle which is represented by the system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha'} = 1, \qquad \mathrm{S}\frac{\beta'}{\rho} = 1.$$

(14.) In the *particular case* when $\beta \parallel \alpha$ (15.), so that the quotient $\beta : \alpha$ is a *scalar*, which must be positive and greater than unity, in order that the plane (5.) may (*really*) *cut* the sphere (6.), and therefore that the circle (7.) and the cone (8.) may be *real*, we may write

$$\beta = a^2\alpha, \qquad a > 1, \qquad \mathrm{T}(\beta : \alpha) = a^2, \qquad \alpha' = \alpha, \qquad \beta' = \beta;$$

and the circle (13.) *coincides* with the circle (7.).

(15.) In the same *case*, the cone is one of *revolution*; every point P of its circular *base* (that is, of the *circumference* thereof) being at one *constant distance* *from the vertex* O, namely at a distance $= a\mathrm{T}\alpha$. For, in the case supposed, the equations (7.) give, by XII.,

$$\mathrm{N}\frac{\rho}{\alpha} = \mathrm{S}\frac{\rho}{\alpha} : \mathrm{S}\frac{\alpha}{\rho} = 1 : \mathrm{S}\frac{\alpha}{\rho} = a^2 : \mathrm{S}\frac{\beta}{\rho} = a^2; \quad \text{or} \quad \mathrm{T}\rho = a\mathrm{T}\alpha.$$

(Compare 145, (12.), and 186, (5.).)

(16.) Conversely, if the cone *be* one of revolution, the equations (7.) must conduct to a result of the form,

$$a^2 = \mathrm{N}\frac{\rho}{\alpha} = \mathrm{S}\frac{\rho}{\alpha} : \mathrm{S}\frac{\alpha}{\rho} = \mathrm{S}\frac{\beta}{\rho} : \mathrm{S}\frac{\alpha}{\rho}, \text{ or (comp. (2.))}, \mathrm{S}\frac{\beta - a^2\alpha}{\rho} = 0;$$

which can only be by the line $\beta - a^2\alpha$ vanishing, or by our having $\beta = a^2\alpha$, as in (14.); since otherwise we should have, by XIV., $\rho \perp \beta - a^2\alpha$, and *all the points of the base* would be situated *in one plane* passing *through the vertex* O, which (for any actual cone) would be absurd.

(17.) Supposing, then, that we have *not* $\beta \parallel \alpha$, and therefore *not* $\alpha' = \alpha$, $\beta' = \beta$, as in (14.), nor even $\alpha' \parallel \alpha$, $\beta' \parallel \beta$, we see that the cone (8.) is *not* a cone of *revolution* (or what is often called a *right cone*); but that it is, on the contrary, an *oblique* (or *scalene*) *cone*, although still a *cyclic* one. And we see that *such* a cone is cut in *two distinct series** *of circular sections*, by planes parallel to the two distinct (and mutually non-parallel) planes, (5.) and (13.); or to *two new planes*, drawn *through the vertex* O, which have been called† the *two Cyclic Planes* of the cone, namely, the two following:

$$\mathrm{S}\frac{\rho}{\alpha} = 0; \qquad \mathrm{S}\frac{\rho}{\beta} = 0;$$

while the *two lines* from the vertex, OA and OB, which are *perpendicular to these two planes* respectively, may be said to be the *two Cyclic Normals*.

(18.) Of these two *lines*, α and β, the *second* has been seen to be a *diameter* of the sphere (6.), which may be said to be *circumscribed* to the cone (8.), when that cone is considered as having the circle (7.) for its *base*; the *second cyclic plane* (17.) is therefore the *tangent plane at the vertex* of the cone, to that *first circumscribed sphere* (6.).

(19.) The sphere (13.) may in like manner be said to be circumscribed to

* These *two series* of *sub-contrary* (or *antiparallel*) but *circular sections* of a cyclic cone, appear to have been first discovered by Apollonius: see the Fifth Proposition of his First Book, in which he says, καλείσθω δὲ ἡ τοιαύτη τομὴ ὑπεναντία (page 22 of Halley's Edition).

† By M. Chasles.

the cone, if the latter be considered as resting on the new circle (13.), or as terminated by *that* circle as its *new base*; and the diameter of this *new sphere* is the line OB′, or β', which has by (12.) the *direction* of the line α, or of the *first cyclic normal* (17.); so that (comp. (18.)) the *first cyclic plane* is the *tangent plane* at the vertex, to the *second circumscribed sphere* (13.).

(20.) *Any other sphere* through the vertex, which *touches the first cyclic plane*, and which therefore has its *diameter from the vertex* = $b'\beta'$, where b' is some scalar co-efficient, is represented by the equation,

$$\mathrm{S}\frac{b'\beta'}{\rho} = 1, \quad \text{or} \quad \mathrm{S}\frac{\beta'}{\rho} = \frac{1}{b'};$$

it therefore *cuts the cone in a circle*, of which (by (12.)) the *equation of the plane* is

$$\mathrm{S}\frac{\rho}{\alpha'} = b', \quad \text{or} \quad \mathrm{S}\frac{\rho}{b'\alpha'} = 1,$$

so that the *perpendicular from the vertex* is $b'\alpha' \parallel \beta$ (comp. (5.)); and consequently this *plane of section* of sphere and cone is *parallel to the second cyclic plane* (17.).

(21.) In like manner *any sphere*, such as

$$\mathrm{S}\frac{b\beta}{\rho} = 1, \text{ where } b \text{ is any scalar,}$$

which *touches the second cyclic plane* at the vertex, *intersects the cone* (8.) in a *circle*, of which the plane has for equation,

$$\mathrm{S}\frac{\rho}{b\alpha} = 1,$$

and is therefore *parallel to the first cyclic plane.*

(22.) The equation of the cone (by IX., X., XVI.) may also be thus written:

$$\mathrm{SU}\frac{\rho}{\alpha}\,.\,\mathrm{SU}\frac{\beta}{\rho} = \mathrm{T}\frac{\alpha}{\beta}; \quad \text{or,} \quad \cos\angle\frac{\rho}{\alpha}\,.\,\cos\angle\frac{\rho}{\beta} = \mathrm{T}\frac{\alpha}{\beta};$$

it expresses, therefore, that *the product of the cosines of the inclinations, of any variable side* (ρ) *of an oblique cyclic cone, to two fixed lines* (α and β), namely to the *two cyclic normals* (17.), is *constant*; or that the *product of the sines of the inclinations*, of the same variable side (or *ray*, ρ) of the cone, *to two fixed planes*, namely to the *two cyclic planes*, is thus a constant quantity.

(23.) The *two great circles*, in which the *concentric sphere* $T\rho = 1$ is *cut by the two cyclic planes*, have been called the *two Cyclic Arcs** of the *Spherical Conic* (11.), in which that sphere is cut by the cone. It follows (by (22.)) that the *product of the sines of the (arcual) perpendiculars, let fall from any point* P *of a given spherical conic, on its two cyclic arcs, is constant.*

(24.) These properties of *cyclic cones*, and of *spherical conics*, are not put forward as *new*; but they are of importance enough, and have been here deduced with sufficient facility, to show that we are already in possession of a *Calculus*, with its own *Rules*† *of Transformation*, whereby *one enunciation* of a geometrical theorem, or problem, or construction, can be *translated* into several others, of which some may be clearer, or simpler, or more elegant than the one first proposed.

197. Let α, β, γ be any three co-initial vectors, OA, &c., and let $\text{OD} = \delta = \gamma + \beta$, so that OBDC is a parallelogram (6); then, if we write

$$\beta : \alpha = q, \quad \gamma : \alpha = q', \quad \text{and} \quad \delta : \alpha = q'' = q' + q \text{ (106)},$$

and suppose that B′, C′, D′ are the feet of perpendiculars let fall from the points B, C, D on the line OA, we shall have, by 196, XIX., the expressions,

$$(\text{OB}' =)\ \beta' = \alpha \mathrm{S}q, \quad \gamma' = \alpha \mathrm{S}q', \quad \delta' = \alpha \mathrm{S}q'' = \alpha \mathrm{S}\ (q' + q).$$

But also OB = CD, and therefore OB′ = C′D′, the *similar projections* of *equal lines* being *equal*; hence (comp. 11) the *sum of the projections* of the lines β, γ must be equal to the *projection of the sum*, or in symbols,

$$\text{OD}' = \text{OC}' + \text{OB}', \quad \delta' = \gamma' + \beta', \quad \delta' : \alpha = (\gamma' : \alpha) + (\beta' : \alpha).$$

Hence, generally, for *any two quaternions*, q and q', we have the formula:

$$\text{I.} \ldots \mathrm{S}\ (q' + q) = \mathrm{S}q' + \mathrm{S}q\,;$$

or in words, the *scalar of the sum* is equal to the *sum of the scalars*. It is easy to extend this result to the case of any *three* (or more) quaternions, with their respective scalars; thus, if q'' be a third *arbitrary* quaternion, we may write

$$\mathrm{S}\{q'' + (q' + q)\} = \mathrm{S}q'' + \mathrm{S}\ (q' + q) = \mathrm{S}q'' + (\mathrm{S}q' + \mathrm{S}q)\,;$$

where, on account of the *scalar* character of the summands, the last parentheses may be omitted. We may therefore write, generally,

$$\text{II.} \ldots \mathrm{S}\Sigma q = \Sigma \mathrm{S}q, \quad \text{or briefly,} \quad \mathrm{S}\Sigma = \Sigma\mathrm{S}\,;$$

where Σ is used as a *sign of Summation*: and may say that *the Operation of*

* By M. Chasles. † Comp. 145, (10.), &c.

taking the Scalar of a Quaternion is a Distributive Operation (comp. 13). As to the general *Subtraction* of any one quaternion from any other, there is no difficulty in reducing it, by the method of Art. 120, to the second general formula of 106; nor in proving that the *Scalar of the Difference** is always equal to the *Difference of the Scalars*. In symbols,

$$\text{III.} \ldots \mathrm{S}\,(q' - q) = \mathrm{S}q' - \mathrm{S}q;$$

or briefly,

$$\text{IV.} \ldots \mathrm{S}\Delta q = \Delta \mathrm{S}q, \qquad \mathrm{S}\Delta = \Delta\mathrm{S};$$

when Δ is used as the characteristic of the operation of *taking a difference*, by *subtracting* one quaternion, or one scalar, from another.

(1.) It has not *yet* been proved (comp. 195) that the *Addition* of *any number* of Quaternions, $q, q', q'', \ldots$ is an *associative* and a *commutative operation* (comp. 9). But we see, already, that the *scalar of the sum* of any such set of quaternions has a *value*, which is *independent of their order*, and of the mode of *grouping them*.

(2.) If the summands be all *right* quaternions (132), the scalar of *each* separately vanishes, by 196, VII.; wherefore the scalar of their *sum* vanishes also, and that sum is consequently *itself*, by 196, XIV., a *right* quaternion: a result which it is easy to verify. In fact, if $\beta \perp \alpha$ and $\gamma \perp \alpha$, then $\gamma + \beta \perp \alpha$, because α is then perpendicular to the plane of β and γ; hence, by 106, the *sum of any two right quaternions is a right quaternion, and therefore also the sum of any number* of such quaternions.

(3.) Whatever two quaternions q and q' may be, we have always, as in algebra, the two *identities* (comp. 191, (7.)):

$$\text{V.} \ldots (q' - q) + q = q'; \qquad \text{VI.} \ldots (q' + q) - q = q'.$$

198. Without yet entering on the general theory of *scalars of products* or *quotients* of quaternions, we may observe here that because, by 196, XV., *the scalar of a quaternion depends only on the tensor and the angle*, and is *independent of the axis*, we are at liberty to write generally (comp. 173, 178, and 191, (1.), (5.)),

$$\text{I.} \ldots \mathrm{S}qq' = \mathrm{S}q'q; \qquad \text{II.} \ldots \mathrm{S}\,.\,q\,(q' : q) = \mathrm{S}q';$$

the *two products*, qq' and $q'q$, having thus always *equal scalars*, although they have been seen to have *unequal axes*, for the general case of *diplanarity* (168, 191). It may also be noticed that, in virtue of what was shown in 193,

* Examples have already occurred in 196, (2.), (5.), (16.).

respecting the quotient, and in 194 respecting the product, of any two *right* quaternions (132), in connexion with their *indices* (133), we may now establish, for any *such* quaternions, the formulæ:

$$\text{III.} \ldots \mathrm{S}(q' : q) = \mathrm{S}(\mathrm{I}q' : \mathrm{I}q) = \mathrm{T}(q' : q) \,.\, \cos \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q);$$

$$\text{IV.} \ldots \mathrm{S}q'q = \mathrm{S}(q' \,.\, q) = \mathrm{S}\left(\mathrm{I}q' : \mathrm{I}\frac{1}{q}\right) = -\mathrm{T}q'q \,.\, \cos \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q);$$

where the new symbol $\mathrm{I}q$ is used, as a temporary abridgment, to denote the *Index* of the quaternion q, supposed here (as above) to be a *right* one. With the same supposition, we have therefore also these other and shorter formulæ:

$$\text{V.} \ldots \mathrm{SU}(q' : q) = + \cos \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q);$$

$$\text{VI.} \ldots \mathrm{SU}q'q = - \cos \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q);$$

which may, by 126, XVI., be interpreted as expressing that, under the same condition of *rectangularity* of q and q',

$$\text{VII.} \ldots \angle (q' : q) = \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q);$$

$$\text{VIII.} \ldots \angle q'q = \pi - \angle (\mathrm{Ax} \,.\, q' : \mathrm{Ax} \,.\, q).$$

In words, *the Angle of the Quotient of two Right Quaternions is equal to the Angle of their Axes*; but the *Angle of the Product*, of two such quaternions, is equal to the *Supplement of the Angle of the Axes*. There is no difficulty in proving these results otherwise, by constructions such as that employed in Art. 193; nor in illustrating them by the consideration of isosceles quadrantal triangles, upon the surface of a sphere.

199. Another important *case* of the scalar of a *product* is the case of the *scalar of the square* of a quaternion. On referring to Art. 149, and to fig. 42 [p. 133], we see that while we have always $\mathrm{T}(q^2) = (\mathrm{T}q)^2$, as in 190, and $\mathrm{U}(q^2) = (\mathrm{U}q)^2$, as in 161, we have also,

$$\text{I.} \ldots \angle (q^2) = 2 \angle q, \quad \text{and} \quad \mathrm{Ax} \,.\, (q^2) = \mathrm{Ax} \,.\, q, \quad \text{if} \quad \angle q < \frac{\pi}{2};$$

but, by the adopted definitions of $\angle q$ (130), and of $\mathrm{Ax} \,.\, q$ (127, 128),

$$\text{II.} \ldots \angle (q^2) = 2(\pi - \angle q), \quad \mathrm{Ax} \,.\, (q^2) = - \mathrm{Ax} \,.\, q, \quad \text{if} \quad \angle q > \frac{\pi}{2}.$$

In *each* case, however, by 196, XVI., we may write,

$$\text{III.} \ldots \mathrm{SU}(q^2) = \cos \angle (q^2) = \cos 2 \angle q;$$

a formula which holds even when $\angle q$ is 0, or $\frac{\pi}{2}$, or π, and which gives,

$$\text{IV.} \ldots \mathrm{SU}(q^2) = 2(\mathrm{SU}q)^2 - 1.$$

Hence, generally, the scalar of q^2 may be put under either of the two following forms:

$$\text{V.}\ldots \text{S}(q^2) = \text{T}q^2 . \cos 2\angle q; \quad \text{VI.}\ldots \text{S}(q^2) = 2(\text{S}q)^2 - \text{T}q^2;$$

where we see that it would not be safe to *omit the parentheses*, without some *convention* previously made, and to write simply $\text{S}q^2$, without first deciding whether this last symbol shall be understood to signify the *scalar of the square*, or the *square of the scalar* of q: these two things being generally *unequal*. The *latter* of them, however, occurring rather *oftener* than the former, it appears convenient to fix on *it* as that which is to be understood by $\text{S}q^2$, while the *other* may occasionally be written with a *point* thus, $\text{S} . q^2$; and then, with these conventions respecting *notation*,* we may write:

$$\text{VII.}\ldots \text{S}q^2 = (\text{S}q)^2; \qquad \text{VIII.}\ldots \text{S} . q^2 = \text{S}(q^2).$$

But the *square of the conjugate* of any quaternion is easily seen to be the *conjugate of the square*; so that we have generally (comp. 190, II.) the formula:

$$\text{IX.}\ldots \text{K}q^2 = \text{K}(q^2) = (\text{K}q)^2 = \text{T}q^2 : \text{U}q^2.$$

(1.) A quaternion, like a positive scalar, may be said to have in general *two opposite square roots*; because the *squares of opposite quaternions* are always *equal* (comp. (3.)). But of these two *roots* the *principal* (or *simpler*) one, and that which we shall denote by the symbol $\sqrt{q}$ or $\surd q$, and shall call by eminence *the Square Root* of q, is that which has its *angle acute*, and not *obtuse*. We shall therefore write, generally,

$$\text{X.}\ldots \angle\sqrt{q} = \tfrac{1}{2}\angle q; \qquad \text{Ax} . \sqrt{q} = \text{Ax} . q;$$

with the reservation that, when $\angle q = 0$, or $= \pi$, this common *axis* of q and $\surd q$ becomes (by 131, 149) an *indeterminate* unit-line.

(2.) Hence,

$$\text{XI.}\ldots \text{S}\surd q > 0, \quad \text{if} \quad \angle q < \pi;$$

while this *scalar of the square root* of a quaternion may, by VI., be thus transformed:

$$\text{XII.}\ldots \text{S}\surd q = \surd\{\tfrac{1}{2}(\text{T}q + \text{S}q)\};$$

a formula which holds good, even at the limit $\angle q = \pi$.

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* As, in the *Differential Calculus*, it is usual to write dx^2 instead of $(dx)^2$; while $d(x^2)$ is sometimes written as $d . x^2$. But as d^2x denotes a *second differential*, so it seems safest *not* to denote the square of $\text{S}q$ by the symbol S^2q, which *properly* signifies $\text{SS}q$, or $\text{S}q$, as in 196, VI.; the *second scalar* (like the *second tensor*, 187, (9.), or the *second versor*, 160) being equal to the *first*. Still every calculator will of course use his own discretion; and the employment of the notation S^2q for $\text{S}(q)^2$, as $\cos^2 x$ is often written for $(\cos x)^2$, may sometimes cause a *saving of space*.

(3.) The principle* (1.) that, in quaternions, as in algebra, the equation,

$$\text{XIII.} \ldots (-q)^2 = q^2,$$

is an *identity*, may be illustrated by conceiving that, in fig. 42, a point B′ is determined by the equation $\text{OB}' = \text{BO}$; for then we shall have (comp. fig. 33, *bis* [p. 122]),

$$(-q)^2 = \left(\frac{\text{OB}'}{\text{OA}}\right)^2 = \frac{\text{OC}}{\text{OA}} = q^2, \text{ because } \Delta\,\text{AOB}' \propto \text{B}'\text{OC}.$$

200. Another useful connexion between *scalars* and *tensors* (or *norms*) of quaternions may be derived as follows. In any plane triangle AOB, we have† the relation,

$$(\text{T}\,.\,\text{AB})^2 = (\text{T}\,.\,\text{OA})^2 - 2\,(\text{T}\,.\,\text{OA})\,.\,(\text{T}\,.\,\text{OB})\,.\,\cos \text{AOB} + (\text{T}\,.\,\text{OB})^2;$$

in which the symbols T . OA, &c., denote (by 185, 186) the *lengths* of the sides OA, &c.; but if we still write $q = \text{OB} : \text{OA}$, we have $q - 1 = \text{AB} : \text{OA}$; dividing therefore by $(\text{T}\,.\,\text{OA})^2$, the formula becomes (by 196, &c.),

$$\text{I.} \ldots \text{T}\,(q-1)^2 = 1 - 2\text{T}q\,.\,\text{SU}q + \text{T}q^2 = \text{T}q^2 - 2\text{S}q + 1;$$

or

$$\text{II.} \ldots \text{N}\,(q-1) = \text{N}q - 2\text{S}q + 1.$$

But q is here a perfectly *general* quaternion; we may therefore change its *sign*, and write,

$$\text{III.} \ldots \text{T}\,(1+q)^2 = 1 + 2\text{S}q + \text{T}q^2; \quad \text{IV.} \ldots \text{N}\,(1+q) = 1 + 2\text{S}q + \text{N}q.$$

And since it is easy to prove (by 106, 107) that

$$\text{V.} \ldots \left(\frac{q'}{q} + 1\right) q = q' + q,$$

whatever two quaternions q and q' may be, while

$$\text{VI.} \ldots \text{S}\frac{q'}{q}\,.\,\text{N}q = \text{S}\,.\,q'\text{K}q = \text{S}\,.\,q\text{K}q',$$

we easily infer this other general formula,

$$\text{VII.} \ldots \text{N}\,(q' + q) = \text{N}q' + 2\text{S}\,.\,q\text{K}q' + \text{N}q;$$

which gives, if x be any scalar,

$$\text{VIII.} \ldots \text{N}\,(q + x) = \text{N}q + 2x\text{S}q + x^2.$$

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* Compare the Note to page 162.

† By the Second Book of Euclid, or by plane trigonometry.

(1.) We are now prepared to effect, by *rules** *of transformation*, some other *passages* from one mode of *expression* to another, of the kind which has been alluded to, and partly exemplified, in former sub-articles. Take, for example, the formula,

$$\mathrm{T}\frac{\rho + a}{\rho - a} = 1, \text{ of 187, (2.)};$$

or the equivalent formula,

$$\mathrm{T}(\rho + a) = \mathrm{T}(\rho - a), \text{ of 186, (6.)};$$

which has been seen, on *geometrical* grounds, to represent a certain *locus*, namely the plane through O, perpendicular to the line OA; and therefore the *same locus* as that which is represented by the equation

$$\mathrm{S}\frac{\rho}{a} = 0, \text{ of 196, (1.)},$$

To *pass* now from the former equations to the latter, by *calculation*, we have only to denote the quotient $\rho : a$ by q, and to observe that the first or second form, as just now cited, becomes then,

$$\mathrm{T}(q+1) = \mathrm{T}(q-1); \quad \text{or} \quad \mathrm{N}(q+1) = \mathrm{N}(q-1);$$

or finally, by II. and IV.,

$$\mathrm{S}q = 0,$$

which gives the third form of equation, as required.

(2.) Conversely, from $\mathrm{S}\frac{\rho}{a} = 0$, we can *return*, by the same general formulæ II. and IV., to the equation $\mathrm{N}\left(\frac{\rho}{a} - 1\right) = \mathrm{N}\left(\frac{\rho}{a} + 1\right)$, or by I. and III. to $\mathrm{T}\left(\frac{\rho}{a} - 1\right) = \mathrm{T}\left(\frac{\rho}{a} + 1\right)$, or to $\mathrm{T}(\rho - a) = \mathrm{T}(\rho + a)$, or to $\mathrm{T}\frac{\rho + a}{\rho - a} = 1$, as above; and generally,

$$\mathrm{S}q = 0 \text{ gives } \mathrm{T}(q-1) = \mathrm{T}(q+1), \text{ or } \mathrm{T}\frac{q+1}{q-1} = 1;$$

while the latter equations, in turn, involve, as has been seen, the former.

(3.) Again, if we take the Apollonian Locus, 145, (8.), (9.), and employ the *first* of the two forms 186, (5.) of its equation, namely,

$$\mathrm{T}(\rho - a^2 a) = a\mathrm{T}(\rho - a),$$

---

* Compare 145, (10): and several subsequent sub-articles.

where a is a given positive scalar different from unity, we may write it as

$$\mathrm{T}(q - a^2) = a\mathrm{T}(q - 1), \quad \text{or as} \quad \mathrm{N}(q - a^2) = a^2\mathrm{N}(q - 1);$$

or by VIII.,

$$\mathrm{N}q - 2a^2\mathrm{S}q + a^4 = a^2(\mathrm{N}q - 2\mathrm{S}q + 1);$$

or, after suppressing $-2a^2\,\mathrm{S}q$, transposing, and dividing by $a^2 - 1$,

$$\mathrm{N}q = a^2; \quad \text{or,} \quad \mathrm{N}\rho = a^2\mathrm{N}\alpha; \quad \text{or,} \quad \mathrm{T}\rho = a\mathrm{T}\alpha;$$

which last is the *second form* 186, (5.), and is thus *deduced from the first, by calculation alone*, without any immediate appeal to *geometry*, or the construction of any *diagram*.

(4.) Conversely if we take the equation,

$$\mathrm{N}\frac{\rho}{\alpha} = a^2, \text{ of 145, (12.),}$$

which was there seen to represent the same locus, considered as a spheric surface, with O for centre, and $a\alpha$ for one of its radii, and write it as $\mathrm{N}q = a^2$, we can then *by calculation* return to the form

$$\mathrm{N}(q - a^2) = a^2\mathrm{N}(q - 1), \quad \text{or} \quad \mathrm{T}(q - a^2) = a\mathrm{T}(q - 1),$$

or finally,

$$\mathrm{T}(\rho - a^2\alpha) = a\mathrm{T}(\rho - \alpha), \text{ as in 186, (5.);}$$

this *first form* of that sub-article being thus deduced *from the second*, namely from $\mathrm{T}\rho = a\mathrm{T}\alpha$, or $\mathrm{T}\frac{\rho}{\alpha} = a$.

(5.) It is far from being the intention of the foregoing remarks, to *discourage attention* to the *geometrical interpretation* of the various *forms of expression*, and general *rules of transformation*, which thus offer themselves in working with quaternions; on the contrary, one main object of the present Chapter has been to establish a firm *geometrical basis*, for all such forms and rules. But *when* such a *foundation* has once been laid, it is, as we see, *not necessary* that we should continually *recur* to the examination of it, in building up the *superstructure*. That *each* of the *two forms*, in 186, (5.), *involves* the *other* may be *proved*, as above, *by calculation*; but it is interesting to inquire what is the *meaning* of this result: and in seeking to *interpret* it, we should be led anew to the theorem of the *Apollonian Locus*.

(6.) The result (4.) of calculation, that

$$\mathrm{N}(q - a^2) = a^2\mathrm{N}(q - 1), \text{ if } \mathrm{N}q = a^2,$$

may be expressed under the form of an *identity*, as follows:

$$\text{IX.} \ldots \mathrm{N}(q - \mathrm{N}q) = \mathrm{N}q \,.\, \mathrm{N}(q - 1);$$

in which q may be *any quaternion*.

(7.) Or, by 191, VII., because it will soon be seen that

$$q\,(q-1) = q^2 - q, \text{ as in algebra,}$$

we may write it as this other identity:

$$\text{X.} \ldots \text{N}\,(q - \text{N}q) = \text{N}\,(q^2 - q).$$

(8.) If $\text{T}\,(q-1) = 1$, then $\text{S}\,\frac{1}{q} = \frac{1}{2}$; and conversely, the former equation follows from the latter; because each may be put under the form (comp. 196, XII.), $\text{N}q = 2\text{S}q$.

(9.) Hence, if $\text{T}\,(\rho - \alpha) = \text{T}\alpha$, then $\text{S}\,\frac{2\alpha}{\rho} = 1$, and reciprocally. In fact (comp. 196, (6.)), each of these two equations expresses that the locus of P is the sphere which passes through O, and has its centre at A; or which has OB $= 2\alpha$ for a diameter.

(10.) By changing q to $q + 1$ in (8), we find that

$$\text{if } \text{T}q = 1, \quad \text{then} \quad \text{S}\,\frac{q-1}{q+1} = 0, \text{ and reciprocally.}$$

(11.) Hence if $\text{T}\rho = \text{T}\alpha$, then $\text{S}\,\frac{\rho - \alpha}{\rho + \alpha} = 0$, and reciprocally; because (by 106)

$$\frac{\rho - \alpha}{\rho + \alpha} = \frac{\rho - \alpha}{\alpha} : \frac{\rho + \alpha}{\alpha} = \left(\frac{\rho}{\alpha} - 1\right) : \left(\frac{\rho}{\alpha} + 1\right).$$

(12.) Each of these two equations (11.) expresses that the locus of P is the sphere through A, which has its centre at O; and their proved agreement is a recognition, by quaternions, of the elementary geometrical theorem, that the angle in a semicircle is a right angle.

SECTION 13.

On the Right Part (or Vector Part) of a Quaternion; and on the Distributive Property of the Multiplication of Quaternions.

201. A given *vector* OB can always be decomposed, in one but in only one way, into two component vectors, of which it is the *sum* (6); and of which *one*, as OB′ in fig. 50, is *parallel* (15) to another given vector OA, while the *other*, as OB″ in the same figure, is *perpendicular* to that given line OA; namely, by letting fall the perpendicular BB′ on OA, and drawing OB″ = B′B, so that OB′BB″ shall be a rectangle. In other words, if α and β be any two given, actual, and co-initial vectors, it is always possible to deduce from them, in one definite way, two other

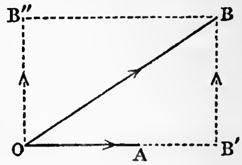

Fig. 50.

co-initial vectors, β' and β'', which need not however *both* be actual (1); and which shall satisfy (comp. 6, 15, 129) the conditions,

$$\beta = \beta' + \beta'' = \beta'' + \beta', \qquad \beta' \parallel \alpha, \qquad \beta'' \perp \alpha;$$

β' vanishing, when $\beta \perp \alpha$; and β'' being null, when $\beta \parallel \alpha$; but both being (what we may call) *determinate vector-functions* of α and β. And of these two functions, it is evident that β' is the orthographic *projection of* β *on the line* α; and that β'' is the corresponding *projection of* β *on the plane* through O, which is *perpendicular to* α.

202. Hence it is easy to infer, that there is always one, but only one way, of *decomposing a given quaternion*,

$$q = \text{OB} : \text{OA} = \beta : \alpha,$$

into two parts or summands (195), of which *one* shall be, as in 196, a *scalar*, while the *other* shall be a *right quotient* (132). Of these two parts, the *former* has been already called (196) *the scalar part*, or simply *the Scalar* of the Quaternion, and has been denoted by the symbol Sq; so that, with reference to the recent figure 50, we have

$$\text{I.} \ldots \text{S}q = \text{S}\,(\text{OB} : \text{OA}) = \text{OB}' : \text{OA}; \quad \text{or,} \quad \text{S}\,(\beta : \alpha) = \beta' : \alpha.$$

And we now propose to call the *latter* part the RIGHT PART* of the same quaternion, and to denote it by the new symbol

$$\text{V}q;$$

writing thus, in connexion with the same figure,

$$\text{II.} \ldots \text{V}q = \text{V}\,(\text{OB} : \text{OA}) = \text{OB}'' : \text{OA}; \quad \text{or,} \quad \text{V}\,(\beta : \alpha) = \beta'' : \alpha.$$

The *System of Notations*, peculiar to the present Calculus, will thus have been completed; and we shall have the following general *Formula of Decomposition of a Quaternion into two Summands* (comp. 188), of the *Scalar* and *Right* kinds:

$$\text{III.} \ldots q = \text{S}q + \text{V}q = \text{V}q + \text{S}q,$$

or, briefly and symbolically,

$$\text{IV.} \ldots 1 = \text{S} + \text{V} = \text{V} + \text{S}.$$

(1.) In connexion with the same fig. 50, we may write also,

$$\text{V}\,(\text{OB} : \text{OA}) = \text{B}'\text{B} : \text{OA},$$

because, by construction, $\text{B}'\text{B} = \text{OB}''$.

* This *Right Part*, Vq, will come to be also called *the Vector Part*, or simply *the* VECTOR, of the Quaternion; because it will be found possible and useful to identify such part with its own Index-Vector (133). Compare the Notes to pages 121, 137, 175 [and Art. 286].

(2.) In like manner, for fig. 36 [p. 115], we have the equation,

$$\mathrm{V}\,(\mathrm{OB} : \mathrm{OA}) = \mathrm{A'B} : \mathrm{OA}.$$

(3.) Under the recent conditions,

$$\mathrm{V}\,(\beta' : \alpha) = 0, \quad \text{and} \quad \mathrm{S}\,(\beta'' : \alpha) = 0.$$

(4.) In general, it is evident that

$$\mathrm{V.}\ldots q = 0, \quad \text{if} \quad \mathrm{S}q = 0, \quad \text{and} \quad \mathrm{V}q = 0; \text{ and reciprocally.}$$

(5.) More generally,

$$\mathrm{VI.}\ldots q' = q, \quad \text{if} \quad \mathrm{S}q' = \mathrm{S}q, \quad \text{and} \quad \mathrm{V}q' = \mathrm{V}q; \text{ with the converse.}$$

(6.) Also $$\mathrm{VII.}\ldots \mathrm{V}q = 0, \quad \text{if} \quad \angle\, q = 0, \quad \text{or} \quad = \pi;$$

or $$\mathrm{VIII.}\ldots \mathrm{V}\,(\beta : \alpha) = 0, \quad \text{if} \quad \beta \parallel \alpha;$$

the *right part* of a *scalar* being zero.

(7.) On the other hand,

$$\mathrm{IX.}\ldots \mathrm{V}q = q, \quad \text{if} \quad \angle\, q = \frac{\pi}{2};$$

a *right quaternion* being *its own right part.*

203. We had (196, XIX.) a formula which may now be written thus,

$$\mathrm{I.}\ldots \mathrm{OB'} = \mathrm{S}\,(\mathrm{OB} : \mathrm{OA}) \,.\, \mathrm{OA}, \quad \text{or} \quad \beta' = \mathrm{S}\frac{\beta}{\alpha} \,.\, \alpha,$$

to express the *projection of* OB *on* OA, or of the vector β on α; and we have evidently, by the definition of the new symbol $\mathrm{V}q$, the analogous formula,

$$\mathrm{II.}\ldots \mathrm{OB''} = \mathrm{V}\,(\mathrm{OB} : \mathrm{OA}) \,.\, \mathrm{OA}, \quad \text{or} \quad \beta'' = \mathrm{V}\frac{\beta}{\alpha} \,.\, \alpha,$$

to express the *projection of* β *on the plane* (through O), which is drawn so as to be *perpendicular to* α; and which has been considered in several former sub-articles (comp. 186, (6.), and 196, (1.)). It follows (by 186, &c.) that

$$\mathrm{III.}\ldots \mathrm{T}\beta'' = \mathrm{TV}\frac{\beta}{\alpha} \,.\, \mathrm{T}\alpha = \textit{perpendicular distance of } \mathrm{B} \textit{ from } \mathrm{OA};$$

this perpendicular being *here* considered with reference to its *length* alone, as the characteristic T of the *tensor* implies. It is to be observed that because the *factor*, $\mathrm{V}\frac{\beta}{\alpha}$, in the recent formula II. for the projection β'', is *not a scalar*, we must write that factor as a *multiplier*, and *not* as a *multiplicand*; although we were at liberty, in consequence of a general convention (15), respecting the *multiplication of vectors and scalars*, to denote the *other* projection β' under the form,

$$\text{I}'. \ldots \beta' = \alpha \text{S} \frac{\beta}{\alpha} \text{ (196, XIX.)}.$$

(1.) The equation,

$$\text{V}\frac{\rho}{\alpha} = 0,$$

expresses that the locus of P is the *indefinite right line* OA.

(2.) The equation,

$$\text{V}\frac{\rho - \beta}{\alpha} = 0 \quad \text{or} \quad \text{V}\frac{\rho}{\alpha} = \text{V}\frac{\beta}{\alpha},$$

expresses that the locus of P is the indefinite right line BB″, in fig. 50, which is drawn through the point B, parallel to the line OA.

(3.) The equation

$$\text{S}\frac{\rho - \beta}{\alpha} = 0, \quad \text{or} \quad \text{S}\frac{\rho}{\alpha} = \text{S}\frac{\beta}{\alpha}, \text{ of 196, (2.)},$$

has been seen to express that the locus of P is the *plane* through B, perpendicular to the line OA; if then we *combine* it with the recent equation (2.), we shall express that the point P is situated at the *intersection* of the two last mentioned loci; or that it *coincides* with the *point* B.

(4.) Accordingly, whether we take the two first or the two last of these recent forms (2.), (3.), namely,

$$\text{V}\frac{\rho - \beta}{\alpha} = 0, \quad \text{S}\frac{\rho - \beta}{\alpha} = 0, \quad \text{or} \quad \text{V}\frac{\rho}{\alpha} = \text{V}\frac{\beta}{\alpha}, \quad \text{S}\frac{\rho}{\alpha} = \text{S}\frac{\beta}{\alpha},$$

we can infer this position of the point P: in the first case by inferring, through 202, V., that $\frac{\rho - \beta}{\alpha} = 0$, whence $\rho - \beta = 0$, by 142; and in the second case by inferring, through 202, VI., that $\frac{\rho}{\alpha} = \frac{\beta}{\alpha}$; so that we have in each case (comp. 104), or as a consequence from each system, the equality $\rho = \beta$, or OP = OB; or finally (comp. 20) the *coincidence*, P = B.

(5.) The equation

$$\text{TV}\frac{\rho}{\alpha} = \text{TV}\frac{\beta}{\alpha},$$

expresses that the locus of the point P is the *cylindric surface of revolution*, which passes through the point B, and has the line OA for its axis; for it expresses, by III., that the *perpendicular distances* of P and B, *from this latter line*, are *equal*.

(6.) The system of the two equations,

$$\mathrm{TV}\frac{\rho}{\alpha} = \mathrm{TV}\frac{\beta}{\alpha}, \quad \mathrm{S}\frac{\rho}{\gamma} = 0,$$

expresses that the locus of P is the (generally) *elliptic section* of the cylinder (5.), made by the plane through O, which is perpendicular to the line OC.

(7.) If we employ an analogous decomposition of ρ, by supposing that

$$\rho = \rho' + \rho'', \qquad \rho' \parallel \alpha, \qquad \rho'' \perp \alpha,$$

the three rectilinear or plane loci, (1.), (2.), (3.), may have their equations thus briefly written:

$$\rho'' = 0\,; \qquad \rho'' = \beta''; \qquad \rho' = \beta':$$

while the combination of the two last of these gives $\rho = \beta$, as in (4.).

(8.) The equation of the cylindric locus, (5.), takes at the same time the form

$$\mathrm{T}\rho'' = \mathrm{T}\beta'';$$

which last equation expresses that the projection P″ of the point P, on the plane through O perpendicular to OA, falls somewhere on the circumference of a circle, with O for centre, and OB″ for radius: and this *circle* may accordingly be considered as the *base of the right cylinder*, in the sub-article last cited.

204. From the mere circumstance that $\mathrm{V}q$ is always a *right quotient* (132), whence $\mathrm{UV}q$ is a *right versor* (153), of which the *plane* (119), and the *axis* (127), coincide with those of q, several general consequences easily follow. Thus we have generally, by principles already established, the relations:

$$\mathrm{I.}\ldots \angle\, \mathrm{V}q = \frac{\pi}{2}; \quad \mathrm{II.}\ldots \mathrm{Ax}\,.\mathrm{V}q = \mathrm{Ax}\,.\,\mathrm{UV}q = \mathrm{Ax}\,.\,q\,;$$

$$\mathrm{III.}\ldots \mathrm{KV}q = -\mathrm{V}q, \quad \text{or} \quad \mathrm{KV} = -\mathrm{V}\,(144)\,;$$

$$\mathrm{IV.}\ldots \mathrm{SV}q = 0, \quad \text{or} \quad \mathrm{SV} = 0\,(196,\ \mathrm{VII.})\,;$$

$$\mathrm{V.}\ldots (\mathrm{UV}q)^2 = -1\,(153,\ 159)\,;$$

and therefore, $\mathrm{VI.}\ldots (\mathrm{V}q)^2 = -(\mathrm{TV}q)^2 = -\mathrm{NV}q,$*

because, by the general decomposition (188) of a quaternion into *factors*, we have

$$\mathrm{VII.}\ldots \mathrm{V}q = \mathrm{TV}q\,.\,\mathrm{UV}q.$$

We have also (comp. 196, VI.),

$$\mathrm{VIII.}\ldots \mathrm{VS}q = 0, \quad \text{or} \quad \mathrm{VS} = 0\,(202,\ \mathrm{VII.})\,;$$

* Compare the Note to page 132.

$$\text{IX.} \ldots \mathrm{VV}q = \mathrm{V}q, \quad \text{or} \quad \mathrm{V}^2 = \mathrm{VV} = \mathrm{V}\ (202,\ \text{IX.});$$

and

$$\text{X.} \ldots \mathrm{VK}q = -\mathrm{V}q, \quad \text{or} \quad \mathrm{VK} = -\mathrm{V},$$

because *conjugate quaternions* have *opposite right parts*, by the definitions in 137, 202, and by the construction of fig. 36 [p. 115]. For the same reason, we have this other general formula,

$$\text{XI.} \ldots \mathrm{K}q = \mathrm{S}q - \mathrm{V}q, \quad \text{or} \quad \mathrm{K} = \mathrm{S} - \mathrm{V};$$

but we had

$$q = \mathrm{S}q + \mathrm{V}q, \quad \text{or} \quad 1 = \mathrm{S} + \mathrm{V}, \text{ by 202, III., IV.};$$

hence not only, by addition,

$$q + \mathrm{K}q = 2\mathrm{S}q, \quad \text{or} \quad 1 + \mathrm{K} = 2\mathrm{S}, \text{ as in 196, I.},$$

but also, by subtraction,

$$\text{XII.} \ldots q - \mathrm{K}q = 2\mathrm{V}q, \quad \text{or} \quad 1 - \mathrm{K} = 2\mathrm{V};$$

whence the *Characteristic*, V, *of the Operation of taking the Right Part of a Quaternion* (comp. 132, (6); 137; 156; 187; 196), may be *defined* by either of the two following symbolical equations:

$$\text{XIII.} \ldots \mathrm{V} = 1 - \mathrm{S}\ (202,\ \text{IV.}); \qquad \text{XIV.} \ldots \mathrm{V} = \tfrac{1}{2}(1 - \mathrm{K});$$

whereof the former connects it with the characteristic S, and the latter with the characteristic K; while the dependence of K on S and V is expressed by the recent formula XI.; and that of S on K by 196, II′. Again, if the line OB, in fig. 50, be multiplied (15) by any scalar coefficient, the perpendicular BB′ is evidently multiplied by the same; hence, generally,

$$\text{XV.} \ldots \mathrm{V}xq = x\mathrm{V}q, \text{ if } x \text{ be any scalar};$$

and therefore, by 188, 191,

$$\text{XVI.} \ldots \mathrm{V}q = \mathrm{T}q \,.\, \mathrm{VU}q, \quad \text{and} \quad \text{XVII.} \ldots \mathrm{TV}q = \mathrm{T}q \,.\, \mathrm{TVU}q.$$

But the consideration of the right-angled triangle, OB′B, in the same figure, shows that

$$\text{XVIII.} \ldots \mathrm{TV}q = \mathrm{T}q \,.\, \sin \angle q,$$

because, by 202, II., we have

$$\mathrm{TV}q = \mathrm{T}(\mathrm{OB}'' : \mathrm{OA}) = \mathrm{T} \,.\, \mathrm{OB}'' : \mathrm{T} \,.\, \mathrm{OA},$$

and

$$\mathrm{T} \,.\, \mathrm{OB}'' = \mathrm{T} \,.\, \mathrm{OB} \,.\, \sin \mathrm{AOB};$$

we arrive then thus at the following general and useful formula, connecting *quaternions* with *trigonometry* anew:

$$\text{XIX.} \ldots \mathrm{TVU}q = \sin \angle q;$$

by combining which with the formula,

$$\mathrm{SU}q = \cos \angle q \text{ (196, XVI.)},$$

we arrive at the general relation:

$$\text{XX.} \ldots (\mathrm{SU}q)^2 + (\mathrm{TVU}q)^2 = 1;$$

which may also (by XVII., and by 196, IX.) be written thus:

$$\text{XXI.} \ldots (\mathrm{S}q)^2 + (\mathrm{TV}q)^2 = (\mathrm{T}q)^2;$$

and might have been immediately deduced, *without sines and cosines*, from the right-angled triangle, by the property of the square of the hypotenuse, under the form,

$$(\mathrm{T} \,.\, \mathrm{OB}')^2 + (\mathrm{T} \,.\, \mathrm{B'B})^2 = (\mathrm{T} \,.\, \mathrm{OB})^2.$$

The same important relation may be expressed in various other ways; for example, we may write,

$$\text{XXII.} \ldots \mathrm{N}q = \mathrm{T}q^2 = \mathrm{S}q^2 - \mathrm{V}q^2,$$

where it is assumed, as an abridgment of *notation* (comp. 199, VII., VIII.), that

$$\text{XXIII.} \ldots \mathrm{V}q^2 = (\mathrm{V}q)^2, \quad \text{but that} \quad \text{XXIV.} \ldots \mathrm{V} \,.\, q^2 = \mathrm{V}(q^2),$$

the import of this last symbol remaining to be examined. And because, by the definition of a *norm*, and by the properties of $\mathrm{S}q$ and $\mathrm{V}q$,

$$\text{XXV.} \ldots \mathrm{NS}q = \mathrm{S}q^2, \quad \text{but} \quad \text{XXVI.} \ldots \mathrm{NV}q = -\mathrm{V}q^2,$$

we may write also,

$$\text{XXVII.} \ldots \mathrm{N}q = \mathrm{N}(\mathrm{S}q + \mathrm{V}q) = \mathrm{NS}q + \mathrm{NV}q;$$

a result which is indeed included in the formula 200, VIII., since that equation gives, generally,

$$\text{XXVIII.} \ldots \mathrm{N}(q + x) = \mathrm{N}q + \mathrm{N}x, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

x being, as usual, any scalar. It may be added that because (by 106, 143) we have, as in algebra, the identity,

$$\text{XXIX.} \ldots -(q' + q) = -q' - q,$$

the *opposite of the sum* of any two quaternions being thus equal to the *sum of the opposites*, we may (by XI.) establish this other general formula:

$$\text{XXX.} \ldots -\mathrm{K}q = \mathrm{V}q - \mathrm{S}q;$$

the *opposite of the conjugate* of any quaternion q having thus the *same right part* as that quaternion, but an *opposite scalar part*.

(1.) From the last formula it may be inferred, that

$$\text{if} \quad q' = -\mathrm{K}q, \quad \text{then} \quad \mathrm{V}q' = +\mathrm{V}q, \quad \text{but} \quad \mathrm{S}q' = -\mathrm{S}q;$$

and therefore that

$$\mathrm{T}q' = \mathrm{T}q, \quad \text{and} \quad \mathrm{Ax}.q' = \mathrm{Ax}.q, \quad \text{but} \quad \angle q' = \pi - \angle q;$$

which two last relations might have been deduced from 138 and 143, without the introduction of the characteristics S and V.

(2.) The equation,

$$\left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = \left(\mathrm{V}\frac{\beta}{\alpha}\right)^2, \quad \text{or (by XXVI.),} \quad \mathrm{NV}\frac{\rho}{\alpha} = \mathrm{NV}\frac{\beta}{\alpha},$$

like the equation of 203, (5.), expresses that the locus of P is the *right cylinder*, or cylinder of revolution, with OA for its axis, which passes through the point B.

(3.) The system of the two equations,

$$\left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = \left(\mathrm{V}\frac{\beta}{\alpha}\right)^2, \quad \mathrm{S}\frac{\rho}{\gamma} = 0,$$

like the corresponding system in 203, (6.), represents generally an *elliptic section* of the same right cylinder; but if it happened that $\gamma \parallel \alpha$, the section then becomes *circular*.

(4.) The system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha} = x, \quad \left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = x^2 - 1, \quad \text{with} \quad x > -1, \quad x < 1,$$

represents the *circle*,* in which the cylinder of revolution, with OA for axis, and with $(1 - x^2)^{\frac{1}{2}}\mathrm{T}\alpha$ for radius, is perpendicularly cut by a plane at a distance $= \pm x\mathrm{T}\alpha$ from O; the vector of the centre of this circular section being $x\alpha$.

(5.) While the scalar x increases (algebraically) from -1 to 0, and thence to $+1$, the connected scalar $\sqrt{(1 - x^2)}$ at first increases from 0 to 1, and then decreases from 1 to 0; the *radius* of the circle (4.) at the same time enlarging from zero to a maximum $= \mathrm{T}\alpha$, and then again diminishing to zero; while the position of the *centre* of the circle varies continuously, in one constant direction, from a *first limit-point* A′, if OA′ $= -\alpha$, to the point A, as a *second limit.*

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* By the word "circle," in these pages, is usually meant a *circumference*, and *not* an *area*; and in like manner, the words "sphere," "cylinder," "cone," &c., are usually here employed to denote *surfaces*, and *not volumes.*

(6.) The *locus* of all such *circles* is the *sphere*, with AA′ for a diameter, and therefore with O for centre; namely, the sphere which has already been represented by the equation $T\rho = T\alpha$ of 186, (2.); or by $T\frac{\rho}{\alpha} = 1$, of 187, (1.); or by

$$S\frac{\rho-\alpha}{\rho+\alpha} = 0, \text{ of 200, (11.)};$$

but which now presents itself under the new form,

$$\left(S\frac{\rho}{\alpha}\right)^2 - \left(V\frac{\rho}{\alpha}\right)^2 = 1,$$

obtained by *eliminating* x between the two recent equations (4.).

(7.) It is easy, however, to *return* from the last form to the second, and thence to the first, or to the third, by rules of calculation already established, or by the general relations between the symbols used. In fact, the last equation (6.) may be written, by XXII., under the form,

$$N\frac{\rho}{\alpha} = 1; \quad \text{whence} \quad T\frac{\rho}{\alpha} = 1, \text{ by 190, VI.};$$

and therefore also $T\rho = T\alpha$, by 187, and $S\frac{\rho-\alpha}{\rho+\alpha} = 0$, by 200, (11.).

(8.) Conversely, the sphere through A, with O for centre, might already have been seen, by the first definition and property of a *norm*, stated in 145, (11.), to admit (comp. 145, (12.)) of being represented by the equation $N\frac{\rho}{\alpha} = 1$; and therefore, by XXII., under the recent form (6.); in which if we write x to denote the variable scalar $S\frac{\rho}{\alpha}$, as in the first of the two equations (4.), we recover the second of those equations: and thus might be led to consider, as in (6.), the *sphere* in question as the *locus of a variable circle*, which is (as above) the *intersection* of a *variable cylinder*, with a *variable plane* perpendicular to its axis.

(9.) The same sphere may also, by XXVII., have its equation written thus,

$$N\left(S\frac{\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1; \quad \text{or} \quad T\left(S\frac{\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1.$$

(10.) If, in each variable plane represented by the first equation (4.), we conceive the radius of the circle, or that of the variable cylinder, to be multiplied by any constant and positive scalar a, the centre of the circle and the axis of the cylinder remaining unchanged, we shall pass thus to a *new system of circles*, represented by this new system of equations,

$$S\frac{\rho}{\alpha} = x, \quad \left(V\frac{\rho}{a\alpha}\right)^2 = x^2 - 1.$$

(11.) The *locus* of these *new* circles will evidently be a *Spheroid of Revolution*; the *centre* of this new surface being the centre O, and the *axis* of the same surface being the *diameter* AA′, of the *sphere* lately considered: which sphere is therefore either *inscribed* or *circumscribed* to the spheroid, according as the constant $a >$ or < 1; because the *radii* of the new circles are in the first case *greater*, but in the second case *less*, than the radii of the old circles; or because the *radius of the equator* of the spheroid $= a\,\mathrm{T}\alpha$, while the radius of the sphere $= \mathrm{T}\alpha$.

(12.) The equations of the *two co-axal cylinders* of revolution, which *envelope* respectively the sphere and spheroid (or are *circumscribed* thereto) are:

$$\left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = -1, \quad \left(\mathrm{V}\frac{\rho}{a\alpha}\right)^2 = -1; \quad \text{or} \quad \mathrm{NV}\frac{\rho}{\alpha} = 1, \quad \mathrm{NV}\frac{\rho}{\alpha} = a^2;$$

or

$$\mathrm{TV}\frac{\rho}{\alpha} = 1, \quad \mathrm{TV}\frac{\rho}{\alpha} = a.$$

(13.) The system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha} = x, \quad \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = x^2 - 1, \quad \text{with } \beta \textit{ not } \parallel \alpha,$$

represents (comp. (3.)) a *variable ellipse*, if the scalar x be still treated as a variable.

(14.) The result of the elimination of x between the two last equations, namely this new equation,

$$\left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 1; \quad \text{or} \quad \mathrm{NS}\frac{\rho}{\alpha} + \mathrm{NV}\frac{\rho}{\beta} = 1, \text{ by XXV., XXVI.};$$

or

$$\mathrm{N}\left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right) = 1, \text{ by XXVII.; or finally, } \mathrm{T}\left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right) = 1, \text{ by 190, VI.},$$

represents the *locus of all such ellipses* (13.), and will be found to be an adequate representation, through quaternions, of the *general* ELLIPSOID (with *three unequal axes*): that celebrated surface being here referred to its *centre*, as the *origin* O of vectors to its points; and the *six scalar* (or algebraic) *constants*, which enter into the usual *algebraic equation* (by co-ordinates) of such a *central ellipsoid*, being here virtually included in the *two independent vectors*, α and β, which may be called its *two Vector-Constants*.*

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* It will be found, however, that *other pairs* of *vector-constants*, for the *central ellipsoid*, may occasionally be used with advantage.

(15.) The equation (comp. (12.)),

$$\left(V\frac{\rho}{\beta}\right)^2 = -1, \quad \text{or} \quad NV\frac{\rho}{\beta} = 1, \quad \text{or} \quad TV\frac{\rho}{\beta} = 1,$$

represents a *cylinder of revolution, circumscribed to the ellipsoid*, and touching it *along the ellipse* which answers to the value $x = 0$, in (13.); so that the *plane* of this *ellipse of contact* is represented by the equation,

$$S\frac{\rho}{\alpha} = 0;$$

the *normal* to this *plane* being thus (comp. 196, (17.)) the vector α, or OA; while the *axis* of the lately mentioned enveloping *cylinder* is β, or OB.

(16.) Postponing any further discussion of the recent *quaternion equation of the ellipsoid* (14.), it may be noted here that we have generally, by XXII., the two following useful transformations for the *squares*, of the *scalar* Sq, and of the *right part* Vq, of any quaternion q:

$$\text{XXXI.} \ldots Sq^2 = Tq^2 + Vq^2; \qquad \text{XXXII.} \ldots Vq^2 = Sq^2 - Tq^2.$$

(17.) In referring briefly to these, and to the connected formula XXII., upon occasion, it may be somewhat safer to write,

$$(S)^2 = (T)^2 + (V)^2, \quad (V)^2 = (S)^2 - (T)^2, \quad (T)^2 = (S)^2 - (V)^2,$$

than $S^2 = T^2 + V^2$, &c.; because these last forms of notation, S^2, &c., have been otherwise interpreted already, in analogy to the known *Functional Notation*, or Notation of the *Calculus of Functions*, or of *Operations* (comp. 187, (9.)); 196, VI.; and 204, IX.).

(18.) In pursuance of the same analogy, *any scalar* may be denoted by the *general symbol*,

$$V^{-1}0;$$

because *scalars* are the *only* quaternions of which the *right parts vanish*.

(19.) In like manner, a *right quaternion*, generally, may be denoted by the symbol,

$$S^{-1}0;$$

and since this includes (comp. 204, I.) the *right part* of *any* quaternion, we may establish this *general symbolic transformation of a Quaternion*:

$$q = V^{-1}0 + S^{-1}0.$$

(20.) With this form of notation, we should have generally, at least for *real** quaternions, the inequalities,

$$(\mathrm{V}^{-1}0)^2 > 0; \quad (\mathrm{S}^{-1}0)^2 < 0;$$

so that a (geometrically *real*) *Quaternion* is generally of the form:

Square-root of a Positive, plus Square-root of a Negative.

(21.) The equations 196, XVI., and 204, XIX., give, as a new link between quaternions and *trigonometry*, the formula:

$$\text{XXXIII.} \ldots \tan \angle q = \mathrm{TVU}q : \mathrm{SU}q = \mathrm{TV}q : \mathrm{S}q.$$

(22.) It may not be entirely in accordance with the *theory* of that *Functional* (or *Operational*) *Notation* to which allusion has lately been made, but it will be found to be convenient in *practice*, to write this last result under one or other of the *abridged forms*: †

$$\text{XXXIV.} \ldots \tan \angle q = \frac{\mathrm{TV}}{\mathrm{S}} \cdot q; \quad \text{or} \quad \text{XXXIV}' \ldots \tan \angle q = (\mathrm{TV} : \mathrm{S})\, q;$$

which have the advantage of *saving the repetition of the symbol of the quaternion*, when that symbol happens to be a *complex expression*, and not, as here, a single letter, q.

(23.) The transformation 194, for the *index* of a right quotient, gives generally, by II., for any quaternion q, the formulæ:

$$\text{XXXV.} \ldots \mathrm{IV}q = \mathrm{TV}q \,.\, \mathrm{Ax} \,.\, q; \quad \text{XXXVI.} \ldots \mathrm{IUV}q = \mathrm{Ax} \,.\, q;$$

so that we may establish generally the symbolical ‡ equation,

$$\text{XXXVI}' \ldots \mathrm{IUV} = \mathrm{Ax}.$$

(24.) And because $\mathrm{Ax} \,.\, (1 : \mathrm{V}q) = - \mathrm{Ax} \,.\mathrm{V}q$, by 135, and therefore $= - \mathrm{Ax} \,.\, q$, by II., we may write also, by XXXV.,

$$\text{XXXV}' \ldots \mathrm{I}\,(1 : \mathrm{V}q) = - \mathrm{Ax} \,.\, q : \mathrm{TV}q.$$

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* Compare Art. 149; and the Notes to pages 87, 135.

† Compare the Note to Art. 199.

‡ At a later stage [286] it will be found possible (comp. the Note to page 175, &c.), to write, generally,

$$\mathrm{IV}q = \mathrm{V}q, \qquad \mathrm{IUV}q = \mathrm{UV}q;$$

and then (comp. the Note in page 120 to Art. 129) the recent equations, XXXVI., XXXVI′., will take these shorter forms [291]:

$$\mathrm{Ax} \,.\, q = \mathrm{UV}q; \quad \mathrm{Ax} \,. = \mathrm{UV}.$$

205. If any parallelogram OBDC (comp. 197) be projected on the plane through O, which is perpendicular to OA, the projected figure OB″D″C″ (comp. 11) is still a parallelogram; so that

$$\mathrm{OD}'' = \mathrm{OC}'' + \mathrm{OB}''\ (6), \quad \text{or} \quad \delta'' = \gamma'' + \beta'';$$

and therefore, by 106,

$$\delta'' : \alpha = (\gamma'' : \alpha) + (\beta'' : \alpha).$$

Hence, by 120, 202, for *any two quaternions*, q and q', we have the general formula,

$$\text{I.} \ldots \mathrm{V}(q' + q) = \mathrm{V}q' + \mathrm{V}q;$$

with which it is easy to connect this other,

$$\text{II.} \ldots \mathrm{V}(q' - q) = \mathrm{V}q' - \mathrm{V}q.$$

Hence also, for any *three* quaternions, q, q', q'',

$$\mathrm{V}\{q'' + (q' + q)\} = \mathrm{V}q'' + \mathrm{V}(q' + q) = \mathrm{V}q'' + (\mathrm{V}q' + \mathrm{V}q);$$

and similarly for any greater number of summands: so that we may write generally (comp. 197, II.),

$$\text{III.} \ldots \mathrm{V}\Sigma q = \Sigma \mathrm{V}q, \quad \text{or briefly} \quad \text{III}'. \ldots \mathrm{V}\Sigma = \Sigma\mathrm{V};$$

while the formula II. (comp. 197, IV.) may, in like manner, be thus written,

$$\text{IV.} \ldots \mathrm{V}\Delta q = \Delta\mathrm{V}q, \quad \text{or} \quad \text{IV}'. \ldots \mathrm{V}\Delta = \Delta\mathrm{V};$$

the *order* of the terms added, and the mode of *grouping* them, in III., being *as yet* supposed to remain unaltered, although both those restrictions will soon be removed. We conclude then, that the *characteristic* V, of the operation of *taking the right part* (202, 204) of a quaternion, like the characteristic S of *taking the scalar* (196, 197), and the characteristic K of *taking the conjugate* (137, 195*), is a *Distributive Symbol*, or represents a *distributive operation*: whereas the characteristics, Ax., ∠, N, U, T, of the operations of taking respectively the *axis* (128, 129), the *angle* (130), the *norm* (145, (11.)), the *versor* (156), and the *tensor* (187), are *not* thus distributive symbols (comp. 186, (10.), and 200, VII.); or do *not* operate upon a *whole* (or *sum*), by operating on its *parts* (or *summands*).

(1.) We may now recover the symbolical equation $\mathrm{K}^2 = 1$ (145), under the form (comp. 196, VI.; 202, IV.; and 204, IV., VIII., IX., XI.):

$$\text{VIII.} \ldots \mathrm{K}^2 = (\mathrm{S} - \mathrm{V})^2 = \mathrm{S}^2 - \mathrm{SV} - \mathrm{VS} + \mathrm{V}^2 = \mathrm{S} + \mathrm{V} = 1.$$

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* Indeed, it has only been proved as yet (comp. 195, (1.)), that $\mathrm{K}\Sigma q = \Sigma\mathrm{K}q$, for the case of *two* summands; but this result will soon be extended [207].

(2.) In like manner we can recover each of the expressions for S^2, V^2 from the other, under the forms (comp. again 202, IV.):

$$\text{VI.} \ldots \mathrm{S}^2 = (1-\mathrm{V})^2 = 1 - 2\mathrm{V} + \mathrm{V}^2 = 1 - \mathrm{V} = \mathrm{S}, \text{ as in 196, VI.};$$

$$\text{VII.} \ldots \mathrm{V}^2 = (1-\mathrm{S})^2 = 1 - 2\mathrm{S} + \mathrm{S}^2 = 1 - \mathrm{S} = \mathrm{V}, \text{ as in 204, IX.};$$

or thus (comp. 196, II'., and 204, XIV.), from the expressions for S and V in terms of K:

$$\text{VIII.} \ldots \mathrm{S}^2 = \tfrac{1}{4}(1+\mathrm{K})^2 = \tfrac{1}{4}(1 + 2\mathrm{K} + \mathrm{K}^2) = \tfrac{1}{2}(1+\mathrm{K}) = \mathrm{S};$$

$$\text{IX.} \ldots \mathrm{V}^2 = \tfrac{1}{4}(1-\mathrm{K})^2 = \tfrac{1}{4}(1 - 2\mathrm{K} + \mathrm{K}^2) = \tfrac{1}{2}(1-\mathrm{K}) = \mathrm{V}.$$

(3.) Similarly,

$$\text{X.} \ldots \mathrm{SV} = \tfrac{1}{4}(1+\mathrm{K})(1-\mathrm{K}) = \tfrac{1}{4}(1-\mathrm{K}^2) = 0, \text{ as in 204, IV.};$$

and $$\text{XI.} \ldots \mathrm{VS} = \tfrac{1}{4}(1-\mathrm{K})(1+\mathrm{K}) = \tfrac{1}{4}(1-\mathrm{K}^2) = 0, \text{ as in 204, VIII.*}$$

206. As regards the *addition* (or subtraction) of such *right parts*, $\mathrm{V}q$, $\mathrm{V}q'$, or generally of any two right quaternions (132), we may *connect* it with the addition (or subtraction) of their *indices* (133), as follows. Let OBDC be again any parallelogram (197, 205), but let OA be now an unit-vector (129) perpendicular to its plane; so that

$$\mathrm{T}\alpha = 1, \quad \angle(\beta : \alpha) = \angle(\gamma : \alpha) = \angle(\delta : \alpha) = \frac{\pi}{2}, \quad \delta = \gamma + \beta.$$

Let OB'D'C' be another parallelogram in the same plane, obtained by a positive rotation of the former, through a right angle, round OA as an axis; so that

$$\angle(\beta' : \beta) = \angle(\gamma' : \gamma) = \angle(\delta' : \delta) = \frac{\pi}{2};$$

$$\mathrm{Ax}.(\beta' : \beta) = \mathrm{Ax}.(\gamma' : \gamma) = \mathrm{Ax}.(\delta' : \delta) = \alpha.$$

Then the three right quotients, $\beta : \alpha$, $\gamma : \alpha$, and $\delta : \alpha$, may represent *any two right quaternions*, q, q', and their *sum*, $q' + q$, which is always (by 197, (2.)) *itself* a *right* quaternion; and the *indices* of these three right quotients are (comp. 133, 193) the three lines β', γ', δ', so that we may write, under the foregoing conditions of construction,

$$\beta' = \mathrm{I}(\beta : \alpha), \quad \gamma' = \mathrm{I}(\gamma : \alpha), \quad \delta' = \mathrm{I}(\delta : \alpha).$$

* [It may be instructive to the student to form symbolical equations analogous to those in 161 (3.) from the six symbols S, V, K and T, U, R. He may compare the equations obtained from the distributive symbols S, V and K, with those obtained from T, U and R, and may notice the pairs of symbols commutative in order of operation, &c. It is well to combine the symbols as in a multiplication table.]

But this third index is (by the second parallelogram) the *sum* of the two former indices, or in symbols, $\delta' = \gamma' + \beta'$; we may therefore write,

$$\text{I.} \ldots \mathrm{I}(q' + q) = \mathrm{I}q' + \mathrm{I}q, \quad \text{if} \quad \angle q = \angle q' = \frac{\pi}{2};$$

or in words *the Index of the Sum* of any two Right Quaternions is equal to the Sum of their Indices.* Hence, generally, for any two quaternions, q and q', we have the formula,

$$\text{II.} \ldots \mathrm{IV}(q' + q) = \mathrm{IV}q' + \mathrm{IV}q,$$

because $\mathrm{V}q$, $\mathrm{V}q'$ are *always* right quotients (202, 204), and $\mathrm{V}(q' + q)$ is always their *sum* (205, I.); so that the *index of the right part of the sum of any two quaternions* is the *sum of the indices of the right parts.* In like manner, there is no difficulty in proving that

$$\text{III.} \ldots \mathrm{I}(q' - q) = \mathrm{I}q' - \mathrm{I}q, \quad \text{if} \quad \angle q' = \angle q = \frac{\pi}{2};$$

and generally, that

$$\text{IV.} \ldots \mathrm{IV}(q' - q) = \mathrm{IV}q' - \mathrm{IV}q;$$

the *Index of the Difference* of any two right quotients, or of the right parts of any two quaternions, being thus equal to the *Difference of the Indices.** We may then *reduce* the *addition* or *subtraction* of any two such quotients, or parts, to the addition or subtraction of their *indices*; a right quaternion being always (by 133) determined, when its index is given, or known.

207. We see, then, that as the MULTIPLICATION *of any two Quaternions* was (in 191) *reduced* to (1st) the *arithmetical operation* of *multiplying their tensors,* and (IInd) the *geometrical operation* of *multiplying their versors,* which latter was *constructed* by a certain *composition of rotations,* and was *represented* (in either of two distinct but connected ways, 167, 175) by sides or angles of a *spherical triangle*: so the ADDITION *of any two Quaternions* may be reduced (by 197, I., and 206, II.) to, Ist, *the algebraical addition of their scalar parts,* considered as two positive or negative numbers (16.); and, IInd, *the geometrical addition of the indices of their right parts,* considered as certain *vectors* (1.): this latter *Addition of Lines* being performed according to the *Rule of the Parallelogram* (6.).† In like manner, as the general *Division of Quaternions*

* Compare the Note to page 175.

† It does not fall within the plan of these Notes to allude often to the history of the subject; but it ought to be distinctly stated that this celebrated *Rule,* for what may be called *Geometrical Addition of right lines,* considered as *analogous* to *composition of motions* (or of *forces*), had occurred to several writers *before* the invention of the quaternions: although the method adopted, in the present and in a former work, of deducing that rule, by algebraical analogies, from the *symbol* B − A (1.) for the line AB, may possibly not have been anticipated. The reader may compare the Notes to the Preface to the author's Volume of Lectures on Quaternions (Dublin, 1853).

was seen (in 191) to admit of being reduced to an *arithmetical division of tensors*, and a *geometrical division of versors*, so we may now (by 197, III., and 206, IV.) reduce, generally, the *Subtraction of Quaternions* to (Ist) an *algebraical subtraction of scalars*, and (IInd) a *geometrical subtraction of vectors*: this last operation being again constructed by a parallelogram, or even by a *plane triangle* (comp. Art. 4, and fig. 2). And because the *sum* of any given set of *vectors* was early seen to have a *value* (9.), which is independent of their *order*, and of the mode of *grouping* them, we may now infer that the *Sum of any number of given Quaternions* has, in like manner, a *Value* (comp. 197, (I.)), which is *independent of the Order, and of the Grouping of the Summands*: or in other words, that *the general Addition of Quaternions is a Commutative** *and an Associative Operation.*

(1.) The formula, $\mathrm{V}\Sigma q = \Sigma \mathrm{V} q$, of 205, III.,

is now seen to hold good, for *any number* of quaternions, independently of the *arrangement* of the terms in each of the two sums, and of the manner in which they may be *associated.*

(2.) We can infer anew that

$$\mathrm{K}\,(q' + q) = \mathrm{K}q' + \mathrm{K}q, \text{ as in 195, II.,}$$

under the form of the equation or identity,

$$\mathrm{S}\,(q' + q) - \mathrm{V}\,(q' + q) = (\mathrm{S}q' - \mathrm{V}q') + (\mathrm{S}q - \mathrm{V}q).$$

(3.) More generally, it may be proved, in the same way, that

$$\mathrm{K}\Sigma q = \Sigma \mathrm{K} q, \quad \text{or} \quad \text{briefly}, \quad \mathrm{K}\Sigma = \Sigma\mathrm{K},$$

whatever the number of the summands may be.

208. As regards the *quotient or product of the right parts*, $\mathrm{V}q$ and $\mathrm{V}q'$, of any two quaternions, let t and t' denote the *tensors* of those two parts, and let x denote the *angle of their indices*, or of their *axes*, or the mutual *inclination* of the axes, or of the *planes*,† of the two quaternions q and q' *themselves*, so that (by 204, XVIII.),

$$t = \mathrm{TV}q = \mathrm{T}q \,.\, \sin \angle\, q, \quad t' = \mathrm{TV}q' = \mathrm{T}q' \,.\, \sin \angle\, q',$$

and $$x = \angle\,(\mathrm{IV}q' : \mathrm{IV}q) = \angle\,(\mathrm{Ax}\,.\,q' : \mathrm{Ax}\,.\,q).$$

* Compare the Note to page 176.

† *Two planes*, of course, make with each other, in general, *two* unequal and supplementary *angles*: but we here suppose that these are mutually *distinguished*, by taking account of the *aspect* of each plane, as distinguished from the *opposite* aspect: which is most easily done (111), by considering the *axes* as above.

Then, by 193, 194, and by 204, XXXV., XXXV'.,

$$\text{I.} \ldots \mathrm{V}q' : \mathrm{V}q = \mathrm{IV}q' : \mathrm{IV}q = + (\mathrm{TV}q' : \mathrm{TV}q) \, . \, (\mathrm{Ax} \, . \, q' : \mathrm{Ax.} \, q) \, ;$$

$$\text{II.} \ldots \mathrm{V}q' \, . \, \mathrm{V}q = \mathrm{IV}q' : \mathrm{I}\frac{1}{\mathrm{V}q} = - (\mathrm{TV}q' \, . \, \mathrm{TV}q) \, . \, (\mathrm{Ax} \, . \, q' : \mathrm{Ax.} \, q) \, ;$$

and therefore (comp. 198), with the temporary abridgments proposed above,

$$\text{III.} \ldots \mathrm{S}(\mathrm{V}q' : \mathrm{V}q) = t't^{-1} \cos x ; \quad \text{IV.} \ldots \mathrm{SU}(\mathrm{V}q' : \mathrm{V}q) = + \cos x ;$$

$$\text{V.} \ldots \mathrm{S}(\mathrm{V}q' \, . \, \mathrm{V}q) = - t't \cos x ; \quad \text{VI.} \ldots \mathrm{SU}(\mathrm{V}q' \, . \, \mathrm{V}q) = - \cos x ;$$

$$\text{VII.} \ldots \angle(\mathrm{V}q' \, . \, \mathrm{V}q) = x ; \qquad \text{VIII.} \ldots \angle(\mathrm{V}q' \, . \, \mathrm{V}q) = \pi - x.$$

We have also generally (comp. 204, XVIII., XIX.),

$$\text{IX.} \ldots \mathrm{TV}(\mathrm{V}q' : \mathrm{V}q) = t't^{-1} \sin x ; \quad \text{X.} \ldots \mathrm{TVU}(\mathrm{V}q' : \mathrm{V}q) = \sin x ;$$

$$\text{XI.} \ldots \mathrm{TV}(\mathrm{V}q' \, . \, \mathrm{V}q) = t't \sin x ; \quad \text{XII.} \ldots \mathrm{TVU}(\mathrm{V}q' \, . \, \mathrm{V}q) = \sin x ;$$

and in particular,

$$\text{XIII.} \ldots \mathrm{V}(\mathrm{V}q' : \mathrm{V}q) = 0, \quad \text{and} \quad \text{XIV.} \ldots \mathrm{V}(\mathrm{V}q' \, . \, \mathrm{V}q) = 0,$$

$$\text{if } q' \parallel\!| \, q \; (123) ;$$

because (comp. 191, (6.), and 204, VI.) the quotient or product of the right parts of two *complanar* quaternions (supposed here to be both *non-scalar* (108), so that t and t' are each > 0) degenerates (131) into a *scalar*, which may be thus expressed:

$$\text{XV.} \ldots \mathrm{V}q' : \mathrm{V}q = + t't^{-1}, \quad \text{and} \quad \text{XVI.} \ldots \mathrm{V}q' \, . \, \mathrm{V}q = - t't, \quad \text{if } x = 0 ;$$

but

$$\text{XVII.} \ldots \mathrm{V}q' : \mathrm{V}q = - t't^{-1}, \text{ and } \text{XVIII.} \ldots \mathrm{V}q' \, . \, \mathrm{V}q = + t't, \quad \text{if } x = \pi ;$$

the first case being that of *coincident*, and the second case that of *opposite axes*. In the more general case of *diplanarity* (119), if we denote by δ the unit-line which is *perpendicular to both their axes*, and therefore *common to their two planes*, or in which those planes *intersect*, and which is so directed that the *rotation* round it from $\mathrm{Ax} \, . \, q$ to $\mathrm{Ax} \, . \, q'$ is positive (comp. 127, 128), the recent formulæ I., II., give easily,

$$\text{XIX.} \ldots \mathrm{Ax} \, . \, (\mathrm{V}q' : \mathrm{V}q) = + \delta ; \quad \text{XX.} \ldots \mathrm{Ax} \, . \, (\mathrm{V}q' \, . \, \mathrm{V}q) = - \delta ;$$

and therefore (by IX., XI., and by 204, XXXV.), the *indices of the right parts*, of the quotient and product of the right parts of any two diplanar quaternions, may be expressed as follows:

$$\text{XXI.} \ldots \mathrm{IV}(\mathrm{V}q' : \mathrm{V}q) = + \delta \, . \, t't^{-1} \sin x ;$$

$$\text{XXII.} \ldots \mathrm{IV}(\mathrm{V}q' \, . \, \mathrm{V}q) = - \delta \, . \, t't \sin x.$$

(1.) Let ABC be any triangle upon the unit-sphere (128), of which the spherical *angles* and the *corners* may be denoted by the same letters A, B, C, while the *sides* shall as usual be denoted by a, b, c; and let it be supposed that the *rotation* (comp. 177) round A from C to B, and therefore that round B from A to C, &c., is *positive*, as in fig. 43 [p. 144]. Then writing, as we have often done,

$$q = \beta : \alpha, \quad \text{and} \quad q' = \gamma : \beta, \quad \text{where} \quad \alpha = \text{OA}, \ \&\text{c.},$$

we easily obtain the following expressions for the three scalars t, t', x, and for the vector δ:

$$t = \sin c; \quad t' = \sin a; \quad x = \pi - \text{B}; \quad \delta = -\beta.$$

(2.) In fact we have here,

$$\text{T}q = \text{T}q' = 1, \quad \angle\, q = c, \quad \angle\, q' = a;$$

whence t and t' are as just stated. Also if A′, B′, C′ be (as in 175) the *positive poles* of the three *successive sides* BC, CA, AB, of the given triangle, and therefore the points A, B, C the *negative poles* (comp. 180, (2.)) of the new arcs B′C′, C′A′, A′B′, then

$$\text{Ax}\,.\,q = \text{OC}', \quad \text{Ax}\,.\,q' = \text{OA}';$$

but x and δ are the *angle* and the *axis* of the *quotient* of these two axes, or of the quaternion which is *represented* (162) by the *arc* C′A′; therefore x is, as above stated, the *supplement* of the *angle* B, and δ is directed to the point upon the sphere, which is diametrically *opposite* to the *point* B.

(3.) Hence, by III. V. VII. VIII. IX. XI., for any triangle ABC on the unit-sphere, with $\alpha = \text{OA}$, &c., we have the formulæ:

$$\text{XXIII.}\ldots \text{S}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = -\sin a \operatorname{cosec} c \cos \text{B};$$

$$\text{XXIV.}\ldots \text{S}\left(\text{V}\frac{\gamma}{\beta} \,.\, \text{V}\frac{\beta}{\alpha}\right) = +\sin a \sin c \cos \text{B};$$

$$\text{XXV.}\ldots \angle\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = \pi - \text{B}; \qquad \text{XXVI.}\ldots \angle\left(\text{V}\frac{\gamma}{\beta} \,.\, \text{V}\frac{\beta}{\alpha}\right) = \text{B};$$

$$\text{XXVII.}\ldots \text{TV}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = +\sin a \operatorname{cosec} c \sin \text{B};$$

$$\text{XXVIII.}\ldots \text{TV}\left(\text{V}\frac{\gamma}{\beta} \,.\, \text{V}\frac{\beta}{\alpha}\right) = +\sin a \sin c \sin \text{B}.$$

(4.) Also, by XIX. XX. XXI. XXII., if the rotation round B from A to C be still *positive*,

$$\text{XXIX.} \ldots \mathrm{Ax.}\left(\mathrm{V}\frac{\gamma}{\beta} : \mathrm{V}\frac{\beta}{\alpha}\right) = -\beta; \quad \text{XXX.} \ldots \mathrm{Ax.}\left(\mathrm{V}\frac{\gamma}{\beta} \cdot \mathrm{V}\frac{\beta}{\alpha}\right) = +\beta;$$

$$\text{XXXI.} \ldots \mathrm{IV}\left(\mathrm{V}\frac{\gamma}{\beta} : \mathrm{V}\frac{\beta}{\alpha}\right) = -\beta \sin a \operatorname{cosec} c \sin \mathrm{B};$$

$$\text{XXXII.} \ldots \mathrm{IV}\left(\mathrm{V}\frac{\gamma}{\beta} \cdot \mathrm{V}\frac{\beta}{\alpha}\right) = +\beta \sin a \sin c \sin \mathrm{B}.$$

(5.) If, on the other hand, the rotation round B from A to C were *negative*, then writing for a moment $\alpha_1 = -\alpha$, $\beta_1 = -\beta$, $\gamma_1 = -\gamma$, we should have a new and *opposite triangle*, $\mathrm{A_1B_1C_1}$, in which the rotation round $\mathrm{B_1}$ from $\mathrm{A_1}$ to $\mathrm{C_1}$ would be *positive*, but the angle at $\mathrm{B_1}$ equal in magnitude to that at B; so that by treating (as usual) all the angles of a spherical triangle as positive, we should have $\mathrm{B_1} = \mathrm{B}$, as well as $c_1 = c$, and $a_1 = a$; and therefore, for example, by XXXI.,

$$\mathrm{IV}\left(\mathrm{V}\frac{\gamma_1}{\beta_1} : \mathrm{V}\frac{\beta_1}{\alpha_1}\right) = -\beta_1 \sin a_1 \operatorname{cosec} c_1 \sin \mathrm{B_1},$$

$$\text{or} \quad \mathrm{IV}\left(\mathrm{V}\frac{\gamma}{\beta} : \mathrm{V}\frac{\beta}{\alpha}\right) = +\beta \sin a \operatorname{cosec} c \sin \mathrm{B};$$

the four formulæ of (4.) would therefore still subsist, provided that, for this new direction of rotation in the given triangle, we were to *change the sign of* β, *in the second member of each.*

(6.) Abridging, generally $\mathrm{IV}q : \mathrm{S}q$ to $(\mathrm{IV} : \mathrm{S})\, q$, as $\mathrm{TV}q : \mathrm{S}q$ was abridged, in 204, XXXIV'., to $(\mathrm{TV} : \mathrm{S})\, q$, we have by (5.), and by XXIV., XXXII., this other general formula, for any three unit-vectors α, β, γ, considered still as terminating at the corners of a spherical triangle ABC:

$$\text{XXXIII.} \ldots (\mathrm{IV} : \mathrm{S})\left(\mathrm{V}\frac{\gamma}{\beta} \cdot \mathrm{V}\frac{\beta}{\alpha}\right) = \pm\beta \tan \mathrm{B};$$

the upper or the lower sign being taken, according as the rotation round B from A to C, or that round β from α to γ, which might perhaps be denoted by the symbol $\alpha\hat{\beta}\gamma$, and which in quantity is equal to the spherical angle B, is positive or negative.

209. When the *planes* of any *three* quaternions q, q', q'', considered as all passing through the origin O (119), contain any *common line*, those three may then be said to be *Collinear* Quaternions*; and because the *axis* of each is then

* Quaternions of which the planes are *parallel* to any common line may also be said to be *collinear.* Compare the first Note to page 116.

perpendicular to that line, it follows that *the Axes of Collinear Quaternions are Complanar*: while conversely, the *complanarity of the axes* insures the *collinearity of the quaternions,* because the *perpendicular to the plane* of the axes is a line common to the planes of the quaternions.

(1.) Complanar quaternions are always collinear; but the converse proposition does not hold good, collinear quaternions being not necessarily complanar.

(2.) Collinear quaternions, considered as *fractions* (101), can always be reduced to a *common denominator* (120); and conversely, if three or more quaternions *can* be so reduced, as to appear under the form of fractions with a common denominator ϵ, those quaternions must be *collinear*: because the line ϵ is then common to all their planes.

(3.) *Any two quaternions* are *collinear with any scalar*; the *plane of* a scalar being *indeterminate** (131).

(4.) Hence the *scalar and right parts*, Sq, Sq', Vq, Vq', of *any two quaternions*, are always *collinear* with each other.

(5.) The *conjugates* of collinear quaternions are themselves collinear.

210. Let q, q', q'' be any three collinear quaternions; and let α denote a line common to their planes. Then we may determine (comp. 120) three other lines β, γ, δ, such that

$$q = \frac{\beta}{\alpha}, \qquad q' = \frac{\gamma}{\alpha}, \qquad q'' = \frac{\alpha}{\delta};$$

and thus may conclude that (as in algebra),

$$\text{I.} \ldots (q' + q)\, q'' = q'q'' + qq'',$$

because, by 106, 107,

$$\left(\frac{\gamma}{\alpha} + \frac{\beta}{\alpha}\right)\frac{\alpha}{\delta} = \frac{\gamma+\beta}{\alpha} \cdot \frac{\alpha}{\delta} = \frac{\gamma+\beta}{\delta} = \frac{\gamma}{\delta} + \frac{\beta}{\delta} = \frac{\gamma}{\alpha}\frac{\alpha}{\delta} + \frac{\beta}{\alpha}\frac{\alpha}{\delta}.$$

In like manner, at least under the same condition of collinearity,† it may be proved that

$$\text{II.} \ldots (q' - q)\, q'' = q'q'' - qq''.$$

Operating by the characteristic K upon these two equations, and attending to 192, II., and 195, II., we find that

$$\text{III.} \ldots Kq''. (Kq' + Kq) = Kq''. Kq' + Kq''. Kq;$$

$$\text{IV.} \ldots Kq''. (Kq' - Kq) = Kq''. Kq' - Kq''. Kq;$$

where (by 209, (5.)) the three *conjugates* of arbitrary collinears, Kq, Kq', Kq'',

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* Compare the Note to page 117.

† It will soon be seen, however, that this condition is unnecessary.

may represent *any three* collinear quaternions. We have, therefore, with the same degree of generality as before,

$$\text{V.} \ldots q''(q'+q) = q''q' + q''q; \qquad \text{VI.} \ldots q''(q'-q) = q''q' - q''q.$$

If, then, q, q', q'', q''' be *any four collinear quaternions*, we may establish the formula (again agreeing with algebra):

$$\text{VII.} \ldots (q'''+q'')(q'+q) = q'''q' + q''q' + q'''q + q''q;$$

and similarly for any greater number, so that we may write briefly,

$$\text{VIII.} \ldots \Sigma q' \,.\, \Sigma q = \Sigma q'q,$$

where $\qquad \Sigma q' = q_1 + q_2 + \ldots + q_m, \qquad \Sigma q' = q'_1 + q'_2 + \ldots + q'_n,$

and $\qquad \Sigma q'q = q'_1 q_1 + \ldots q'_1 q_m + q'_2 q_1 + \ldots + q'_n q_m,$

m and n being any positive whole numbers. In words (comp. 13.), *the Multiplication of Collinear* Quaternions is a Doubly Distributive Operation.*

(1.) Hence, by 209, (4.), and 202, III., we have this general transformation, for the *product of any two quaternions*:

$$\text{IX.} \ldots q'q = \text{S}q' \,.\, \text{S}q + \text{V}q' \,.\, \text{S}q + \text{S}q' \,.\, \text{V}q + \text{V}q' \,.\, \text{V}q.$$

(2.) Hence also, for the *square of any quaternion*, we have the transformation† (comp. 126; 199, VII.; and 204, XXIII.):

$$\text{X.} \ldots q^2 = \text{S}q^2 + 2\text{S}q \,.\, \text{V}q + \text{V}q^2.$$

(3.) *Separating* the scalar and right *parts* of this last expression, we find these other general formulæ:

$$\text{XI.} \ldots \text{S} \,.\, q^2 = \text{S}q^2 + \text{V}q^2; \qquad \text{XII.} \ldots \text{V} \,.\, q^2 = 2\text{S}q \,.\, \text{V}q;$$

whence also, dividing by $\text{T}q^2$, we have

$$\text{XIII.} \ldots \text{SU}(q^2) = (\text{SU}q)^2 + (\text{VU}q)^2; \qquad \text{XIV.} \ldots \text{VU}(q^2) = 2\text{SU}q \,.\, \text{VU}q.$$

(4.) By supposing $q' = \text{K}q$, in IX., and therefore $\text{S}q' = \text{S}q$, $\text{V}q' = -\text{V}q$, and transposing the two conjugate and therefore complanar factors (comp. 191, (1.)), we obtain this general transformation for a *norm*, or for the *square* of a *tensor* (comp. 190, V.; 202, III.; and 204, XI.):

$$\text{XV.} \ldots \text{T}q^2 = \text{N}q = q\text{K}q = (\text{S}q + \text{V}q)(\text{S}q - \text{V}q) = \text{S}q^2 - \text{V}q^2;$$

which had indeed presented itself before (in 204, XXII.), but is now obtained

* This *distributive property of multiplication* will soon be found (compare the last Note) to extend to the more general case, in which the quaternions are *not collinear*.

† [By means of the formulæ of 204 many different transformations involving K, S, V, and T may be effected on a square or product.]

in a new way, and *without any employment of sines, or cosines,* or even of the well-known theorem respecting the *square of the hypotenuse.*

(5.) Eliminating Vq^2, by XV., from XI., and dividing by Tq^2, we find that

$$\text{XVI.} \ldots \mathrm{S}\,.\,q^2 = 2\mathrm{S}q^2 - \mathrm{T}q^2; \quad \text{XVII.} \ldots \mathrm{SU}(q^2) = 2(\mathrm{SU}q)^2 - 1;$$

agreeing with 199, VI. and IV., but obtained here without any use of the known formula for the *cosine of the double* of an angle.

(6.) Taking the scalar and right parts of the expression IX., we obtain these other general expressions:

$$\text{XVIII.} \ldots \mathrm{S}q'q = \mathrm{S}q'\,.\,\mathrm{S}q + \mathrm{S}(\mathrm{V}q'.\mathrm{V}q);$$

$$\text{XIX.} \ldots \mathrm{V}q'q = \mathrm{V}q'\,.\,\mathrm{S}q + \mathrm{V}q\,.\,\mathrm{S}q' + \mathrm{V}(\mathrm{V}q'.\,\mathrm{V}q);$$

in the latter of which we may (by 126) transpose the two factors $\mathrm{V}q'$, $\mathrm{S}q$, or $\mathrm{V}q$, $\mathrm{S}q'$. We may also (by 206, 207) write, instead of XIX., this other formula:

$$\text{XIX}'. \ldots \mathrm{IV}q'q = \mathrm{IV}q'\,.\,\mathrm{S}q + \mathrm{IV}q\,.\,\mathrm{S}q' + \mathrm{IV}(\mathrm{V}q'.\,\mathrm{V}q).$$

(7.) If we suppose, in VII., that $q'' = \mathrm{K}q$, $q''' = \mathrm{K}q'$, and transpose (comp. (4.)) the two complanar (because conjugate) factors, $q' + q$ and $\mathrm{K}(q' + q)$, we obtain the following general expression for the *norm of a sum*:

$$(q' + q)\,\mathrm{K}(q' + q) = q'\mathrm{K}q' + q\mathrm{K}q' + q'\mathrm{K}q + q\mathrm{K}q;$$

or briefly,

$$\text{XX.} \ldots \mathrm{N}(q' + q) = \mathrm{N}q' + 2\mathrm{S}\,.\,q\mathrm{K}q' + \mathrm{N}q, \text{ as in 200, VII.};$$

because

$$q'\mathrm{K}q = \mathrm{K}\,.\,q\mathrm{K}q', \text{ by 192, II., and } (1 + \mathrm{K})\,.\,q\mathrm{K}q' = 2\mathrm{S}\,.\,q\mathrm{K}q', \text{ by 196, II}'.$$

(8.) By changing q' to x in XX., or by forming the product of $q + x$ and $\mathrm{K}q + x$, where x is any scalar, we find that

$$\text{XXI.} \ldots \mathrm{N}(q + x) = \mathrm{N}q + 2x\mathrm{S}q + x^2, \text{ as in 200, VIII.};$$

whence, in particular,

$$\text{XXI}'. \ldots \mathrm{N}(q - 1) = \mathrm{N}q - 2\mathrm{S}q + 1, \text{ as in 200, II.}$$

(9.) Changing q to $\beta : \alpha$, and multiplying by the square of $\mathrm{T}\alpha$, we get, for any two vectors, α and β, the formula,

$$\text{XXII.} \ldots \mathrm{T}(\beta - \alpha)^2 = \mathrm{T}\beta^2 - 2\mathrm{T}\beta\,.\,\mathrm{T}\alpha\,.\,\mathrm{SU}\frac{\beta}{\alpha} + \mathrm{T}\alpha^2,$$

in which $T\alpha^2$ denotes* $(T\alpha)^2$; because (by 190, and by 196, IX.),

$$N\left(\frac{\beta}{\alpha}-1\right)=N\frac{\beta-\alpha}{\alpha}=\left(\frac{T(\beta-\alpha)}{T\alpha}\right)^2, \quad \text{and} \quad S\frac{\beta}{\alpha}=\frac{T\beta}{T\alpha}SU\frac{\beta}{\alpha}.$$

(10.) In any plane triangle, ABC, with sides of which the *lengths* are as usual denoted by a, b, c, let the vertex C be taken as the origin O of vectors; then $\alpha = \text{CA}$, $\beta = \text{CB}$, $\beta - \alpha = \text{AB}$, $T\alpha = b$, $T\beta = a$, $T(\beta - \alpha) = c$, $SU\frac{\beta}{\alpha} = \cos \text{C}$; we *recover* therefore, from XXII., the *fundamental formula of plane trigonometry*, under the form

$$\text{XXIII.} \ldots c^2 = a^2 - 2ab \cos \text{C} + b^2.$$

(11.) It is important to observe that we have not here been arguing in a circle; because although, in Art. 200, we assumed, for the convenience of the student, a previous knowledge of the last written formula, in order to arrive more rapidly at certain applications, yet in these recent deductions from the *distributive property* VIII. *of multiplication* of (at least) collinear quaternions, we have founded nothing on the results of that former Article; and have made no use of any properties of *oblique-angled triangles*, or even of *right-angled* ones, since the theorem of the square of the hypotenuse has been virtually proved anew in (4.): nor is it necessary to the argument, that any *properties* of trigonometric *functions* should be known, beyond the mere *definition* of a *cosine*, as a certain *projecting factor*, from which the formula 196, XVI. was derived, and which justifies us in writing cos C in the last equation (10.). The geometrical Examples, in the sub-articles to 200, may therefore be read again, and their validity be seen anew, *without any appeal to even plane trigonometry* being now supposed.

(12.) The formula XV. gives $Sq^2 = Tq^2 + Vq^2$, as in 204, XXXI.; and we know that Vq^2, as being generally the *square of a right quaternion*, is equal to a *negative scalar* (comp. 204, VI.), so that

$$\text{XXIV.} \ldots Vq^2 < 0, \quad \text{unless} \quad \angle q = 0, \quad \text{or} \quad = \pi,$$

in each of which two cases $Vq = 0$, by 202, (6.), and therefore its square vanishes; hence,

$$\text{XXV.} \ldots Sq^2 < Tq^2, \quad (SUq)^2 < 1,$$

in every other case.

* We are not *yet* at liberty to interpret the symbol $T\alpha^2$ as denoting *also* $T(\alpha^2)$; because we have not yet assigned *any* meaning to the *square of a vector*, or generally to the *product of two vectors*. In the Third Book of these Elements [282 (3.)] it will be shown, that such a square or product can be interpreted as being a *quaternion*: and then it will be found (comp. 190), that

$$T(\alpha^2) = (T\alpha)^2 = T\alpha^2,$$

whatever vector α may be.

(13.) It might therefore have been thus proved, without any use of the transformation $SUq = \cos \angle q$ (196, XVI.), that (for any *real* quaternion q) we may have the inequalities,

$$\text{XXVI.} \ldots SUq < +1, \quad SUq > -1, \quad \text{and} \quad Sq < +Tq, \quad Sq > -Tq,$$

unless it happen that $\angle q = 0$, or $= \pi$; SUq being $= +1$, and $Sq = +Tq$, in the first case; whereas $SUq = -1$, and $Sq = -Tq$, in the second case.

(14.) Since $Tq^2 = Nq$, and $Tq \,.\, Tq' = T \,.\, qKq' = T \,.\, q'Kq = Nq \,.\, T\,(q' : q)$, while $S \,.\, qKq' = S \,.\, q'Kq = Nq \,.\, S\,(q' : q)$, the formula XX. gives, by XXVI.,

$$\text{XXVII.} \ldots (Tq' + Tq)^2 - T\,(q' + q)^2 = 2\,(T - S)\,qKq' = 2Nq \,.\, (T - S)(q' : q) > 0,$$

if we adopt the abridged notation,

$$\text{XXVIII.} \ldots Tq - Sq = (T - S)\,q,$$

and suppose that the quotient $q' : q$ is *not* a positive scalar; hence,

$$\text{XXIX.} \ldots Tq' + Tq > T\,(q' + q), \quad \text{unless} \quad q' = xq, \quad \text{and} \quad x > 0;$$

in which excepted case, each member of this last inequality becomes $= (1 + x)\,Tq$.

(15.) Writing $q = \beta : \alpha$, $q' = \gamma : \alpha$, and multiplying by $T\alpha$, the formula XXIX. becomes,

$$\text{XXX.} \ldots T\gamma + T\beta > T\,(\gamma + \beta), \quad \text{unless} \quad \gamma = x\beta, \quad x > 0;$$

in which latter case, but not in any other, we have $U\gamma = U\beta$ (155). We therefore arrive anew at the results of 186, (9.), (10.), but without its having been necessary to consider any *triangle*, as was done in those former sub-articles.

(16.) On the other hand, with a corresponding abridgment of notation, we have, by XXVI.,

$$\text{XXXI.} \ldots Tq + Sq = (T + S)\,q > 0, \quad \text{unless} \quad \angle q = \pi;$$

also, by XX., &c.,

$$\text{XXXII.} \ldots T\,(q' + q)^2 - (Tq' - Tq)^2 = 2\,(T + S)\,qKq' = 2Nq \,.\, (T + S)(q' : q);$$

hence,

$$\text{XXXIII.} \ldots T\,(q' + q) > \pm\,(Tq' - Tq), \quad \text{unless} \quad q' = -xq, \quad x > 0;$$

where either sign may be taken.

(17.) And hence, on the plan of (15.), for any two vectors β, γ,

$$\text{XXXIV.} \ldots T\,(\gamma + \beta) > \pm\,(T\gamma - T\beta), \quad \textit{unless} \quad U\gamma = -U\beta,$$

whichever sign he adopted; but, on the contrary,

$$\text{XXXV.} \ldots \mathrm{T}(\gamma + \beta) = \pm (\mathrm{T}\gamma - \mathrm{T}\beta), \quad \textit{if} \quad \mathrm{U}\gamma = -\mathrm{U}\beta,$$

the upper or the lower sign being taken, according as $\mathrm{T}\gamma >$ or $< \mathrm{T}\beta$: all which agrees with what was inferred, in 186, (11.), from *geometrical* considerations alone, combined with the definition of $\mathrm{T}\alpha$. In fact, if we make $\beta = \mathrm{OB}$, $\gamma = \mathrm{OC}$, and $-\gamma = \mathrm{OC}'$, then OBC' will be in general *a plane triangle*, in which the length of the side BC' *exceeds* the difference of the lengths of the two other sides; but if it happen that the directions of the two lines OB, OC' coincide, or in other words that the lines OB, OC have opposite directions, then the difference of lengths of these two lines becomes *equal* to the length of the line BC'.

(18.) With the representations of q and q', assigned in 208, (1.), by two sides of a *spherical triangle* ABC, we have the values,

$$\mathrm{S}q = \cos c, \qquad \mathrm{S}q' = \cos a, \qquad \mathrm{S}q'q = \mathrm{S}(\gamma : \alpha) = \cos b;$$

the equation XVIII. gives therefore, by 208, XXIV., *the fundamental formula of spherical trigonometry* (comp. (10.)), as follows:

$$\text{XXXVI.} \ldots \cos b = \cos a \cos c + \sin a \sin c \cos \mathrm{B}.$$

(19.) To interpret, with reference to the same spherical triangle, the connected equation XIX., or XIX′, let it be now supposed, as in 208, (5.), that the rotation round B from C to A is positive, so that B and B′ are situated at the same side of the arc CA, if B′ be still, as in 208, (2.), the positive pole of that arc. Then writing $\alpha' = \mathrm{OA}'$, &c., we have

$$\mathrm{IV}q = \gamma' \sin c; \qquad \mathrm{IV}q' = \alpha' \sin a; \qquad \mathrm{IV}q'q = -\beta' \sin b;$$

and $$\mathrm{IV}(\mathrm{V}q'.\mathrm{V}q) = -\beta \sin a \sin c \sin \mathrm{B} \text{ (comp. 208, (5.))},$$

with the recent values (18), for $\mathrm{S}q$ and $\mathrm{S}q'$; thus the formula XIX′. becomes, by transposition of the two terms last written:

$$\text{XXXVII.} \ldots \beta \sin a \sin c \sin \mathrm{B} = \alpha' \sin a \cos c + \beta' \sin b + \gamma' \sin c \cos a.$$

(20.) Let $\rho = \mathrm{OP}$ be any unit-vector; then, dividing each term of the last equation by ρ, and taking the scalar of each of the four quotients, we have, by 196, XVI., this new equation:

$$\text{XXXVIII.} \ldots \sin a \sin c \sin \mathrm{B} \cos \mathrm{PB} = \sin a \cos c \cos \mathrm{PA}' + \sin b \cos \mathrm{PB}'$$
$$+ \sin c \cos a \cos \mathrm{PC}';$$

where a, b, c are as usual the sides of the spherical triangle ABC, and A′, B′, C′ are still, as in 208, (2.), the positive poles of those sides; but P is an arbitrary point, upon the surface of the sphere. Also cos PA′, cos PB′, cos PC′ are evidently the sines of the arcual perpendiculars let fall from that point upon those sides; being positive when P is, relatively to them, in the same hemispheres as the opposite corners of the triangle, but negative in the contrary case; so that cos AA′, &c., are positive, and are the sines of the three altitudes of the triangle.

(21.) If we place P at B, two of these perpendiculars vanish, and the last formula becomes, by 208, XXVIII.,

$$\text{XXXIX.} \ldots \sin b \cos \mathrm{BB}' = \sin a \sin c \sin \mathrm{B} = \mathrm{TV}\left(\mathrm{V}\frac{\gamma}{\beta} \cdot \mathrm{V}\frac{\beta}{\alpha}\right);$$

such then is the quaternion expression for the product of the sine of the side CA, multiplied by the sine of the perpendicular let fall upon that side, from the opposite vertex B.

(22.) Placing P at A, dividing by $\sin a \cos c$, and then interchanging B and C, we get this other fundamental formula of spherical trigonometry,

$$\text{XL.} \ldots \cos \mathrm{AA}' = \sin c \sin \mathrm{B} = \sin b \sin \mathrm{C};$$

and we see that this is included in the interpretation of the quaternion equation XIX., or XIX′., as the formula XXXVI. was seen in (18.) to be the interpretation of the connected equation XVIII.

(23.) By assigning other positions to P, other formulæ of spherical trigonometry may be deduced, from the recent equation XXXVIII. Thus if we suppose P to coincide with B′, and observe that (by the supplementary* triangle),

$$\mathrm{B'C'} = \pi - \mathrm{A}, \quad \mathrm{C'A'} = \pi - \mathrm{B}, \quad \mathrm{A'B'} = \pi - \mathrm{C},$$

while

$$\cos \mathrm{BB}' = \sin a \sin \mathrm{C} = \sin c \sin \mathrm{A}, \text{ by XL.},$$

we easily deduce the formula,

$$\text{XLI.} \ldots \sin a \sin c \sin \mathrm{A} \sin \mathrm{B} \sin \mathrm{C} = \sin \mathrm{B} - \cos c \cos \mathrm{C} \sin \mathrm{A} - \cos a \cos \mathrm{A} \sin \mathrm{C};$$

which obviously agrees, at the plane limit, with the elementary relation,

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = \pi.$$

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* *No previous knowledge of Spherical Trigonometry*, properly so called, is here supposed; the supplementary relations of two polar triangles to each other forming rather a part, and a very elementary one, of *spherical geometry*.

(24.) Again, by placing P at A′, the general equation becomes,

$$\text{XLII.} \ldots \sin a \cos c = \sin b \cos \mathrm{C} + \sin c \cos a \cos \mathrm{B};$$

with the verification that, at the plane limit,

$$a = b \cos \mathrm{C} + c \cos \mathrm{B}.$$

But we cannot here delay on such deductions, or verifications: although it appeared to be worth while to point out, that the whole of spherical trigonometry may thus be developed, from the fundamental equation of multiplication of quaternions (107), when that equation is operated on by the two characteristics S and V, and the results interpreted as above.

211. It may next be proved, as follows, that the distributive formula I. of the last Article holds good, when the three quaternions, q, q', q'', which enter into it, without being now necessarily *collinear*, are *right*; in which case their *reciprocals* (135), and their *sums* (197, (2.)), will be right also. Let then

$$\angle q = \angle q' = \angle q'' = \frac{\pi}{2}, \quad qq_{,} = 1;$$

and therefore,

$$\angle q_{,} = \angle (q'' + q') = \frac{\pi}{2}.$$

We shall then have, by 106, 194, 206,

$$(q'' + q')\, q = \mathrm{I}\,(q'' + q') : \mathrm{I}q_{,}$$

$$= (\mathrm{I}q'' : \mathrm{I}q_{,}) + (\mathrm{I}q' : \mathrm{I}q_{,}) = q''q + q'q;$$

and the distributive property in question is proved.

(1.) By taking conjugates, as in 210, it is easy hence to infer, that the *other* distributive formula, 210, V., holds good for any three right quaternions; or that

$$q\,(q'' + q') = qq'' + qq', \quad \text{if} \quad \angle q = \angle q' = \angle q'' = \frac{\pi}{2}.$$

(2.) For *any three* quaternions, we have therefore the two equations:

$$(\mathrm{V}q'' + \mathrm{V}q') \,.\, \mathrm{V}q = \mathrm{V}q''.\mathrm{V}q + \mathrm{V}q'.\mathrm{V}q;$$

$$\mathrm{V}q \,.\, (\mathrm{V}q'' + \mathrm{V}q') = \mathrm{V}q \,.\, \mathrm{V}q'' + \mathrm{V}q \,.\, \mathrm{V}q'.$$

(3.) The quaternions q, q', q'' being still arbitrary, we have thus, by 210, IX.,

$$(q''+q')\,q = (\mathrm{S}q''+\mathrm{S}q')\,.\,\mathrm{S}q + (\mathrm{V}q''+\mathrm{V}q')\,.\,\mathrm{S}q + \mathrm{V}q\,.\,(\mathrm{S}q''+\mathrm{S}q') + (\mathrm{V}q''+\mathrm{V}q')\,.\,\mathrm{V}q$$

$$= (\mathrm{S}q''.\mathrm{S}q + \mathrm{V}q''.\mathrm{S}q + \mathrm{V}q\,.\,\mathrm{S}q'' + \mathrm{V}q''.\mathrm{V}q) + (\mathrm{S}q'.\,\mathrm{S}q + \mathrm{V}q'.\,\mathrm{S}q + \mathrm{V}q\,.\,\mathrm{S}q' + \mathrm{V}q'.\mathrm{V}q)$$

$$= q''q + q'q;$$

so that the formula 210, I., and therefore also (by conjugates) the formula 210, V., is valid *generally*.

212. The *General* Multiplication of Quaternions* is therefore (comp. 13, 210) a *Doubly Distributive Operation*; so that we may *extend*, to quaternions *generally*, the formula (comp. 210, VIII.),

$$\text{I.} \ldots \Sigma q' . \Sigma q = \Sigma q'q :$$

however many the summands of each set may be, and whether they be, or be not, *collinear* (209), or *right* (211).

(1.) Hence, as an extension of 210, XX., we have now,

$$\text{II.} \ldots \mathrm{N}\Sigma q = \Sigma \mathrm{N} q + 2\Sigma \mathrm{S} q \mathrm{K} q' ;$$

where the second sign of summation refers to all possible binary combinations of the quaternions $q, q', \ldots$

(2.) And, as an extension of 210, XXIX., we have the inequality,

$$\text{III.} \ldots \Sigma \mathrm{T} q > \mathrm{T}\Sigma q,$$

unless *all* the quaternions $q, q', \ldots$ bear *scalar* and *positive* ratios to each other, in which case the two members of this inequality become equal: so that the *sum of the tensors*, of any set of quaternions, is *greater than the tensor of the sum*, in every other case.

(3.) In general, as an extension of 210, XXVII.,

$$\text{IV.} \ldots (\Sigma \mathrm{T} q)^2 - (\mathrm{T}\Sigma q)^2 = 2\Sigma\, (\mathrm{T} - \mathrm{S})\, q \mathrm{K} q' .$$

(4.) The formulæ, 210, XVIII., XIX., admit easily of analogous extensions.

(5.) We have also (comp. 168) the general equation,

$$\text{V.} \ldots (\Sigma q)^2 - \Sigma\, (q^2) = \Sigma\, (qq' + q'q) ;$$

in which, by 210, IX.,

$$\text{VI.} \ldots qq' + q'q = 2\,(\mathrm{S}q \,.\, \mathrm{S}q' + \mathrm{V}q \,.\, \mathrm{S}q' + \mathrm{V}q' . \mathrm{S}q + \mathrm{S}\,(\mathrm{V}q' . \mathrm{V}q)) ;$$

because, by 208, we have generally

$$\text{VII.} \ldots \mathrm{V}\,(\mathrm{V}q' . \mathrm{V}q) = -\mathrm{V}\,(\mathrm{V}q \,.\, \mathrm{V}q') ;$$

or $$\text{VIII.} \ldots \mathrm{V}q'q = -\mathrm{V}qq', \quad \text{if} \quad \angle q = \angle q' = \frac{\pi}{2} .$$

(Comp. 191, (2.), and 204, X.)

* Compare the Notes to pages 211 and 212. [On page (35) of the Preface to the "Lectures on Quaternions," Hamilton refers to an early speculation of his (1831) on the multiplication of lines for which the product of sums was *not* equal to the sum of products. When addition is not commutative, multiplication even by a scalar is not distributive. See 180 (3.)]

213. Besides the advantage which the Calculus of Quaternions gains, from the general establishment (212) of the *Distributive Principle*, or *Distributive Property of Multiplication*, by being, so far, *assimilated to Algebra*, in processes which are of continual occurrence, this principle or property will be found to be of great importance, in applications of that calculus to *Geometry*; and especially in questions respecting the (real or ideal*) *intersections of right lines with spheres*, or other surfaces of the second order, including *contacts* (real or ideal), as *limits* of such intersections. The following examples may serve to give some notion, how the general distributive principle admits of being applied to such questions: in some of which however the less general principle (210), respecting the multiplication of *collinear* quaternions (209), would be sufficient. And first we shall take the case of *chords of a sphere*, drawn from a given point upon its surface.

(1.) From a point A, of a sphere with O for centre, let it be required to draw a chord AP, which shall be parallel to a given line OB; or more fully, to *assign the vector*, ρ = OP, of the extremity of the chord so drawn, *as a function of the two given vectors*, α = OA, and β = OB; or rather of α and $U\beta$, since it is evident that the *length* of the line β cannot affect the result of the construction, which fig. 51 may serve to illustrate.

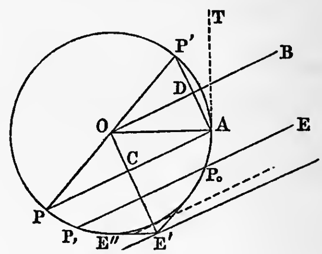

Fig. 51.

(2.) Since AP ∥ OB, or $\rho - \alpha \parallel \beta$, we may begin by writing the expression,

$$\rho = \alpha + x\beta \; (15),$$

which may be considered (comp. 23, 99) as a form of the *equation of the right line* AP; and in which it remains to determine the scalar coefficient x, so as to satisfy the *equation of the sphere*,

$$T\rho = T\alpha \; (186, (2.)).$$

In short, we are to seek to satisfy the equation,

$$T(\alpha + x\beta) = T\alpha,$$

by some scalar x which shall be (in general) different from zero; and then to substitute this scalar in the expression $\rho = \alpha + x\beta$, in order to determine the required vector ρ.

* Compare the Notes to pages 87, 88, &c.

(3.) For this purpose, an obvious process is, after dividing both sides by $T\beta$, to square, and to employ the formula 210, XXI., which had indeed occurred before, as 200, VIII., but not then as a consequence of the distributive property of multiplication. In this manner we are conducted to a quadratic equation, which admits of division by x, and gives then,

$$x = -2S\frac{a}{\beta}; \qquad \rho = a - 2\beta S\frac{a}{\beta};$$

the problem (1.) being thus resolved, with the verification that β may be replaced by $U\beta$, in the resulting expression for ρ.

(4.) As a mere exercise of calculation, we may vary the last process (3.), by dividing the last equation (2.) by $T\alpha$, instead of $T\beta$, and then going on as before. This last procedure gives

$$1 = N\left(1 + x\frac{\beta}{a}\right) = 1 + 2xS\frac{\beta}{a} + x^2 N\frac{\beta}{a};$$

and therefore

$$x = -2S\frac{\beta}{a} : N\frac{\beta}{a} = -2S\frac{a}{\beta} \text{ (by 196, XII'.), as before.}$$

(5.) In general, by 196, II'.,

$$1 - 2S = -K;$$

hence, by (3.),

$$\frac{\rho}{\beta} = -K\frac{a}{\beta};$$

and finally,

$$\rho = -K\frac{a}{\beta}.\beta;$$

a new expression for ρ, in which it is not permitted generally, as it was in (3.), to treat the vector β as the multiplier,* instead of the multiplicand.

(6.) It is now easy to see that the second equation of (2.) is satisfied; for the expression (5.) for ρ gives (by 186, 187, &c.),

$$T\rho = T\frac{a}{\beta}.T\beta = Ta,$$

as was required.

(7.) To interpret the solution (3.), let C in fig. 51 be the middle point of the chord AP, and let D be the foot of the perpendicular let fall from A on OB; then the expression (3.) for ρ gives, by 196, XIX.,

$$CA = \tfrac{1}{2}(a - \rho) = \beta S\frac{a}{\beta} = OD;$$

and accordingly, OCAD is a parallelogram.

* Compare the Note to page 159.

(8.) To interpret the expression (5.), which gives

$$\frac{-\rho}{\beta} = \mathrm{K}\frac{\alpha}{\beta}, \quad \text{or} \quad \frac{\mathrm{OP}'}{\mathrm{OB}} = \mathrm{K}\frac{\mathrm{OA}}{\mathrm{OB}}, \quad \text{if} \quad \mathrm{OP}' = \mathrm{PO},$$

we have only to observe (comp. 138) that the angle AOP′ is bisected internally, or the supplementary angle AOP externally, by the indefinite right line OB (see again, fig. 51).

(9.) Conversely, the *geometrical considerations* which have thus served in (7.) and (8.) to *interpret* or to *verify* the two forms of solution (3.), (5.), might have been employed to *deduce* those two forms, if we had not seen how to obtain them, by *rules of calculation*, from the proposed conditions of the question. (Comp. 145, (10.), &c.)

(10.) It is evident, from the nature of that question, that α ought to be deducible from β and ρ, by exactly the same processes as those which have served us to deduce ρ from β and α. Accordingly, the form (3.) of ρ gives

$$\mathrm{S}\frac{\rho}{\beta} = -\mathrm{S}\frac{\alpha}{\beta}, \qquad \alpha = \rho + 2\beta\mathrm{S}\frac{\alpha}{\beta} = \rho - 2\beta\mathrm{S}\frac{\rho}{\beta};$$

and the form (5.) gives

$$\mathrm{K}\frac{\rho}{\beta} = -\frac{\alpha}{\beta}, \qquad \alpha = -\mathrm{K}\frac{\rho}{\beta}\cdot\beta.$$

And since the first form can be recovered from the second, we see that each leads us back to the parallelism, $\rho - \alpha \parallel \beta$ (2.).

(11.) The solution (3.) for x shows that

$$x = 0, \quad \rho = \alpha, \quad \mathrm{P} = \mathrm{A}, \quad \text{if} \quad \mathrm{S}(\alpha : \beta) = 0, \quad \text{or if} \quad \beta \perp \alpha.$$

And the geometrical meaning of this result is obvious; namely, that a right line drawn at the extremity of a radius OA of a sphere, so as to be perpendicular to that radius, does not (in strictness) *intersect* the sphere, but *touches* it: its *second* point of meeting the surface *coinciding*, in this case, as a *limit*, with the *first*.

(12.) Hence we may infer that the plane represented by the equation,

$$\mathrm{S}\frac{\rho - \alpha}{\alpha} = 0, \quad \text{or} \quad \mathrm{S}\frac{\rho}{\alpha} = 1,$$

is the *tangent plane* (comp. 196, (5.)) to the sphere here considered, at the point A.

(13.) Since β may be replaced by any vector parallel thereto, we may substitute for it $\gamma - \alpha$, if $\gamma = \mathrm{OC}$ be the vector of *any given point* C *upon the chord*

AP, whether (as in fig. 51) the middle point, or not; we may therefore write, by (3.) and (5.),

$$\rho = a - 2(\gamma - a)\,\mathrm{S}\frac{a}{\gamma - a} = -\mathrm{K}\frac{a}{\gamma - a}\,.\,(\gamma - a).$$

214. In the examples of the foregoing Article, there was no room for the occurrence of *imaginary roots* of an equation, or for *ideal intersections* of line and surface. To give now a case in which such imaginary intersections may occur, we shall proceed to consider the question of drawing a *secant* to a sphere, in a given direction, from a given *external point*; the recent figure 51 still serving us for illustration.

(1.) Suppose then that ε is the vector of any given point E, through which it is required to draw a chord or secant $\mathrm{EP_0P_1}$, parallel to the same given line β as before. We have now, if $\rho_0 = \mathrm{OP}_0$,

$$\rho_0 = \varepsilon + x_0\beta, \qquad \mathrm{T}a = \mathrm{T}\rho_0 = \mathrm{T}(\varepsilon + x_0\beta),$$

$$x_0^2 + 2x_0\mathrm{S}\frac{\varepsilon}{\beta} + \mathrm{N}\frac{\varepsilon}{\beta} - \mathrm{N}\frac{a}{\beta} = 0,$$

$$x_0 = -\mathrm{S}\frac{\varepsilon}{\beta} \mp \sqrt{\left\{\left(\mathrm{T}\frac{a}{\beta}\right)^2 + \left(\mathrm{V}\frac{\varepsilon}{\beta}\right)^2\right\}},$$

x_0 being a new scalar; and similarly, if $\rho_1 = \mathrm{OP}_1$,

$$\rho_1 = \varepsilon + x_1\beta, \qquad x_1 = -\mathrm{S}\frac{\varepsilon}{\beta} \pm \sqrt{\left\{\left(\mathrm{T}\frac{a}{\beta}\right)^2 + \left(\mathrm{V}\frac{\varepsilon}{\beta}\right)^2\right\}},$$

by transformations* which will easily occur to any one who has read recent articles with attention. And the points P_0, P_1 will be together *real*, or together *imaginary*, according as the quantity under the radical sign is positive or negative; that is, according as we have one or other of the two following inequalities,

$$\mathrm{T}\frac{a}{\beta} > \quad \text{or} \quad < \mathrm{TV}\frac{\varepsilon}{\beta}.$$

(2.) The equation (comp. 203, (5.)),

$$\mathrm{TV}\frac{\rho}{\beta} = \mathrm{T}\frac{a}{\beta} \quad \text{or} \quad \left(\mathrm{T}\frac{a}{\beta}\right)^2 + \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 0,$$

represents a cylinder of revolution, with OB for its axis, and with $\mathrm{T}a$ for the radius of its base. If E be a point of this cylindric surface, the quantity

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* It does not seem to be necessary, at the present stage, to supply so many references to former Articles, or sub-articles, as it has hitherto been thought useful to give; but such may still, from time to time, be given.

under the radical sign (1.) vanishes; and the two roots x_0, x_1 of the quadratic become *equal*. In this case, then, the line through E, which is parallel to OB, *touches* the given sphere; as is otherwise evident geometrically, since the cylinder *envelopes* the sphere (comp. 204, (12.)), and the line is one of its generatrices. If E be *internal* to the cylinder, the intersections P_0, P_1 are *real*; but if E be *external* to the same surface, those intersections are *ideal*, or *imaginary*.

(3.) In this last case, if we make, for abridgment,

$$s = -\mathrm{S}\frac{\varepsilon}{\beta}, \quad \text{and} \quad t = \sqrt{\left\{\left(\mathrm{TV}\frac{\varepsilon}{\beta}\right)^2 - \left(\mathrm{T}\frac{\alpha}{\beta}\right)^2\right\}},$$

s and t being thus two given and *real scalars*, we may write,

$$x_0 = s - t\sqrt{-1}; \qquad x_1 = s + t\sqrt{-1};$$

where $\sqrt{-1}$ is the *old* and *ordinary imaginary symbol* of *Algebra*, and is *not invested here* with any sort of *Geometrical Interpretation*.* We merely express thus the *fact of calculation*, that (with these meanings of the symbols α, β, ε, s and t) the formula $\mathrm{T}\alpha = \mathrm{T}(\varepsilon + x\beta)$, (1.), *when treated by the rules of quaternions, conducts to the quadratic equation*,

$$(x - s)^2 + t^2 = 0,$$

which has *no real root*; the reason being that *the right line* through E is, in the present case, *wholly external to the sphere*, and therefore *does not really intersect it at all*; although, for the sake of *generalization of language*, we may agree to *say*, as usual, that the line intersects the sphere in *two imaginary points*.

(4.) We must however agree, then, for *consistency of symbolical expression*, to consider these two ideal points as having *determinate* but *imaginary vectors*, namely, the two following:

$$\rho_0 = \varepsilon + s\beta - t\beta\sqrt{-1}; \qquad \rho_1 = \varepsilon + s\beta + t\beta\sqrt{-1};$$

in which it is easy to prove, Ist, that the *real part* $\varepsilon + s\beta$ is the *vector* ε' *of the foot* E′ *of the perpendicular* let fall from the centre O on the line through E which is drawn (as above) parallel to OB; and IInd, that the *real tensor* $t\mathrm{T}\beta$ *of the coefficient of* $\sqrt{-1}$ in the *imaginary part* of each expression, represents the *length of a tangent* E′E″ to the sphere, drawn from that external point, or foot, E′.

* Compare again the Note to page 87, and Art. 149.

(5.) In fact, if we write $OE' = \epsilon' = \epsilon + s\beta$, we shall have

$$E'E = \epsilon - \epsilon' = -s\beta = \beta S\frac{\epsilon}{\beta} = \text{projection of } OE \text{ on } OB\,;$$

which proves the Ist assertion (4.), whether the points P_0, P_1 be real or imaginary. And because

$$\left(T\frac{\epsilon'}{\beta}\right)^2 = N\frac{\epsilon'}{\beta} = N\left(\frac{\epsilon}{\beta} + s\right) = N\frac{\epsilon}{\beta} + 2sS\frac{\epsilon}{\beta} + s^2$$
$$= \left(T\frac{\epsilon}{\beta}\right)^2 - \left(S\frac{\epsilon}{\beta}\right)^2 = \left(TV\frac{\epsilon}{\beta}\right)^2 = t^2 + \left(T\frac{\alpha}{\beta}\right)^2,$$

we have, for the case of imaginary intersections,

$$tT\beta = \sqrt{(T\epsilon'^2 - T\alpha^2)} = T\,.\,E'E'',$$

and the IInd assertion (4.) is justified.

(6.) An expression of the form (4.), or of the following,

$$\rho' = \beta + \gamma\sqrt{-1},$$

in which β and γ are *two real vectors*, while $\sqrt{-1}$ is the (scalar) *imaginary of algebra*, and *not* a symbol for a *geometrically real right versor* (149, 153), may be said to be a BIVECTOR.

(7.) In like manner, an expression of the form (3.), or $x' = s + t\sqrt{-1}$, where s and t are *two real scalars*, but $\sqrt{-1}$ is still the ordinary imaginary of *algebra*, may be said by analogy to be a BISCALAR. *Imaginary roots of algebraic equations* are thus, in general, *biscalars*.

(8.) And if a *bivector* (6.) be *divided by a* (*real*) *vector*, the *quotient*, such as

$$q' = \frac{\rho'}{\alpha} = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\sqrt{-1} = q_0 + q_1\sqrt{-1},$$

in which q_0 and q_1 are *two real quaternions*, but $\sqrt{-1}$ is, as before, *imaginary*, may be said to be a BIQUATERNION.*

215. The same distributive principle (212) may be employed in investigations respecting *circumscribed cones*, and the *tangents* (real or ideal), which can be drawn to a given sphere from a given point.

(1.) Instead of conceiving that O, A, B are three given points, and that *limits of position* of the point E are sought, as in 214, (2.), which shall allow the points of intersection P_0, P_1 to be real, we may suppose that O, A, E (which may be assumed to be collinear, without loss of generality, since α enters only by its tensor) are now the data of the question; and that *limits of direction* of

* Compare the second Note to page 133. [This word is used in a different sense by W. K. Clifford.]

the line OB are to be assigned, which shall permit the same reality: EP_0P_1 being still drawn parallel to OB, as in 214, (1.).

(2.) Dividing the equation $T\alpha = T(\varepsilon + x\beta)$ by $T\varepsilon$, and squaring, we have

$$N\frac{\alpha}{\varepsilon} = \left(N\left(1 + x\frac{\beta}{\varepsilon}\right) = \right) 1 + 2xS\frac{\beta}{\varepsilon} + x^2 N\frac{\beta}{\varepsilon};$$

the quadratic in x may therefore be thus written,

$$\left(xT\frac{\beta}{\varepsilon} + SU\frac{\beta}{\varepsilon}\right)^2 = \left(T\frac{\alpha}{\varepsilon}\right)^2 + \left(VU\frac{\beta}{\varepsilon}\right)^2;$$

and its roots are real and unequal, or real and equal, or imaginary, according as

$$TVU\frac{\beta}{\varepsilon} < \text{or} = \text{or} > T\frac{\alpha}{\varepsilon};$$

that is, according as

$$\sin EOB < \text{ or} = \text{or} > T \,.\, OA : T \,.\, OE.$$

(3.) If E be *interior* to the sphere, then $T\varepsilon < T\alpha$, $T(\alpha : \varepsilon) > 1$; but $TVUq$ can never exceed unity (by 204, XIX., or by 210, XV., &c.); we have, therefore, in this case, the *first* of the three recent alternatives, and the two roots of the quadratic are *necessarily* real and unequal, whatever the direction of β may be. Accordingly it is evident, geometrically, that *every* indefinite right line, drawn through an internal point, must cut the spheric surface in two distinct and real points.

(4.) If the point E be *superficial*, so that $T\varepsilon = T\alpha$, $T(\alpha : \varepsilon) = 1$, then the first alternative (2.) still exists, except at the limit for which $\beta \perp \varepsilon$, and therefore $TVU(\beta : \varepsilon) = 1$, in which case we have the second alternative. *One* root of the quadratic in x is now $= 0$, for *every* direction of β; and the other root, namely $x = -2S(\varepsilon : \beta)$, is likewise always *real*, but *vanishes* for the case when the angle EOB is right. In short, we have here the same system of chords and of tangents, from a point upon the surface, as in 213; the only difference being, that we now write E for A, or ε for α.

(5.) But finally, if E be an *external* point, so that $T\varepsilon > T\alpha$, and $T(\alpha : \varepsilon) < 1$, then $TVU(\beta : \varepsilon)$ may either fall short of this last tensor, or equal, or exceed it; so that any one of the three alternatives (2.) may come to exist, according to the varying direction of β.

(6.) To illustrate geometrically the law of passage from one such alternative to another, we may observe that the equation

$$TVU\frac{\rho}{\varepsilon} = T\frac{\alpha}{\varepsilon},$$

or

$$\sin EOP = T \,.\, OA : T \,.\, OE,$$

represents (when E is thus external) a *real cone* of revolution, with its vertex at the centre O of the sphere; and according as the line OB lies *inside* this cone, or *on* it, or *outside* it, the first or the second or the third of the three alternatives (2.) is to be adopted; or in other words, the line through E, drawn parallel (as before) to OB, either *cuts* the sphere, or *touches* it, or does not (really) *meet* it at all. (Compare the annexed fig. 52.)

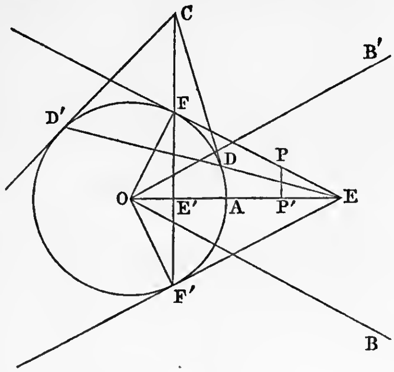

Fig. 52.

(7.) If E be still an external point, the *cone of tangents* which can be drawn from it to the sphere is *real*; and the equation of this *enveloping* or *circumscribed cone*, with its vertex at E, may be obtained from that of the recent cone (6.), by simply changing ρ to $\rho - \varepsilon$; it is, therefore, or at least one form of it is,

$$\mathrm{TVU}\frac{\rho - \varepsilon}{\varepsilon} = \mathrm{T}\frac{\alpha}{\varepsilon}; \quad \text{or} \quad \sin \mathrm{OEP} = \mathrm{T}\,.\,\mathrm{OA} : \mathrm{T}\,.\,\mathrm{OE}.$$

(8.) In general, if q be any quaternion, and x any scalar,

$$\mathrm{VU}\,(q + x) = \mathrm{V}q : \mathrm{T}\,(q + x);$$

the recent equation (7.) may therefore be thus written:

$$\mathrm{T}\frac{\mathrm{V}(\rho : \varepsilon)\,.\,\varepsilon}{\rho - \varepsilon} = \mathrm{T}\frac{\alpha}{\varepsilon};$$

or $$\mathrm{T}\,.\,\mathrm{P'P} : \mathrm{T}\,.\,\mathrm{EP} = \mathrm{T}\,.\,\mathrm{OA} : \mathrm{T}\,.\,\mathrm{OE},$$

if P′ be the foot of the perpendicular let fall from P on OE; and in fact the first quotient is evidently = sin OEP.

(9.) We may also write,

$$\mathrm{TV}\frac{\rho}{\varepsilon} = \mathrm{T}\frac{\alpha}{\varepsilon}\,.\,\mathrm{T}\left(\frac{\rho}{\varepsilon} - 1\right); \quad \text{or} \quad 0 = \left(\mathrm{S}\frac{\rho}{\varepsilon}\right)^2 - \mathrm{N}\frac{\rho}{\varepsilon} + \mathrm{N}\frac{\alpha}{\varepsilon}\left(\mathrm{N}\frac{\rho}{\varepsilon} - 2\mathrm{S}\frac{\rho}{\varepsilon} + 1\right);$$

or $$\left(\mathrm{S}\frac{\rho}{\varepsilon} - \mathrm{N}\frac{\alpha}{\varepsilon}\right)^2 = \left(1 - \mathrm{N}\frac{\alpha}{\varepsilon}\right)\left(\mathrm{N}\frac{\rho}{\varepsilon} - \mathrm{N}\frac{\alpha}{\varepsilon}\right),$$

as another form of the equation of the circumscribed cone.

(10.) If then we make also

$$\mathrm{N}\frac{\rho}{\alpha} = 1, \quad \text{or} \quad \mathrm{N}\frac{\rho}{\varepsilon} = \mathrm{N}\frac{\alpha}{\varepsilon},$$

to express that the point P is *on the enveloped sphere*, as well as on the enveloping *cone*, we find the following equation of the *plane of contact*, or of what is called the *polar plane* of the point E, with respect to the given sphere:

$$\left(S\frac{\rho}{\varepsilon} - N\frac{\alpha}{\varepsilon}\right)^2 = 0; \quad \text{or} \quad S\frac{\rho}{\varepsilon} - N\frac{\alpha}{\varepsilon} = 0;$$

while the fact that it *is* a plane of *contact** is exhibited by the occurrence of the exponent 2, or by its equation entering through its *square*.

(11.) The vector,

$$\varepsilon' = \varepsilon S\frac{\rho}{\varepsilon} = \varepsilon N\frac{\alpha}{\varepsilon} = \text{OE}',$$

is that of the point E′ in which the polar plane (10.) of E cuts perpendicularly the right line OE; and we see that

$$T\varepsilon \,.\, T\varepsilon' = T\alpha^2, \quad \text{or} \quad T \,.\, \text{OE} \,.\, T \,.\, \text{OE}' = (T \,.\, \text{OA})^2,$$

as was to be expected from elementary theorems, of spherical or even of plane geometry.

(12.) The equation (10.), of the polar plane of E, may easily be thus transformed:

$$S\frac{\varepsilon}{\rho} = \left(S\frac{\rho}{\varepsilon} \,.\, N\frac{\varepsilon}{\rho} = \right) N\frac{\alpha}{\rho}, \quad \text{or} \quad S\frac{\varepsilon}{\rho} - N\frac{\alpha}{\rho} = 0;$$

it continues therefore to hold good, when ε and ρ are *interchanged.* If then we take, as the vertex of a *new enveloping cone*, any point C external to the sphere, and situated on the polar plane FF′.. of the former external point E, the *new plane of contact*, or the polar plane DD′.. of the new point C, will pass through the former vertex E: a geometrical relation of *reciprocity*, or of *conjugation*, between the two points C and E, which is indeed well known, but which it appeared useful for our purpose to prove by quaternions† anew.

(13.) In general, each of the two connected equations,

$$S\frac{\rho'}{\rho} = N\frac{\alpha}{\rho}, \quad S\frac{\rho}{\rho'} = N\frac{\alpha}{\rho'},$$

which may also be thus written,

$$1 = \left(S\frac{\rho'}{\alpha}\frac{\alpha}{\rho} \,.\, N\frac{\rho}{\alpha} = \right) S \,.\, \frac{\rho'}{\alpha} K\frac{\rho}{\alpha}, \quad 1 = S \,.\, \frac{\rho}{\alpha} K\frac{\rho'}{\alpha},$$

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* In fact a modern geometer would say, that we have here a case of *two coincident planes* of intersection, merged into a single plane of contact.

† In fact, it will easily be seen that the investigations in recent sub-articles are put forward, almost entirely, as exercises in the Language and Calculus of Quaternions, and not as offering any geometrical novelty of result.

may be said to be a form of the *Equation of Conjugation* between any two points P and P′ (not those so marked in fig. 52), of which the vectors satisfy it: because it expresses that those two points are, in a well-known sense, *conjugate* to each other, with respect to the given sphere, $T\rho = T\alpha$.

(14.) If *one* of the two points, as P′, be *given* by its vector ρ', while the *other* point P and vector ρ are *variable*, the equation then represents a *plane locus*; namely, what is still called the *polar plane* of the given point, whether that point be external or internal, or on the surface of the sphere.

(15.) Let P, P′ be thus two conjugate points; and let it be proposed to find the points S, S′, in which the right line PP′ intersects the sphere. Assuming (comp. 25) that

$$\text{OS} = \sigma = x\rho + y\rho', \quad x + y = 1, \quad T\sigma = T\alpha,$$

and attending to the equation of conjugation (13.), we have, by 210, XX., or by 200, VII., the following quadratic equation in $y : x$,

$$(x+y)^2 = N\left(x\frac{\rho}{\alpha} + y\frac{\rho'}{\alpha}\right) = x^2 N\frac{\rho}{\alpha} + 2xy + y^2 N\frac{\rho'}{\alpha};$$

which gives

$$x^2\left(N\frac{\rho}{\alpha} - 1\right) = y^2\left(1 - N\frac{\rho'}{\alpha}\right).$$

(16.) Hence it is evident that, if the points of intersection S, S′ are to be *real*, one of the two points P, P′ must be interior, and the other must be exterior to the sphere; because, of the *two norms* here occurring, one must be greater and the other less than unity. And because the *two roots* of the quadratic, or the two values of $y : x$, differ *only* by their *signs*, it follows (by 26) that the right line PP′ is *harmonically divided* (as indeed it is well known to be), at the two points S, S′ at which it meets the sphere: or that in a notation already several times employed (25, 31, &c.), we have the *harmonic formula*,

$$(\text{PSP}'\text{S}') = -1.$$

(17.) From a real but *internal point* P, we can still *speak* of a *cone of tangents*, as being drawn to the sphere: but if so, we must say that those tangents are *ideal*, or *imaginary*;* and must consider them as terminating on an *imaginary circle of contact*; of which the *real* but wholly *external plane* is, by quaternions, as by modern geometry, recognised as being (comp. (14.)) the polar plane of the supposed internal point.

* Compare again the Note to page 88, and others formerly referred to.

216. Some readers may find it useful, or at least interesting, to see here a few examples of the application of the General Distributive Principle (212) of multiplication to the *Ellipsoid*, of which some forms of the Quaternion Equation were lately assigned (in 204, (14.)); especially as those forms have been found to conduct* to a Geometrical Construction, previously unknown, for that celebrated and important Surface: or rather to several such constructions. In what follows, it will be supposed that any such reader has made himself already sufficiently familiar with the chief formulæ of the preceding Articles; and therefore comparatively few references† will be given, at least upon the present subject.

(1.) To prove, first, that the locus of the variable ellipse,

$$\text{I.} \ldots \mathrm{S}\frac{\rho}{\alpha} = x, \qquad \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = x^2 - 1, \qquad 204, (13.)$$

which locus is represented by the equation,

$$\text{II.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 1, \qquad 204, (14.)$$

the two constant vectors α, β being supposed to be real, and to be inclined to each other at some acute or obtuse (but not right‡) angle, is a *surface of the second order*, in the sense that it is cut by an arbitrary rectilinear transversal in *two* (real or imaginary) points, and in *no more* than two, let us assume two points L, M, or their vectors $\lambda = \text{OL}$, $\mu = \text{OM}$, as given; and let us seek to determine the points P (real or imaginary), in which the indefinite right line LM intersects the locus II.; or rather the *number* of such intersections, which will be sufficient for the present purpose.

(2.) Making then $\rho = \dfrac{y\lambda + z\mu}{y + z}$ (25), we have, for $y : z$, the following quadratic equation,

$$\text{III.} \ldots \left(y\mathrm{S}\frac{\lambda}{\alpha} + z\mathrm{S}\frac{\mu}{\alpha}\right)^2 - \left(y\mathrm{V}\frac{\lambda}{\beta} + z\mathrm{V}\frac{\mu}{\beta}\right)^2 = (y + z)^2;$$

without proceeding to *resolve* which, we see already, by its mere *degree*, that

* See the Proceedings of the Royal Irish Academy for the year 1846.

† Compare the Note to page 223.

‡ If $\beta \perp \alpha$, the system I. represents (not an ellipse but) a *pair of right lines*, real or ideal, in which the *cylinder of revolution*, denoted by the second equation of that system, is cut by a *plane parallel to its axis*, and represented by the first equation.

the number sought is *two*; and therefore that the locus II. is, as above stated, a surface of the *second* order.

(3.) The equation II. remains unchanged, when $-\rho$ is substituted for ρ; the surface has therefore a *centre*, and this centre is at the *origin* o of vectors.

(4.) It has been seen that the equation of the surface may also be thus written:

$$\text{IV.} \ldots \mathrm{T}\left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right) = 1; \qquad 204,\ (14.)$$

it gives therefore, for the reciprocal of the radius vector from the centre, the expression,

$$\text{V.} \ldots \frac{1}{\mathrm{T}\rho} = \mathrm{T}\left(\mathrm{S}\frac{\mathrm{U}\rho}{\alpha} + \mathrm{V}\frac{\mathrm{U}\rho}{\beta}\right);$$

and this expression has a real value, which never vanishes,* whatever real value may be assigned to the versor $\mathrm{U}\rho$, that is, whatever direction may be assigned to ρ: the surface is therefore *closed*, and *finite*.

(5.) Introducing two new constant and auxiliary vectors, determined by the two expressions,

$$\text{VI.} \ldots \gamma = \frac{2\beta}{\beta + \alpha} \cdot \alpha, \qquad \delta = \frac{2\beta}{\beta - \alpha} \cdot \alpha,$$

which give (by 125) these other expressions,

$$\text{VI}'. \ldots \gamma = \frac{2\alpha}{\beta + \alpha} \cdot \beta, \qquad \delta = \frac{2\alpha}{\beta - \alpha} \cdot \beta,$$

we have

$$\text{VII.} \ldots \frac{\gamma}{\alpha} + \frac{\gamma}{\beta} = 2, \qquad \frac{\delta}{\alpha} - \frac{\delta}{\beta} = 2;$$

$$\text{VII}'. \ldots \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} = 1, \qquad \frac{\beta}{\gamma} - \frac{\beta}{\delta} = 1$$

and under these conditions, γ is said to be the *harmonic mean* between the two former vectors, α and β; and in like manner, δ is the harmonic mean between α and $-\beta$; while 2α is the corresponding mean between γ, δ; and 2β is so, between γ and $-\delta$.

(6.) Under the same conditions, for any arbitrary vector ρ, we have the transformations,

$$\text{VIII.} \ldots \frac{\rho}{\gamma} = \tfrac{1}{2}\left(\frac{\rho}{\alpha} + \frac{\rho}{\beta}\right); \quad \frac{\rho}{\delta} = \tfrac{1}{2}\left(\frac{\rho}{\alpha} - \frac{\rho}{\beta}\right);$$

$$\text{IX.} \ldots \frac{\rho}{\gamma} + \mathrm{K}\frac{\rho}{\delta} = \mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta};$$

---

* It is to be remembered that we have excluded in (1.) the case where $\beta \perp \alpha$; in which case it can be shown that the equation II. represents an *elliptic cylinder*.

the equation IV. of the surface may therefore be thus written :

$$\text{X.}\ldots \mathrm{T}\left(\frac{\rho}{\gamma}+\mathrm{K}\frac{\rho}{\delta}\right)=1;\quad \text{or thus,}\quad \text{X}'\ldots \mathrm{T}\left(\frac{\rho}{\delta}+\mathrm{K}\frac{\rho}{\gamma}\right)=1;$$

the geometrical meaning of which new forms will soon be seen.

(7.) The system of the two planes through the origin, which are respectively perpendicular to the new vectors γ and δ, is represented by the equation,

$$\text{XI.}\ldots \mathrm{S}\frac{\rho}{\gamma}\,\mathrm{S}\frac{\rho}{\delta}=0,\quad \text{or}\quad \text{XII.}\ldots\left(\mathrm{S}\frac{\rho}{\alpha}\right)^2=\left(\mathrm{S}\frac{\rho}{\beta}\right)^2;$$

combining which with the equation II. we get

$$\text{XIII.}\ldots 1=\left(\mathrm{S}\frac{\rho}{\beta}\right)^2-\left(\mathrm{V}\frac{\rho}{\beta}\right)^2=\mathrm{N}\frac{\rho}{\beta};\quad \text{or,}\quad \text{XIV.}\ldots \mathrm{T}\rho=\mathrm{T}\beta.$$

These two diametral planes therefore cut the surface in *two circular sections*, with $\mathrm{T}\beta$ for their common radius; and the normals γ and δ, to the same two planes, may be called (comp. 196, (17.)) the *cyclic normals* of the surface; while the planes themselves may be called its *cyclic planes*.

(8.) Conversely, if we seek the intersection of the surface with the concentric sphere XIV., of which the radius is $\mathrm{T}\beta$, we are conducted to the equation XII. of the system of the two cyclic planes, and therefore to the two circular sections (7.); so that every radius vector of the surface, which is *not* drawn in one or other of these two planes, has a length either greater or less than the radius $\mathrm{T}\beta$ of the sphere.

(9.) By all these marks, it is clear that the locus II., or 204, (14.), is (as above asserted) an *Ellipsoid*; its *centre* being at the origin (3.), and its *mean semiaxis* being $=\mathrm{T}\beta$; while $\mathrm{U}\beta$ has, by 204, (15.), the direction of the *axis* of a *circumscribed cylinder of revolution*, of which cylinder the *radius* is $\mathrm{T}\beta$; and α is, by the last cited sub-article, perpendicular to the plane of the *ellipse of contact*.

(10.) Those who are familiar with modern geometry, and who have caught the notations of quaternions, will easily see that this ellipsoid, II. or IV., is a *deformation* of what may be called the *mean sphere* XIV., and is *homologous* thereto; the infinitely distant point in the direction of β being a *centre of homology*, and either of the two planes XI. or XII. being a *plane of homology* corresponding.

217. The recent form, X. or X'., of the quaternion equation of the ellipsoid admits of being *interpreted* in such a way as to conduct (comp. 216) to

a simple *construction* of that surface; which we shall first investigate by calculation, and then illustrate by geometry.

(1.) Carrying on the Roman numerals from the sub-articles to 216, and observing that (by 190, &c.),

$$\frac{\rho}{\gamma} = \mathrm{K}\frac{\gamma}{\rho}.\mathrm{N}\frac{\rho}{\gamma}, \quad \text{and} \quad \mathrm{K}\frac{\rho}{\delta} = \frac{\delta}{\rho}.\mathrm{N}\frac{\rho}{\delta},$$

the equation X. takes the form,

$$\mathrm{XV.}\ldots 1 = \mathrm{T}\left\{\left(\frac{\delta}{\mathrm{T}\delta^2} + \mathrm{K}\frac{\gamma}{\rho}.\frac{\rho}{\mathrm{T}\gamma^2}\right) : \frac{\rho}{\mathrm{T}\rho^2}\right\};$$

or

$$\mathrm{XVI.}\ldots \frac{t^2}{\mathrm{T}\rho} = \mathrm{T}\left(\iota + \mathrm{K}\frac{\kappa}{\rho}.\rho\right),$$

if we make

$$\mathrm{XVII.}\ldots \frac{\delta}{\mathrm{T}\delta^2} = \frac{\iota}{t^2} \quad \text{and} \quad \frac{\gamma}{\mathrm{T}\gamma^2} = \frac{\kappa}{t^2},$$

when ι and κ are two new constant vectors, and t is a new constant scalar, which we shall suppose to be positive, but of which the value may be chosen at pleasure.

(2.) The comparison of the forms X. and X′. shows that γ and δ may be interchanged, or that they enter symmetrically into the equation of the ellipsoid, although they may not at first seem to do so; it is therefore allowed to assume that

$$\mathrm{XVIII.}\ldots \mathrm{T}\gamma > \mathrm{T}\delta, \quad \text{and therefore that} \quad \mathrm{XVIII'.}\ldots \mathrm{T}\iota > \mathrm{T}\kappa;$$

for the supposition $\mathrm{T}\gamma = \mathrm{T}\delta$ would give, by VI.,

$$\mathrm{T}(\beta + \alpha) = \mathrm{T}(\beta - \alpha), \quad \text{and} \therefore \text{(by 186, (6.) \&c.)} \quad \beta \perp \alpha,$$

which latter case was excluded in 216, (1.).

(3.) We have thus,

$$\mathrm{XIX.}\ldots \mathrm{U}\iota = \mathrm{U}\delta; \qquad \mathrm{U}\kappa = \mathrm{U}\gamma;$$

$$\mathrm{XX.}\ldots \mathrm{T}\iota = \frac{t^2}{\mathrm{T}\delta}; \qquad \mathrm{T}\kappa = \frac{t^2}{\mathrm{T}\gamma};$$

$$\mathrm{XXI.}\ldots \frac{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}{t^2} = \left(\frac{t}{\mathrm{T}\delta}\right)^2 - \left(\frac{t}{\mathrm{T}\gamma}\right)^2.$$

(4.) Let ABC be a plane triangle, such that

$$\mathrm{XXII.}\ldots \mathrm{CB} = \iota, \qquad \mathrm{CA} = \kappa;$$

let also $\qquad \mathrm{AE} = \rho.$

Then if a sphere, which we shall call the *diacentric sphere*, be described round the point C as centre, with a radius = $T\kappa$, and therefore so as to pass *through the centre* A (here written instead of O) of the ellipsoid, and if D be the point in which the line AE meets this sphere again, we shall have, by 213, (5.), (13.),

$$\text{XXIII.} \ldots \text{CD} = -\text{K}\frac{\kappa}{\rho}.\rho,$$

and therefore

$$\text{XXIII}'. \ldots \text{DB} = \iota + \text{K}\frac{\kappa}{\rho}.\rho;$$

so that the equation XVI. becomes

$$\text{XXIV.} \ldots t^2 = \text{T}.\text{AE}.\text{T}.\text{DB}.$$

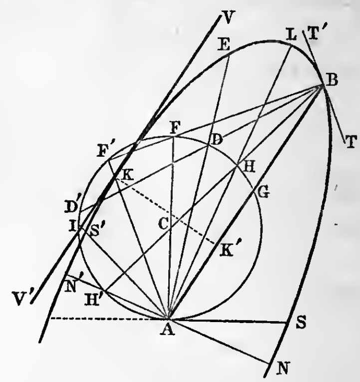

Fig. 53.

(5.) The point B is *external* to the diacentric sphere (4.), by the assumption (2.); a real tangent (or rather cone of tangents) to this sphere can therefore be drawn from that point; and if we select the length of such a tangent as the value (1.) of the scalar t, that is to say, if we make each member of the formula XXI. equal to unity, and denote by D′ the second intersection of the right line BD with the sphere, as in fig. 53, we shall have (by Euclid III.) the elementary relation,

$$\text{XXV.} \ldots t^2 = \text{T}.\text{DB}.\text{T}.\text{BD}';$$

whence follows this *Geometrical Equation of the Ellipsoid*,

$$\text{XXVI.} \ldots \text{T}.\text{AE} = \text{T}.\text{BD}';$$

or in somewhat more familiar notation,

$$\text{XXVII.} \ldots \overline{\text{AE}} = \overline{\text{BD}'};$$

where $\overline{\text{AE}}$ denotes the *length* of the line AE, and similarly for $\overline{\text{BD}'}$.

(6.) The following very simple *Rule of Construction* (comp. the recent fig. 53) results therefore from our quaternion analysis:—

From a fixed point A, *on the surface of a given sphere, draw any chord* AD; *let* D′ *be the second point of intersection of the same spheric surface with the secant* BD, *drawn from a fixed external* point* B; *and take a radius vector* AE, *equal in*

* It is merely to fix the conceptions, that the point B is here supposed to be *external* (5.); the calculations and the construction would be almost the same, if we assumed B to be an *internal* point, or $T\iota < T\kappa$, $T\gamma < T\delta$.

length to the line BD′, *and in direction either coincident with, or opposite to, the chord* AD : *the locus of the point* E *will be an ellipsoid, with* A *for its centre, and with* B *for a point of its surface.*

(7.) Or thus—

If, of a plane but variable quadrilateral ABED′, *of which one side* AB *is given in length and in position, the two diagonals* AE, BD′ *be equal to each other in length, and if their intersection* D *be always situated upon the surface of a given sphere, whereof the side* AD′ *of the quadrilateral is a chord, then the opposite side* BE *is a chord of a given ellipsoid.*

218. From either of the two foregoing statements, of the *Rule of Construction for the Ellipsoid* to which quaternions have conducted, many *geometrical consequences* can easily be inferred, a few of which may be mentioned here, with their proofs by *calculation* annexed : the present Calculus being, of course, still employed.

(1.) That the corner B, of what may be called the *Generating Triangle* ABC, is in fact a point of the generated surface, with the construction 217, (6.), may be proved, by conceiving the variable chord AD of the given diacentric sphere to take the position AG ; where G is the second intersection of the line AB with that spheric surface.

(2.) If D be conceived to approach to A (instead of G), and therefore D′ to G (instead of A), the *direction* of AE (or of AD) then tends to become tangential to the sphere at A, while the *length* of AE (or of BD′) tends, by the construction, to become equal to the length of BG ; the surface has therefore a *diametral* and *circular section*, in a plane which touches the diacentric sphere at A, and with a radius = $\overline{\text{BG}}$.

(3.) Conceive a circular section of the sphere through A, made by a plane perpendicular to BC ; if D move along this circle, D′ will move along a parallel circle through G, and the length of BD′, or that of AE, will again be equal to $\overline{\text{BG}}$; such then is the radius of a *second* diametral and *circular section* of the *ellipsoid*, made by the lately mentioned plane.

(4.) The construction gives us thus *two cyclic planes* through A ; the perpendiculars to which planes, or the *two cyclic normals* (216, (7.)) of the ellipsoid, are seen to have the directions of the *two sides*, CA, CB, of the *generating triangle* ABC (1.).

(5.) Again, since the rectangle

$$\overline{\text{BA}} \cdot \overline{\text{BG}} = \overline{\text{BD}} \cdot \overline{\text{BD}'} = \overline{\text{BD}} \cdot \overline{\text{AE}} = \text{double area of triangle ABE : sin BDE},$$

we have the equation,

$$\text{XXVIII.} \ldots \text{perpendicular distance of E from AB} = \overline{\text{BG}} \,.\, \text{sin BDE};$$

the *third side*, AB, of the generating triangle (1.), is therefore the *axis of revolution* of a *cylinder*, which *envelops* the ellipsoid, and of which the radius has the same length, $\overline{\text{BG}}$, as the radius of each of the two diametral and circular sections.

(6.) For the points of contact of ellipsoid and cylinder, we have the geometrical relation,

$$\text{XXIX.} \ldots \text{BDE} = \text{a right angle}; \quad \text{or} \quad \text{XXIX}' \ldots \text{ADB} = \text{a right angle};$$

the point D is therefore situated on a *second spheric surface*, which has the line AB for a diameter, and intersects the diacentric sphere in a *circle*, whereof the plane passes through A, and cuts the enveloping cylinder in an *ellipse of contact* (comp. 204, (15.), and 216, (9.)), of that cylinder with the ellipsoid.

(7.) Let AC meet the diacentric sphere again in F, and let BF meet it again in F′ (as in fig. 53); the *common plane* of the last-mentioned circle and ellipse (6.) can then be easily proved to cut perpendicularly the plane of the generating triangle ABC in the line AF′; so that the line F′B is *normal to this plane of contact*; and therefore also (by conjugate diameters, &c.) *to the ellipsoid*, at B.

(8.) These *geometrical consequences of the construction* (217), to which many others might be added, can all be shown to be consistent with, and confirmed by, the *quaternion analysis* from which that construction itself was derived. Thus, the two *circular sections* (2.), (3.) had presented themselves in 216, (7.); and their two *cyclic normals* (4.), or the sides CA, CB of the triangle, being (by 217, (4.)) the two vectors κ, ι, have (by 217, (1.) or (3.)) the directions of the two former vectors γ, δ; which again agrees with 216, (7.).

(9.) Again, it will be found that the assumed relations between the *three pairs of constant vectors*, α, β; γ, δ; and ι, κ, any *one* of which *pairs* is sufficient to determine the ellipsoid, conduct to the following expressions (of which the investigation is left to the student, as an exercise):

$$\text{XXX.} \ldots \alpha = \frac{\delta}{\delta + \gamma}\gamma = \frac{\gamma}{\delta + \gamma}\delta = \frac{+t^2}{\text{T}(\iota + \kappa)}\text{U}(\iota + \kappa) = \text{F}'\text{B};$$

$$\text{XXXI.} \ldots \beta = \frac{\delta}{\delta - \gamma}\gamma = \frac{\gamma}{\delta - \gamma}\delta = \frac{-t^2}{\text{T}(\iota - \kappa)}\text{U}(\iota - \kappa) = \text{BG};$$

the letters B, F′, G referring here to fig. 53, while $\alpha\beta\gamma\delta$ retain their former meanings (216), and are not interpreted as vectors of the points ABCD in that figure. Hence the recent geometrical inferences, that AB (or BG) is the axis of revolution of an enveloping cylinder (5.), and that F′B is normal to the plane of the ellipse of contact (7.), agree with the former conclusions (216, (9.), or 204, (15.)), that β is such an axis, and that α is such a normal.

(10.) It is easy to prove, generally, that

$$\mathrm{S}\frac{q-1}{q+1} = \mathrm{S}\frac{(q-1)(\mathrm{K}q+1)}{(q+1)(\mathrm{K}q+1)} = \frac{\mathrm{N}q-1}{\mathrm{N}(q+1)}, \quad \mathrm{S}\frac{q+1}{q-1} = \frac{\mathrm{N}q-1}{\mathrm{N}(q-1)};$$

whence

$$\text{XXXII.} \ldots \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa} = \frac{\mathrm{T}\iota^2-\mathrm{T}\kappa^2}{\mathrm{T}(\iota+\kappa)^2}, \quad \mathrm{S}\frac{\iota+\kappa}{\iota-\kappa} = \frac{\mathrm{T}\iota^2-\mathrm{T}\kappa^2}{\mathrm{T}(\iota-\kappa)^2},$$

whatever two vectors ι and κ may be. But we have here,

$$\text{XXXIII.} \ldots t^2 = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2, \text{ by 217, (5.)};$$

the recent expressions (9.) for α and β become, therefore,

$$\text{XXXIV.} \ldots \alpha = +(\iota+\kappa)\,\mathrm{S}\frac{\iota-\kappa}{\iota+\kappa}; \quad \beta = -(\iota-\kappa)\,\mathrm{S}\frac{\iota+\kappa}{\iota-\kappa}.$$

The last form 204, (14.), of the equation of the ellipsoid, may therefore be now thus written:

$$\text{XXXV.} \ldots \mathrm{T}\left(\mathrm{S}\frac{\rho}{\iota+\kappa} : \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa} - \mathrm{V}\frac{\rho}{\iota-\kappa} : \mathrm{S}\frac{\iota+\kappa}{\iota-\kappa}\right) = 1;$$

in which the sign of the right part may be changed. And thus we verify by calculation the recent result (1.) of the construction, namely, that B is a point of the surface; for we see that the last equation is satisfied, when we suppose

$$\text{XXXVI.} \ldots \rho = \mathrm{AB} = \iota - \kappa = \beta : \mathrm{S}\frac{\beta}{\alpha};$$

a value of ρ which evidently satisfies also the form 216, IV.

(11.) From the form 216, II., combined with the value XXXIV. of α, it is easy to infer that the plane,

$$\text{XXXVII.} \ldots \mathrm{S}\frac{\rho}{\alpha} = 1, \quad \text{or} \quad \text{XXXVII}'. \ldots \mathrm{S}\frac{\rho}{\iota+\kappa} = \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa},$$

which corresponds to the value $x = 1$ in 216, I., *touches the ellipsoid* at the point B, of which the vector ρ has been thus determined (10.); the *normal to the surface*, at that point, has therefore the direction of $\iota + \kappa$, or of α, that is, of FB, or of F′B: so that the last geometrical inference (7.) is thus confirmed, by calculation with quaternions.

219. A few other consequences of the construction (217) may be here noted; especially as regards the geometrical determination of the *three principal semiaxes* of the ellipsoid, and the major and minor semiaxes of any elliptic and *diametral section*; together with the assigning of a certain *system of spherical conics*, of which the *surface* may be considered to be the *locus*.

(1.) Let a, b, c denote the lengths of the greatest, the mean, and the least semiaxes of the ellipsoid, respectively; then if the side BC of the generating triangle cut the diacentric sphere in the points H and H′, the former lying (as in fig. 53) between the points B and C, we have the values,

$$\text{XXXVIII.} \ldots a = \overline{\text{BH}'}; \quad b = \overline{\text{BG}}; \quad c = \overline{\text{BH}};$$

so that the lengths of the sides of the triangle ABC may be thus expressed, in terms of these semiaxes,

$$\text{XXXIX.} \ldots \overline{\text{BC}} = \mathrm{T}\iota = \frac{a+c}{2}; \quad \overline{\text{CA}} = \mathrm{T}\kappa = \frac{a-c}{2}; \quad \overline{\text{AB}} = \mathrm{T}(\iota - \kappa) = \frac{ac}{b};$$

and we may write,

$$\text{XL.} \ldots a = \mathrm{T}\iota + \mathrm{T}\kappa; \quad b = \frac{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}{\mathrm{T}(\iota - \kappa)}; \quad c = \mathrm{T}\iota - \mathrm{T}\kappa.$$

(2.) If, in the respective directions of the two supplementary chords AH, AH′ of the sphere, or in the opposite directions, we set off lines AL, AN, with the lengths of BH′, BH, the points L, N, thus obtained, will be respectively a *major* and a *minor summit* of the surface. And if we erect, at the centre A of that surface a perpendicular AM to the plane of the triangle, with a length $= \overline{\text{BG}}$, the point M (which will be common to the two circular sections, and will be situated on the enveloping cylinder) will be a *mean summit* thereof.

(3.) Conceive that the sphere and ellipsoid are both cut by a plane through A, on which the points B′ and C′ shall be supposed to be the projections of B and C; then C′ will be the centre of the circular section of the sphere; and if the line B′C′ cut this new circle in the points D_1, D_2, of which D_1 may be supposed to be the nearer to B′, the two supplementary chords AD_1, AD_2 of the circle have the *directions* of the *major and minor semiaxes* of the *elliptic section* of the ellipsoid; while the *lengths* of those semiaxes are, respectively, $\overline{\text{BA}} \,.\, \overline{\text{BG}} : \overline{\text{BD}_1}$, and $\overline{\text{BA}} \,.\, \overline{\text{BG}} : \overline{\text{BD}_2}$; or $\overline{\text{BD}'_1}$ and $\overline{\text{BD}'_2}$, if the secants BD_1 and BD_2 meet the sphere again in D'_1 and D'_2.

(4.) If these two semiaxes of the section be called $a_{\prime}$ and $c_{\prime}$, and if we still denote by t the tangent from B to the sphere, we have thus,

$$\text{XLI.} \ldots \overline{\text{BD}_1} = t^2 : a_{\prime} = aca_{\prime}^{-1}; \quad \overline{\text{BD}_2} = t^2 : c_{\prime} = acc_{\prime}^{-1};$$

but if we denote by p_1 and p_2 the inclinations of the plane of the section to the two cyclic planes of the ellipsoid, whereto CA and CB are perpendicular, so that the projections of these two sides of the triangle are

$$\text{XLII.} \ldots \begin{cases} \overline{\text{C}'\text{A}} = \overline{\text{CA}} \,.\, \sin p_1 = \tfrac{1}{2}(a - c) \sin p_1, \\ \overline{\text{C}'\text{B}'} = \overline{\text{CB}} \,.\, \sin p_2 = \tfrac{1}{2}(a + c) \sin p_2, \end{cases}$$

we have

$$\text{XLIII.} \ldots \overline{\text{BD}_2}^2 - \overline{\text{BD}_1}^2 = \overline{\text{B}'\text{D}_2}^2 - \overline{\text{B}'\text{D}_1}^2 = 4\overline{\text{B}'\text{C}'} \,.\, \overline{\text{C}'\text{A}} = (a^2 - c^2) \sin p_1 \sin p_2 ;$$

whence follows the important formula,

$$\text{XLIV.} \ldots c_{\prime}^{-2} - a_{\prime}^{-2} = (c^{-2} - a^{-2}) \sin p_1 \sin p_2 ;$$

or in words, the known and useful theorem, that "*the difference of the inverse squares of the semiaxes, of a plane and diametral section of an ellipsoid, varies as the product of the sines of the inclinations of the cutting plane, to the two planes of circular section.*

(5.) As verifications, if the plane be that of the generating triangle ABC, we have

$$p_1 = p_2 = \frac{\pi}{2}, \quad \text{and} \quad a_{\prime} = a, \quad c_{\prime} = c ;$$

but if the plane be perpendicular to either of the two sides, CA, CB, then either p_1 or $p_2 = 0$, and $c_{\prime} = a_{\prime}$.

(6.) If the ellipsoid be cut by any concentric sphere, *distinct* from the *mean sphere* XIV., so that

$$\text{XLV.} \ldots \overline{\text{AE}} = \text{T}\rho = r \lessgtr b, \text{ where } r \text{ is a given positive scalar;}$$

then

$$\text{XLVI.} \ldots \overline{\text{BD}} = t^2 r^{-1} \lessgtr acb^{-1}, \quad \text{that is,} \quad \lessgtr \overline{\text{BA}} ;$$

so that the *locus* of what may be called the *guide-point* D, through which, by the construction, the variable semidiameter AE of the ellipsoid (or one of its prolongations) passes, and which is still at a constant distance from the given external point B, is now again a *circle* of the diacentric sphere, but one of which the *plane* does *not pass* (as it did in 218, (3.)) *through the centre* A of the ellipsoid. The point E has therefore here, for one locus, the *cyclic cone* which has A for *vertex*, and rests on the last-mentioned circle as its *base*; and since it is also on the concentric *sphere* XLV., it must be on one or other of the *two spherical conics*, in which (comp. 196, (11.)) the cone and sphere last mentioned intersect.

(7.) The *intersection of an ellipsoid with a concentric sphere* is therefore, generally, a *system of two such conics*, varying with the value of the radius r, and becoming, as a *limit*, the *system of the two circular sections*, for the particular value $r = b$; and the ellipsoid itself may be considered as the *locus* of all such spherical conics, including those two circles.

(8.) And we see, by (6.), that the *two cyclic planes* (comp. 196, (17.), &c.) *of any one of the concentric cones*, which rest on any such conic, *coincide with the two cyclic planes of the ellipsoid*: all this resulting, with the greatest ease, from the *construction* (217) to which quaternions had conducted.

(9.) With respect to the figure 53, which was designed to illustrate that construction, the signification of the letters ABCDD′EFF′GHH′LN has been already explained. But as regards the other letters we may here add, Ist, that N′ is a second minor summit of the surface, so that AN′ = NA; IInd, that K is a point in which the chord AF′, of what we may here call the *diacentric circle* AGF, intersects what may be called the *principal ellipse*,* or the *section* NBLEN′ of the ellipsoid, made by the *plane of the greatest and least axes*, that is by the plane of the *generating triangle* ABC, so that the lengths of AK and BF are equal; IIIrd, that the *tangent*, VKV′, to this ellipse at this point, is *parallel to the side* AB *of the triangle*, or to the *axis of revolution of the enveloping cylinder* 218, (5.), being in fact *one side* (or *generatrix*) of that cylinder; IVth, that AK, AB are thus two *conjugate semidiameters* of the ellipse, and therefore the tangent TBT′, at the point B of that ellipse, is parallel to the line AKF′, or perpendicular to the line BFF′; Vth, this latter line is thus the *normal* (comp. 218, (7.), (11.)) to the same elliptic section, and therefore also to the ellipsoid, at B; VIth, that the *least distance* KK′ between the parallels AB, KV, being = the radius b of the cylinder, is equal in length to the line BG, and also to each of the two semidiameters, AS, AS′ of the ellipse, which are *radii of the two circular sections* of the ellipsoid, in planes perpendicular to the plane of the figure; VIIth, that AS touches the circle at A; and VIIIth, that the point S′ is on the chord AI of that circle, which is drawn at right angles to the side BC of the triangle.

220. The reader will easily conceive that the quaternion equation of the ellipsoid admits of being put under several other forms; among which, however, it may here suffice to mention one, and to assign its geometrical interpretation.

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* In the plane of what is called, by many modern geometers, the *focal hyperbola* of the ellipsoid.

(1.) For any three vectors, ι, κ, ρ, we have the transformations,

$$\text{XLVII.} \ldots \mathrm{N}\left(\frac{\iota}{\rho} + \mathrm{K}\frac{\kappa}{\rho}\right) = \mathrm{N}\frac{\iota}{\rho} + \mathrm{N}\frac{\kappa}{\rho} + 2\mathrm{S}\frac{\iota}{\rho}\frac{\kappa}{\rho}$$

$$= \mathrm{N}\frac{\iota}{\kappa}\mathrm{N}\frac{\kappa}{\rho} + \mathrm{N}\frac{\kappa}{\iota}\mathrm{N}\frac{\iota}{\rho} + 2\mathrm{S}\frac{\iota}{\rho}\frac{\kappa}{\rho}\mathrm{T}\frac{\kappa}{\iota}\mathrm{T}\frac{\iota}{\kappa}$$

$$= \mathrm{N}\left(\frac{\iota}{\rho}\mathrm{T}\frac{\kappa}{\iota} + \mathrm{K}\frac{\kappa}{\rho}\mathrm{T}\frac{\iota}{\kappa}\right) = \mathrm{N}\left(\frac{\kappa}{\rho}\mathrm{T}\frac{\iota}{\kappa} + \mathrm{K}\frac{\iota}{\rho}\mathrm{T}\frac{\kappa}{\iota}\right)$$

$$= \mathrm{N}\left(\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho} + \mathrm{K}\frac{\mathrm{U}\kappa\,.\,\mathrm{T}\iota}{\rho}\right) = \mathrm{N}\left(\frac{\mathrm{U}\kappa\,.\,\mathrm{T}\iota}{\rho} + \mathrm{K}\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho}\right);$$

whence follows this other general transformation:

$$\text{XLVIII.} \ldots \mathrm{T}\left(\iota + \mathrm{K}\frac{\kappa}{\rho}\,.\,\rho\right) = \mathrm{T}\left(\mathrm{U}\kappa\,.\,\mathrm{T}\iota + \mathrm{K}\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho}\,.\,\rho\right).$$

(2.) If then we introduce two new auxiliary and constant vectors, ι' and κ', defined by the equations,

$$\text{XLIX.} \ldots \iota' = -\mathrm{U}\kappa\,.\,\mathrm{T}\iota, \quad \kappa' = -\mathrm{U}\iota\,.\,\mathrm{T}\kappa,$$

which give

$$\text{L.} \ldots \mathrm{T}\iota' = \mathrm{T}\iota, \quad \mathrm{T}\kappa' = \mathrm{T}\kappa, \quad \mathrm{T}(\iota' - \kappa') = \mathrm{T}(\iota - \kappa), \quad \mathrm{T}\iota'^2 - \mathrm{T}\kappa'^2 = t^2,$$

we may write the equation XVI. (in 217) of the ellipsoid under the following precisely similar form:

$$\text{LI.} \ldots \frac{t^2}{\mathrm{T}\rho} = \mathrm{T}\left(\iota' + \mathrm{K}\frac{\kappa'}{\rho}\,.\,\rho\right);$$

in which ι' and κ' have simply taken the places of ι and κ.

(3.) Retaining then the *centre* A of the ellipsoid, construct a *new diacentric sphere*, with a new centre C′, and a new *generating triangle*, AB′C′, where B′ is a *new fixed external point*, but the *lengths of the sides* are the *same*, by the conditions,

$$\text{LII.} \ldots \mathrm{AC'} = -\kappa', \quad \mathrm{C'B'} = +\iota', \quad \text{and therefore} \quad \mathrm{AB'} = \iota' - \kappa';$$

draw any secant B′D″D‴ (instead of BDD′), and set off a line AE in the direction of AD″, or in the opposite direction, with a length equal to that of BD‴; *the locus of the point* E *will be the same ellipsoid as before.*

(4.) The only inference which we shall here* draw from this new construction is, that there exists (as is known) a *second enveloping cylinder of*

* If room shall allow, a few additional remarks may be made, on the relations of the constant vectors ι, κ, &c., to the ellipsoid, and on some other constructions of that surface, when, in the following Book, its equation shall come to be put under the new form, $\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$. [See 404.]

revolution, and that its axis is the side AB′ of the new triangle AB′C′; but that the *radius* of this second cylinder is equal to that of the first, namely to the *mean semiaxis*, b, of the ellipsoid; and that the *major semiaxis*, a, or the line AL in fig. 53, *bisects the angle* BAB′, *between the two axes of revolution* of these two circumscribed cylinders: the *plane* of the new ellipse of contact being geometrically determined by a process exactly similar to that employed in 218, (7.); and being perpendicular to the new vector, $\iota' + \kappa'$, as the old plane of contact was (by 218, (11.)) to $\iota + \kappa$,

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SECTION 14.

On the Reduction of the General Quaternion to a Standard Quadrinomial Form; with a First Proof of the Associative Principle of Multiplication of Quaternions.

221. Retaining the significations (181) of the three rectangular unit-lines OI, OJ, OK, as the *axes*, and therefore also the *indices* (159), of three given right versors, i, j, k, in three mutually rectangular planes, we can express the *index* OQ of any *other* right quaternion, such as $\mathrm{V}q$, under the *trinomial form* (comp. 62),

$$\text{I.} \ldots \mathrm{IV}q = \mathrm{OQ} = x \,.\, \mathrm{OI} + y \,.\, \mathrm{OJ} + z \,.\, \mathrm{OK};$$

where xyz are some three scalar coefficients, namely, the three rectangular co-ordinates of the extremity Q of the index, with respect to the three axes OI, OJ, OK. Hence we may write also generally, by 206 and 126,

$$\text{II.} \ldots \mathrm{V}q = xi + yj + zk = ix + jy + kz;$$

and this last form, $ix + jy + kz$, may be said to be a *Standard Trinomial Form*, to which *every right quaternion*, or the *right part* $\mathrm{V}q$ of *any* proposed quaternion q, can be (as above) *reduced*. If then we denote by w the *scalar part*, $\mathrm{S}q$, of the same general quaternion q, we shall have, by 202, the following *General Reduction of a Quaternion to a* STANDARD QUADRINOMIAL FORM (183):

$$\text{III.} \ldots q = (\mathrm{S}q + \mathrm{V}q =)\ w + ix + jy + kz;$$

in which the *four scalars*, $wxyz$, may be said to be the *Four Constituents of the Quaternion*. And it is evident (comp. 202, (5.), and 133), that if we write in like manner,

$$\text{IV.} \ldots q' = w' + ix' + jy' + kz',$$

where ijk denote the same three given right versors (181) as before, then *the equation*

$$\text{V.} \ldots q' = q,$$

between these two quaternions, q and q', *includes the four following scalar equations between the constituents:*

$$\text{VI.} \ldots w' = w, \qquad x' = x, \qquad y' = y, \qquad z' = z;$$

which is a new justification (comp. 112, 116) of the *propriety* of *naming*, as we have done throughout the present Chapter, the *General Quotient of two Vectors* (101) a QUATERNION.

222. When the *Standard Quadrinomial Form* (221) is adopted, we have then not only

$$\text{I.} \ldots \mathrm{S}q = w, \quad \text{and} \quad \mathrm{V}q = ix + jy + kz,$$

as before, but also, by 204, XI.,

$$\text{II.} \ldots \mathrm{K}q = (\mathrm{S}q - \mathrm{V}q =)\ w - ix - jy - kz.$$

And because the *distributive property of multiplication* of quaternions (212), combined with the *laws of the symbols* ijk (182), or with the *General and Fundamental Formula of this whole Calculus* (183), namely with the formula,

$$i^2 = j^2 = k^2 = ijk = -1, \tag{A}$$

gives the transformation,

$$\text{III.} \ldots (ix + jy + kz)^2 = -(x^2 + y^2 + z^2),$$

we have, by 204, &c., the following new expressions:

$$\text{IV.} \ldots \mathrm{NV}q = (\mathrm{TV}q)^2 = -\mathrm{V}q^2 = x^2 + y^2 + z^2;$$

$$\text{V.} \ldots \mathrm{TV}q = \sqrt{(x^2 + y^2 + z^2)};$$

$$\text{VI.} \ldots \mathrm{UV}q = (ix + jy + kz) : \sqrt{(x^2 + y^2 + z^2)};$$

$$\text{VII.} \ldots \mathrm{N}q = \mathrm{T}q^2 = \mathrm{S}q^2 - \mathrm{V}q^2 = w^2 + x^2 + y^2 + z^2;$$

$$\text{VIII.} \ldots \mathrm{T}q = \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{IX.} \ldots \mathrm{U}q = (w + ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{X.} \ldots \mathrm{SU}q = w : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{XI.} \ldots \mathrm{VU}q = (ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{XII.} \ldots \mathrm{TVU}q = \sqrt{\frac{x^2 + y^2 + z^2}{w^2 + x^2 + y^2 + z^2}}.$$

(1.) To prove the recent formula III., we may arrange as follows the steps of the multiplication (comp. again 182):

$$\begin{array}{rl} \mathrm{V}q &= ix + jy + kz, \\ \mathrm{V}q &= ix + jy + kz; \\ \hline ix \,.\, \mathrm{V}q &= -x^2 + kxy - jxz; \\ jy \,.\, \mathrm{V}q &= -y^2 - kyx \qquad + iyz, \\ kz \,.\, \mathrm{V}q &= -z^2 \qquad + jzx - izy; \\ \hline \mathrm{V}q^2 = \mathrm{V}q \,.\, \mathrm{V}q &= -x^2 - y^2 - z^2. \end{array}$$

(2.) We have, therefore,

$$\text{XIII.} \ldots (ix + jy + kz)^2 = -1, \quad \text{if} \quad x^2 + y^2 + z^2 = 1,$$

a result to which we have already alluded,* in connexion with the partial *indeterminateness* of signification, in the present calculus, of the symbol $\sqrt{-1}$, when considered as denoting a *right radial* (149), or a *right versor* (153), of which the *plane* or the *axis* is *arbitrary*.

(3.) If $q'' = q'q$, then $\mathrm{N}q'' = \mathrm{N}q' \,.\, \mathrm{N}q$, by 191, (8); but if $q = w + \&c.$, $q' = w' + \&c.$, $q'' = w'' + \&c.$, then

$$\text{XIV.} \ldots \begin{cases} w'' = w'w - (x'x + y'y + z'z), \\ x'' = (w'x + x'w) + (y'z - z'y), \\ y'' = (w'y + y'w) + (z'x - x'z), \\ z'' = (w'z + z'w) + (x'y - y'x); \end{cases}$$

and conversely these four scalar equations are jointly equivalent to, and may be summed up in, the quaternion formula,

$$\text{XV.} \ldots w'' + ix'' + jy'' + kz'' = (w' + ix' + jy' + kz')(w + ix + jy + kz);$$

we ought therefore, under these conditions XIV., to have the equation,

$$\text{XVI.} \ldots w''^2 + x''^2 + y''^2 + z''^2 = (w'^2 + x'^2 + y'^2 + z'^2)(w^2 + x^2 + y^2 + z^2);$$

which can in fact be verified by so easy an algebraical calculation, that its truth may be said to be obvious upon mere inspection, at least when the terms in the four quadrinomial expressions $w'' \ldots z''$ are arranged† as above.

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* Compare the first Note to page 133; and that to page 162.

† From having somewhat otherwise *arranged* those terms, the author had some little trouble at first, in verifying that the twenty-four *double products*, in the expansion of $w''^2 + \&c.$, destroy each other, leaving only the sixteen *products of squares*, or that XVI. follows from XIV., when he was led to anticipate that result through quaternions, in the year 1843. He believes, however, that the *algebraic theorem* XVI., as distinguished from the *quaternion formula* XV., with which it is here connected, had been discovered by the celebrated EULER.

223. The principal *use* which we shall here make of the standard quadrinomial form (221) is to prove by it the general *associative property of multiplication* of quaternions; which can now with great ease be done, the *distributive** *property* (212) of such multiplication having been already proved. In fact, if we write, as in 222, (3.),

$$\text{I.} \ldots \begin{cases} q = w + ix + jy + kz, \\ q' = w' + ix' + jy' + kz', \\ q'' = w'' + ix'' + jy'' + kz'', \end{cases}$$

without now assuming that the relation $q'' = q'q$, or any other relation, exists between the three quaternions q, q', q'', and inquire whether it be true that the *associative formula*,

$$\text{II.} \ldots q''q' \cdot q = q'' \cdot q'q,$$

holds good, we see, by the distributive principle, that we have only to try whether this last formula is valid when the three quaternion factors q, q', q'' are replaced, in any one common order on both sides of the equation, and with or without repetition, by the three given right versors ijk; but this has already been proved, in Art. 183. We arrive then, thus, at the important conclusion, that *the General Multiplication of Quaternions is an Associative Operation*, as it had been previously seen (212) to be a *Distributive* one: although we had also found (168, 183, 191) that *such Multiplication is not* (in general) *Commutative*: or that *the two products*, $q'q$ and qq', are *generally unequal*. We may therefore *omit the point* (as in 183), and may denote each member of the equation II. by the symbol $q''q'q$.

(1.) Let $v = \mathrm{V}q$, $v' = \mathrm{V}q'$, $v'' = \mathrm{V}q''$; so that v, v', v'' are any three right quaternions, and therefore, by 191, (2.), and 196, 204,

$$\mathrm{K}v'v = vv', \qquad \mathrm{S}v'v = \tfrac{1}{2}(v'v + vv'), \qquad \mathrm{V}v'v = \tfrac{1}{2}(v'v - vv').$$

Let this last right quaternion be called $v_{,}$, and let $\mathrm{S}v'v = s_{,}$, so that $v'v = s_{,} + v_{,}$; we shall then have the equations,

$$2\mathrm{V}v''v_{,} = v''v_{,} - v_{,}v''; \qquad 0 = v''s_{,} - s_{,}v''$$

whence, by addition,

$$\begin{aligned} 2\mathrm{V}v''v_{,} &= v'' \cdot v'v - v'v \cdot v'' \\ &= (v''v' + v'v'')\, v - v'\,(v''v + vv'') \\ &= 2v\mathrm{S}v'v'' - 2v'\mathrm{S}v''v; \end{aligned}$$

* At a later stage [II. III. § 2], a sketch will be given of at least one proof of this *Associative Principle of Multiplication*, which will *not presuppose the Distributive Principle.*

and therefore generally, if v, v', v'' be still *right*, as above,

$$\text{III.} \ldots \mathrm{V} \,.\, v''\mathrm{V}v'v = v\mathrm{S}v'v'' - v'\mathrm{S}v''v\,;$$

a formula with which *the student ought to make himself completely familiar*, on account of its extensive utility.

(2.) With the recent notations,

$$\mathrm{V} \,.\, v''\mathrm{S}v'v = \mathrm{V}v''s_{,} = v''s_{,} = v''\mathrm{S}vv'\,;$$

we have therefore this other very useful formula,

$$\text{IV.} \ldots \mathrm{V} \,.\, v''v'v = v\mathrm{S}v'v'' - v'\mathrm{S}v''v + v''\mathrm{S}vv',$$

where the point in the first member may often for simplicity be dispensed with; and in which it is still supposed that

$$\angle\, v = \angle\, v' = \angle\, v'' = \frac{\pi}{2}.$$

(3.) The formula III. gives (by 206),

$$\text{V.} \ldots \mathrm{IV} \,.\, v''\mathrm{V}v'v = \mathrm{I}v \,.\, \mathrm{S}v'v'' - \mathrm{I}v' \,.\, \mathrm{S}v''v\,;$$

hence this last vector, which is evidently *complanar with the two indices* Iv and Iv', is at the same time (by 208) *perpendicular to the third index* Iv'', and therefore (by (1.)) *complanar with the third quaternion* q''.

(4.) With the recent notations, the vector,

$$\text{VI.} \ldots \mathrm{I}v_{,} = \mathrm{IV}v'v = \mathrm{IV}\,(\mathrm{V}q' \,.\, \mathrm{V}q),$$

is (by 208, XXII.) a line perpendicular to both Iv and Iv'; or *common to the planes* of q and q'; being also such that the *rotation* round it from Iv' to Iv is *positive*: while *its length*,

$$\mathrm{TI}v_{,}, \quad\text{or}\quad \mathrm{T}v_{,}, \quad\text{or}\quad \mathrm{TV} \,.\, v'v, \quad\text{or}\quad \mathrm{TV}\,(\mathrm{V}q' \,.\, \mathrm{V}q),$$

bears to the unit of length the same ratio, as that which the parallelogram under the indices, Iv and Iv', *bears to the unit of area.*

(5.) To interpret (comp. IV.) the *scalar expression*,

$$\text{VII.} \ldots \mathrm{S}v''v'v = \mathrm{S}v''v_{,} = \mathrm{S} \,.\, v''\mathrm{V}v'v,$$

(because $\mathrm{S}v''s_{,} = 0$), we may employ the formula 208, V.; which gives the transformation,

$$\text{VIII.} \ldots \mathrm{S}v''v'v = \mathrm{T}v'' \,.\, \mathrm{T}v_{,} \,.\, \cos(\pi - x)\,;$$

where Tv'' denotes the *length* of the *line* Iv'', and T$v_{,}$ represents by (4.) the *area* (positively taken) of the *parallelogram* under Iv' and Iv; while x is (by

208), the *angle* between the two indices Iv'', $Iv_{\prime}$. This angle will be *obtuse*, and therefore the cosine of its supplement will be *positive*, and equal to the *sine of the inclination of the line* Iv'' *to the plane of* Iv *and* Iv', if the *rotation* round Iv'' from Iv' to Iv be *negative*, that is, if the rotation round Iv from Iv' to Iv'' be *positive*; but that cosine will be equal the negative of this sine, if the direction of this rotation be reversed. We have therefore the important interpretation:

$$\text{IX.} \ldots \mathrm{S}v''v'v = \pm \textit{ volume of parallelepiped under } \mathrm{I}v, \mathrm{I}v', \mathrm{I}v'';$$

the *upper* or the *lower sign* being taken, according as the rotation round Iv, from Iv' to Iv'', is *positively* or *negatively directed.*

(6.) For example, we saw that the ternary products ijk and kji have scalar values, namely,

$$ijk = -1, \qquad kji = +1 \text{ by 183, (1.), (2.)};$$

and accordingly the *parallelepiped of indices* becomes, in this case, an *unit-cube*; while the rotation round the index of i, from that of j to that of k, is *positive* (181).

(7.) In general, for any three *right* quaternions $vv'v''$, we have the formula,

$$\text{X.} \ldots \mathrm{S}vv'v'' = -\mathrm{S}v''v'v;$$

and when the three indices are *complanar*, so that the *volume* mentioned in IX. *vanishes*, then each of these two last scalars becomes *zero*; so that we may write, as a new *Formula of Complanarity*;

$$\text{XI.} \ldots \mathrm{S}v''v'v = 0, \quad \text{if} \quad \mathrm{I}v'' \mathbin{|||} \mathrm{I}v', \mathrm{I}v \text{ (123)}:$$

while, on the other hand, this scalar cannot vanish in any *other* case, if the quaternions (or their indices) be still supposed to be *actual* (1, 144); because it then represents an actual volume.

(8.) Hence also we may establish the following *Formula of Collinearity*, for *any three quaternions*:

$$\text{XII.} \ldots \mathrm{S}(\mathrm{V}q'' . \mathrm{V}q' . \mathrm{V}q) = 0, \quad \text{if} \quad \mathrm{IV}q'' \mathbin{|||} \mathrm{IV}q', \mathrm{IV}q;$$

that is, by 209, if the *planes* of q, q', q'' have any *common line*.

(9.) In general, if we employ the *standard trinomial form*, 221, II., namely,

$$v = \mathrm{V}q = ix + jy + kz, \qquad v' = ix' + \&\text{c.}, \qquad v'' = ix'' + \&\text{c.},$$

the laws (182, 183) of the symbols i, j, k give the transformation,

$$\text{XIII.} \ldots \mathrm{S}v''v'v = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);$$

and accordingly this is the known expression for the volume (with a suitable sign) of the parallelepiped, which has the three lines OP, OP$'$, OP$''$ for three co-initial edges, if the rectangular co-ordinates* of the four corners, O, P, P$'$, P$''$, be 000, xyz, $x'y'z'$, $x''y''z''$.

(10.) Again, as another important consequence of the general associative property of multiplication, it may be here observed, that although products of *more than two* quaternions have *not* generally *equal scalars*, for *all* possible permutations of the factors, since we have just seen a case X. in which such a change of arrangement produces a change of *sign* in the result, yet *cyclical permutation is permitted, under the sign* S; or in symbols, that for *any three quaternions* (and the result is easily extended to *any greater number* of such factors) the following formula holds good:

$$\text{XIV.} \ldots \ \mathrm{S}q''q'q = \mathrm{S}qq''q'.$$

In fact, to prove this equality, we have only to write it thus,

$$\text{XIV}'. \ldots \ \mathrm{S}\,(q''q' \,.\, q) = \mathrm{S}\,(q \,.\, q''q'),$$

and to remember that the scalar of the product of any *two* quaternions remains unaltered (198, I.), when the order of those two factors is changed.

(11.) In like manner, by 192, II., it may be inferred that

$$\text{XV.} \ldots \ \mathrm{K}q''q'q = \mathrm{K}\,(q''.\,q'q) = \mathrm{K}q'q \,.\, \mathrm{K}q'' = \mathrm{K}q \,.\, \mathrm{K}q' \,.\, \mathrm{K}q'',$$

with a corresponding result for any greater number of factors; whence by 192, I., if Πq and $\Pi' q$ denote the *products* of any one set of quaternions taken in two *opposite orders*, we may write,

$$\text{XVI.} \ldots \ \mathrm{K}\Pi q = \Pi'\mathrm{K}q; \qquad \text{XVII.} \ldots \ \mathrm{R}\Pi q = \Pi'\mathrm{R}q.$$

(12.) But if v be *right*, as above, then $\mathrm{K}v = -v$, by 144; hence,

$$\text{XVIII.} \ldots \ \mathrm{K}\Pi v = \pm \Pi' v; \quad \text{XIX.} \ldots \ \mathrm{S}\Pi v = \pm \mathrm{S}\Pi' v; \quad \text{XX.} \ldots \ \mathrm{V}\Pi v = \mp \mathrm{V}\Pi' v;$$

upper or *lower signs* being taken, according as the *number* of the right factors is *even* or *odd*; and under the same conditions,

$$\text{XXI.} \ldots \ \mathrm{S}\Pi v = \tfrac{1}{2}\,(\Pi v \pm \Pi' v); \qquad \text{XXII.} \ldots \ \mathrm{V}\Pi v = \tfrac{1}{2}\,(\Pi v \mp \Pi' v);$$

as was lately exemplified (1.), for the case where the number is *two*.

* This result may serve as an example of the manner in which *quaternions*, although *not based* on any usual doctrine of *co-ordinates*, may yet be employed to *deduce*, or to *recover*, and often with great ease, important co-ordinate expressions.

(13.) For the case where that number is *three*, the four last formulæ give,

$$\text{XXIII.} \ldots \mathrm{S}v''v'v = -\,\mathrm{S}vv'v'' = \tfrac{1}{2}\,(v''v'v - vv'v'');$$

$$\text{XXIV.} \ldots \mathrm{V}v''v'v = +\,\mathrm{V}vv'v'' = \tfrac{1}{2}\,(v''v'v + vv'v'');$$

results which obviously agree with X. and IV.

224. For the case of *Complanar Quaternions* (119), the power of reducing each (120) to the form of a fraction (101) which shall have, at pleasure, for its denominator or for its numerator, any arbitrary line in the given plane, furnishes some peculiar facilities for proving the *commutative* and *associative* properties of *Addition* (207), and the *distributive* and *associative* properties of *Multiplication* (212, 223); while, for this *case* of multiplication of quaternions, we have already seen (191, (1.)) that the *commutative* property *also* holds good, as it does in algebraic multiplication. It may therefore be not irrevelant nor useless to insert here a short Second Chapter on the subject of such *complanars*: in treating briefly of which, while assuming as proved the existence of all the foregoing properties, we shall have an opportunity to say something of Powers and Roots and Logarithms; and of the connexion of Quaternions with Plane Trigonometry, and with Algebraical Equations. After which, in the Third and last Chapter of this Second Book, we propose to resume, for a short time, the consideration of *Diplanar Quaternions*; and especially to show how the *Associative Principle of Multiplication* can be established, for them, *without** *employing the Distributive Principle.*

* Compare the Note to page 245.

CHAPTER II.

ON COMPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN ONE PLANE; AND ON POWERS, ROOTS, AND LOGARITHMS OF QUATERNIONS.

SECTION 1.

On Complanar Proportion of Vectors; Fourth Proportional to Three, Third Proportional to Two, Mean Proportional, Square Root; General Reduction of a Quaternion in a Given Plane, to a Standard Binomial Form.

225. The Quaternions of the present Chapter shall all be supposed to be *complanar* (119); their *common plane* being assumed to coincide with that of the given right versor i (181). And *all the lines*, or vectors, such as α, β, γ, &c., or α_0, α_1, α_2, &c., to be here employed, shall be conceived to be *in that given plane* of i; so that we may write (by 123), for the purposes of this Chapter, the *formulæ of complanarity*:

$$q \parallel\!\mid q' \parallel\!\mid q'' \ldots \parallel\!\mid i; \qquad \alpha \parallel\!\mid i, \qquad \beta \parallel\!\mid i, \qquad \alpha_0 \parallel\!\mid i, \text{ \&c.}$$

226. Under these conditions, we can always (by 103, 117) interpret any symbol of the form $(\beta : \alpha) \cdot \gamma$, as denoting a *line* δ in the given plane; which line may also be denoted (125) by the symbol $(\gamma : \alpha) \cdot \beta$, but *not** (comp. 103) by either of the two apparently equivalent symbols, $(\beta \cdot \gamma) : \alpha$, $(\gamma \cdot \beta) : \alpha$; so that we may write,

$$\text{I.} \ldots \delta = \frac{\beta}{\alpha}\gamma = \frac{\gamma}{\alpha}\beta,$$

and may say that this line δ is the *Fourth Proportional* to the three lines α, β, γ; or to the three lines α, γ, β; the *two Means*, β and γ, of any such

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* In fact the symbols $\beta \cdot \gamma$, $\gamma \cdot \beta$, or $\beta\gamma$, $\gamma\beta$, have not as yet received with us *any* interpretation; and even when they shall come to be interpreted as representing certain quaternions, it will be found (comp. 168) that the two combinations, $\frac{\beta}{\alpha}\gamma$ and $\frac{\beta\gamma}{\alpha}$, have generally different significations.

Complanar Proportion of Four Vectors, admitting thus of being *interchanged*, as in algebra. Under the same conditions we may write also (by 125),

$$\text{II.} \ldots \alpha = \frac{\beta}{\delta}\gamma = \frac{\gamma}{\delta}\beta; \quad \beta = \frac{\alpha}{\gamma}\delta = \frac{\delta}{\gamma}\alpha; \quad \gamma = \frac{\delta}{\beta}\alpha = \frac{\alpha}{\beta}\delta;$$

so that (still as in algebra) the *two Extremes*, α and δ, of any such proportion of four lines α, β, γ, δ, may likewise change places among themselves: while we may also make the *means* become the *extremes*, if we at the same time change the extremes to means. More generally, if $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$ be *any odd number* of vectors in the given plane, we can always find *another* vector ρ in that plane, which shall satisfy the equation,

$$\text{III.} \ldots \quad \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta}\alpha = \rho, \quad \text{or} \quad \text{III}'\ldots \quad \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta}\frac{\alpha}{\rho} = 1;$$

and when such a formula holds good, for any *one* arrangement of the *numerator-lines* $\alpha, \gamma, \varepsilon, \ldots$ and of the *denominator-lines* $\rho, \beta, \delta \ldots$ it can easily be proved to hold good also for any *other* arrangement of the numerators, and any other arrangement of the denominators. For example, whatever four (complanar) vectors may be denoted by $\beta\gamma\delta\varepsilon$, we have the transformations,

$$\text{IV.} \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta} = \frac{\varepsilon}{\delta}\gamma : \beta = \frac{\gamma}{\delta}\varepsilon : \beta = \frac{\gamma}{\delta}\frac{\varepsilon}{\beta},$$

the two numerators being thus interchanged. Again,

$$\text{IV}'\ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta} = \frac{\gamma}{\beta}\frac{\varepsilon}{\delta} = \frac{\varepsilon}{\beta}\frac{\gamma}{\delta}, \text{ by IV.};$$

so that the two denominators also may change places.

227. An interesting *case* of such *proportion* (226) is that in which the *means coincide*; so that only *three distinct lines*, such as α, β, γ, are involved: and that we have (comp. Art. 149, and fig. 42 [p. 133]) an equation of the form,

$$\text{I.} \ldots \gamma = \frac{\beta}{\alpha}\beta, \quad \text{or} \quad \alpha = \frac{\beta}{\gamma}\beta,$$

but *not** $\gamma = \beta\beta : \alpha$, nor $\alpha = \beta\beta : \gamma$. In this case, it is said that the *three lines* $\alpha\beta\gamma$ form a *Continued Proportion*; of which α and γ are now the *Extremes*, and β is the *Mean*: this line β being also said to be *a*† *Mean Proportional*

* Compare the Note to the foregoing Article.

† We say, *a mean proportional*; because we shall shortly see that the *opposite line*, $-\beta$, is in the same sense *another mean*; although a *rule* will presently be given, for distinguishing between them, and for *selecting one* as that which may be called, by eminence, *the mean proportional.*

between the two others, α and γ; while γ is *the Third Proportional* to the two lines α and β; and α is, at the same time, the third proportional to γ and β. Under the same conditions, we have

$$\text{II.} \ldots \beta = \frac{\alpha}{\beta}\gamma = \frac{\gamma}{\beta}\alpha;$$

so that this *mean*, β, between α and γ, is also the *fourth proportional* (226) *to itself, as first*, and to those *two other* lines. We have also (comp. again 149),

$$\text{III.} \ldots \left(\frac{\beta}{\alpha}\right)^2 = \frac{\gamma}{\alpha}, \quad \left(\frac{\beta}{\gamma}\right)^2 = \frac{\alpha}{\gamma};$$

whence it is natural to write,

$$\text{IV.} \ldots \frac{\beta}{\alpha} = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}, \quad \frac{\beta}{\gamma} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}},$$

and therefore by (103),

$$\text{V.} \ldots \beta = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}\alpha, \quad \beta = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\gamma;$$

although we are *not* here to write $\beta = (\gamma\alpha)^{\frac{1}{2}}$, nor $\beta = (\alpha\gamma)^{\frac{1}{2}}$. But because we have always, as in algebra (comp. 199, (3.)), the equation or identity, $(-q)^2 = q^2$, we are equally well entitled to write,

$$\text{VI.} \ldots \left(\frac{-\beta}{\alpha}\right)^2 = \frac{\gamma}{\alpha}, \quad \left(\frac{-\beta}{\gamma}\right)^2 = \frac{\alpha}{\gamma}, \quad -\beta = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}\alpha = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\gamma;$$

the symbol $q^{\frac{1}{2}}$ denoting thus, in general, *either of two opposite quaternions*, whereof however one, namely that one of which the *angle* is *acute*, has been already *selected* in 199, (1.), as that which shall be called by us *the Square Root* of the quaternion q, and denoted by $\sqrt{q}$. We may therefore establish the formula,

$$\text{VII.} \ldots \beta = \pm\sqrt{\left(\frac{\gamma}{\alpha}\right)}\,.\,\alpha = \pm\sqrt{\left(\frac{\alpha}{\gamma}\right)}\,.\,\gamma,$$

if α, β, γ form, as above, a continued proportion; the *upper signs* being taken when (as in fig. 42) the *angle* AOC, between the extreme lines α, γ, is *bisected* by the line OB, or β, *itself*; but the *lower signs*, when that angle is bisected by the *opposite line*, $-\beta$, or when β bisects the *vertically opposite angle* (comp. again 199, (3.)) : but the *proportion of tensors*,

$$\text{VIII.} \ldots T\gamma : T\beta = T\beta : T\alpha,$$

and the resulting formulæ

$$\text{IX.} \ldots T\beta^2 = T\alpha\,.\,T\gamma, \quad T\beta = \sqrt{(T\alpha\,.\,T\gamma)},$$

in *each* case holding good. And when we shall speak simply of *the Mean*

Proportional between two vectors, α and γ, which make any acute, or right, or obtuse angle with each other, we shall always henceforth understand the *former* of these two bisectors; namely, the *bisector* OB of that *angle* AOC *itself*, and *not* that of the *opposite* angle: thus taking upper signs, in the recent formula VII.

(1.) At the limit when the angle AOC *vanishes*, so that $U\gamma = U\alpha$, then $U\beta$ = each of these two unit-lines; and the mean proportional β has the *same* common *direction* as each of the two given extremes. This comes to our agreeing to write,

$$\text{X.} \ldots \sqrt{1} = +1, \quad \text{and generally,} \quad \text{X}' \ldots \sqrt{(a^2)} = +a,$$

if a be any positive scalar.

(2.) At the *other* limit, when $\text{AOC} = \pi$, or $U\gamma = -U\alpha$, the *length* of the mean proportional β is still determined by IX., as the *geometric mean* (in the usual sense) between the lengths of the two given extremes (comp. the two figures 41 [p. 132]); but, even with the supposed restriction (225) on the *plane* in which all the lines are situated, an *ambiguity* arises in this case, from the doubt *which* of the two *opposite perpendiculars* at O, to the line AOC, is to be taken as the direction of the *mean vector*. To remove this ambiguity, we shall suppose that the *rotation* round the axis of i (to which *axis all* the lines considered in this Chapter are, by 225, perpendicular), from the first line OA to the second line OB, is in this case *positive*; which supposition is equivalent to writing, for present purposes,

$$\text{XI.*} \ldots \sqrt{-1} = +i; \quad \text{and} \quad \text{XI}' \ldots \sqrt{(-a^2)} = ia, \text{ if } a > 0.$$

And thus the *mean proportional* between two vectors (in the given plane) becomes, in *all* cases, *determined*: at least if their *order* (as first and third) be given.

(3.) If the *restriction* (225) on the *common plane* of the lines, were *removed*, we might *then*, on the recent plan (227), fix *definitely* the *direction*, as well as the *length*, of the *mean* OB, in *every* case *but one*: this excepted case being that in which, as in (2.), the two *given extremes*, OA, OC, have exactly opposite directions; so that the *angle* ($\text{AOC} = \pi$) between them has *no one definite bisector*. In this case, the sought point B would have *no one determined position*, but only a *locus*: namely the *circumference of a circle*, with O for

* It is to be carefully observed that *this square root of negative unity* is *not*, in any sense, *imaginary*, nor even *ambiguous*, in its geometrical *interpretation*, but denotes a *real and given right versor* (181).

centre, and with a radius equal to the geometric mean between $\overline{\text{OA}}$, $\overline{\text{OC}}$, while its plane would be perpendicular to the given right line AOC. (Comp. again the figures 41; and the remarks in 148, 149, 153, 154, on the *square* of a *right radial*, or versor, and on the partially indeterminate character of the *square root* of a *negative scalar*, when interpreted, on the plan of this Calculus, as a *real* in geometry.)

228. The quotient of any two complanar and right quaternions has been seen (191, (6.)) to be a scalar; since then we here suppose (225) that $q \parallel\!| i$, we are at liberty to write,

$$\text{I.} \ldots \mathrm{S}q = x; \qquad \mathrm{V}q : i = y; \qquad \mathrm{V}q = yi = iy;$$

and consequently may establish the following *Reduction of a Quaternion in the given plane* (of i) to a *Standard Binomial Form** (comp. 221):

$$\text{II.} \ldots q = x + iy, \quad \text{if} \quad q \parallel\!| i;$$

x and y being some *two scalars*, which may be called the *two constituents* (comp. again 221) of this binomial. And then an *equation* between *two* quaternions, considered as binomials of this form, such as the equation,

$$\text{III.} \ldots q' = q, \quad \text{or} \quad \text{III}'. \ldots x' + iy' = x + iy,$$

breaks up (by 202, (5.)) into *two scalar equations* between their respective *constituents*, namely,

$$\text{IV.} \ldots x' = x, \qquad y' = y,$$

notwithstanding the *geometrical reality* of the right versor, i.

(1.) On comparing the recent equations II., III., IV., with those marked as III., V., VI, in 221, we see that, in thus passing from *general* to *complanar* quaternions, we have merely *suppressed the coefficients of j and k*, as being for our present purpose, *null*; and have then written x and y, instead of w and x.

(2.) As the word "binomial" has other meanings in algebra, it may be convenient to call the form II. a COUPLE; and the two constituent scalars x and y, of which the values serve to distinguish one such couple from another, may not unnaturally be said to be the *Co-ordinates* of that *Couple*, for a reason which it may be useful to state.

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* It is *permitted*, by 227, XI., to write this expression as $x + y\sqrt{-1}$; but the form $x + iy$ is shorter, and perhaps less liable to any ambiguity of interpretation.

(3.) Conceive, then, that the plane of fig. 50 [p. 192] coincides with that of i, and that positive rotation round Ax . i is, in that figure, directed towards the *left-hand*; which may be reconciled with our general convention (127), by imagining that this *axis* of i is directed from O towards the *back* of the figure; or *below** it, if horizontal. This being assumed, and perpendiculars BB′, BB″ being let fall (as in the figure) on the indefinite line OA itself, and on a normal to that line at O, which normal we may call OA′, and may suppose it to have a length equal to that of OA, with a left-handed rotation AOA′, so that

$$\text{V.} \ldots \text{OA}' = i \,.\, \text{OA}, \quad \text{or briefly,} \quad \text{V}'. \ldots \alpha' = i\alpha,$$

while $\beta' = \text{OB}'$, and $\beta'' = \text{OB}''$, as in 201, and $q = \beta : \alpha$, as in 202;

then, on whichever side of the indefinite right line OA the point B may be situated, a comparison of the quaternion q with the binomial form II. will give the two equations,

$$\text{VI.} \ldots x\ (= \text{S}q) = \beta' : \alpha; \qquad y\ (= \text{V}q : i = \beta'' : i\alpha) = \beta'' : \alpha';$$

so that *these two scalars*, x and y, are precisely the *two rectangular co-ordinates of the point* B, *referred to the two lines* OA and OA′, *as two rectangular unit-axes*, of the *ordinary* (or Cartesian) kind. And since *every other* quaternion, $q' = x' + iy'$, in the given plane, can be reduced to the form $\gamma : \alpha$, or OC is to OA, where C is a point in that plane, which can be projected into C′ and C″ in the same way (comp. 197, 205), we see that the two *new scalars*, or constituents, x' and y', are simply (for the same reason) the *co-ordinates of the new point* C, referred to the same pair of axes.

(4.) It is evident (from the principles of the foregoing Chapter), that if we thus express as *couples* (2.) *any two* complanar quaternions, q and q', we shall have the following general transformations for their *sum*, *difference*, and *product*:

$$\text{VII.} \ldots q' \pm q = (x' \pm x) + i\,(y' \pm y);$$

$$\text{VIII.} \ldots q' \,.\, q = (x'x - y'y) + i\,(x'y + y'x).$$

(5.) Again, for any *one* such couple, q, we have (comp. 222) not only $\text{S}q = x$, and $\text{V}q = iy$, as above, but also,

$$\text{IX.} \ldots \text{K}q = x - iy; \quad \text{X.} \ldots \text{N}q = x^2 + y^2; \quad \text{XI.} \ldots \text{T}q = \sqrt{(x^2 + y^2)};$$

$$\text{XII.} \ldots \text{U}q = \frac{x + iy}{\sqrt{(x^2 + y^2)}}; \qquad \text{XIII.} \ldots \frac{1}{q} = \frac{x - iy}{x^2 + y^2}; \ \&\text{c.}$$

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* Compare the second Note to page 111.

(6.) Hence, for the *quotient* of any two such couples, we have,

$$\text{XIV.}\ldots\begin{cases}\dfrac{q'}{q}=\dfrac{x'+iy'}{x+iy}=\dfrac{x''+iy''}{x^2+y^2}, & x''+iy''=q'\mathrm{K}q,\\ x''=x'x+y'y, & y''=y'x-x'y.\end{cases}$$

(7.) The *law of the norms* (191, (8.)), or the formula, $\mathrm{N}q'q=\mathrm{N}q'\,.\,\mathrm{N}q$, is expressed here (comp. 222, (3.)) by the well-known algebraic equation, or identity,

$$\text{XV.}\ldots(x'^2+y'^2)(x^2+y^2)=(x'x-y'y)^2+(x'y+y'x)^2;$$

in which $xyx'y'$ may be *any four scalars*.

SECTION 2.

On Continued Proportion of Four or more Vectors; Whole Powers and Roots of Quaternions; and Roots of Unity.

229. The conception of *continued proportion* (227) may easily be extended from the case of *three* to that of *four* or more (complanar) vectors; and thus a theory may be formed of *cubes* and *higher whole powers of quaternions*, with a correspondingly extended theory of *roots* of quaternions, including roots of *scalars*, and in particular of *unity*. Thus if we suppose that the *four* vectors $\alpha\beta\gamma\delta$ form a continued proportion, expressed by the formulæ,

$$\text{I.}\ldots\frac{\delta}{\gamma}=\frac{\gamma}{\beta}=\frac{\beta}{\alpha},\quad\text{whence}\quad\text{II.}\ldots\frac{\delta}{\alpha}=\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=\left(\frac{\beta}{\alpha}\right)^3$$

(by an obvious extension of usual algebraic notation,) we may say that the quaternion $\delta:\alpha$ is *the cube*, or the *third power*, of $\beta:\alpha$; and that the latter quaternion is, conversely, *a cube-root* (or *third root*) of the former; which last relation may naturally be denoted by writing,

$$\text{III.}\ldots\frac{\beta}{\alpha}=\left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}},\quad\text{or}\quad\text{III}'.\ldots\beta=\left(\frac{\delta}{\alpha}\right)^{\cdot}\alpha\ \text{(comp. 227, IV., V.)}.$$

230. But it is important to observe that as the equation $q^2=Q$, in which q is a sought and Q is a given quaternion, was found to be satisfied by *two* opposite quaternions q, of the form $\pm\sqrt{Q}$ (comp. 227, VII.), so the slightly less simple equation $q^3=Q$ is satisfied by *three* distinct and real quaternions, if Q be actual and real; whereof *each*, divided by *either* of the other two, gives for *quotient* a *real* quaternion, which is equal to *one of the cube-roots of*

positive unity. In fact, if we conceive (comp. the annexed fig. 54) that β' and β'' are two other but equally long vectors in the given plane, obtained from β by two successive and positive rotations, each through the third part of a circumference, so that

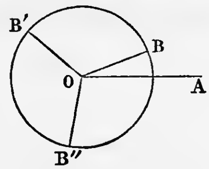

Fig. 54.

$$\text{IV.} \ldots \frac{\beta}{\beta''} = \frac{\beta''}{\beta'} = \frac{\beta'}{\beta},$$

or

$$\text{IV}'. \ldots \frac{\beta}{\beta'} = \frac{\beta'}{\beta''} = \frac{\beta''}{\beta},$$

and therefore

$$\text{V.} \ldots \left(\frac{\beta'}{\beta}\right)^3 = \left(\frac{\beta''}{\beta}\right)^3 = 1, \text{ while V}'. \ldots \frac{\beta''}{\beta} = \left(\frac{\beta'}{\beta}\right)^2, \quad \frac{\beta'}{\beta} = \left(\frac{\beta''}{\beta}\right)^2,$$

we shall have

$$\text{VI.} \ldots \left(\frac{\beta'}{\alpha}\right)^3 = \left(\frac{\beta'}{\beta}\right)^3 \left(\frac{\beta}{\alpha}\right)^3 = \frac{\delta}{\alpha}, \text{ and VI}'. \ldots \left(\frac{\beta''}{\alpha}\right)^3 = \frac{\delta}{\alpha};$$

so that we are equally entitled, at this stage, to write, instead of III. or III′., these other equations:

$$\text{VII.} \ldots \frac{\beta'}{\alpha} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}, \quad \beta' = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}} \alpha;$$

or

$$\text{VII}'. \ldots \frac{\beta''}{\alpha} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}, \quad \beta'' = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}} \alpha.$$

231. A (real and actual) quaternion Q may thus be said to have *three* (real, actual, and) *distinct cube-roots*; of which however only *one* can have an *angle less than sixty degrees*; while *none* can have an angle *equal* to sixty degress, unless the proposed quaternion Q degenerates into a *negative scalar*. In every *other* case, *one of the three* cube-roots of Q, or one of the three values of the symbol $Q^{\frac{1}{3}}$, may be considered as *simpler* than either of the other two, because it has a *smaller angle* (comp. 199, (1.)); and if we, for the present, *denote this one*, which we shall call the *Principal Cube-Root* of the quaternion Q, by the symbol $\sqrt[3]{Q}$, we shall thus be enabled to establish the formula of inequality,

$$\text{VIII.} \ldots \angle \sqrt[3]{Q} < \frac{\pi}{3}, \quad \text{if} \quad \angle Q < \pi.$$

232. At the limit, when Q degenerates, as above, into a negative scalar, *one* of its cube-roots is *itself* a negative scalar, and has therefore its angle $= \pi$; while *each* of the two other roots has its angle $= \frac{\pi}{3}$. In *this* case, among these two roots of which the angles are equal to each other, and are less than that

of the third, we shall consider as *simpler*, and therefore as *principal*, the one which answers (comp. 227, (2.)) to a *positive rotation* through sixty degrees; and so shall be led to write,

$$\text{IX.} \ldots \sqrt[3]{-1} = \frac{1 + i\sqrt{3}}{2}; \quad \text{and} \quad \text{X.} \ldots \angle \sqrt[3]{-1} = \frac{\pi}{3};$$

using thus the *positive sign* for the radical $\sqrt{3}$, by which i is multiplied in the expression IX. for $2\sqrt[3]{-1}$; with the connected formula,

$$\text{IX}'. \ldots \sqrt[3]{(-a^3)} = \frac{a}{2}(1 + i\sqrt{3}), \quad \text{if} \quad a > 0;$$

although it might at first have seemed more natural to adopt as *principal* the *scalar* value, and to write thus,

$$\sqrt[3]{-1} = -1;$$

which latter is in fact *one value* of the symbol, $(-1)^{\frac{1}{3}}$.

(1.) We have, however, on the present plan, as in arithmetic,

$$\text{XI.} \ldots \sqrt[3]{1} = 1; \quad \text{and} \quad \text{XI}'. \ldots \sqrt[3]{(a^3)} = a, \text{ if } a > 0.$$

(2.) The equations,

$$\text{XII.} \ldots \left(\frac{1 + i\sqrt{3}}{2}\right)^3 = -1, \quad \text{and} \quad \text{XIII.} \ldots \left(\frac{-1 + i\sqrt{3}}{2}\right)^3 = +1,$$

can be verified in *calculation*, by actual *cubing*, exactly as in algebra; the only difference being, as regards the *conception* of the subject, that although i satisfies the equation $i^2 = -1$, it is regarded *here* as altogether *real*; namely, as a real *right versor** (181).

233. There is no difficulty in conceiving how the same general principles may be extended (comp. 229) to a *continued proportion* of $n + 1$ complanar vectors,

$$\text{I.} \ldots \alpha, \alpha_1, \alpha_2, \ldots \alpha_n,$$

when n is a whole number greater than three; nor in interpreting, in connexion therewith, the equations,

$$\text{II.} \ldots \frac{\alpha_n}{\alpha} = \left(\frac{\alpha_1}{\alpha}\right)^n; \quad \text{III.} \ldots \frac{\alpha_1}{\alpha} = \left(\frac{\alpha_n}{\alpha}\right)^{\frac{1}{n}}; \quad \text{IV.} \ldots \alpha_1 = \left(\frac{\alpha_n}{\alpha}\right)^{\frac{1}{n}} \alpha.$$

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* *This conception differs fundamentally* from one which had occurred to several able writers, *before* the invention of the quaternions; and according to which the symbols 1 and $\sqrt{-1}$ were interpreted as representing *a pair of equally long and mutually rectangular right lines, in a given plane.* In *Quaternions, no line* is represented by the *number*, ONE, except as regards its *length*; the *reason* being, mainly, that we require, in the present Calculus, to be able to deal with *all possible planes*; and that *no one right line is common to all such.*

Denoting, for the moment, what we shall call the *principal n^{th} root* of a quaternion Q by the symbol $\sqrt[n]{Q}$, we have, on this plan (comp. 231, VIII.),

$$\text{V.} \ldots \angle(\sqrt[n]{Q}) < \frac{\pi}{n}, \quad \text{if} \quad \angle Q < \pi;$$

$$\text{VI.} \ldots \angle(\sqrt[n]{-1}) = \frac{\pi}{n}; \qquad \text{VII.} \ldots \mathrm{V}(\sqrt[n]{-1}) : i > 0;$$

this last condition, namely that there shall be a *positive* (scalar) *coefficient* y of i, in the *binomial* (or *couple*) *form* $x + iy$ (228), for the quaternion $\sqrt[n]{-1}$, thus serving to complete the determination of that *principal n^{th}* root of *negative unity*; or of any *other* negative scalar, since -1 may be changed to $-a$, if $a > 0$, in each of the two last formulæ. And as to the *general n^{th} root of a quaternion*, we may write, on the same principles,

$$\text{VIII.} \ldots Q^{\frac{1}{n}} = 1^{\frac{1}{n}} \,.\, \sqrt[n]{Q};$$

where the factor $1^{\frac{1}{n}}$, representing the *general n^{th} root of positive unity*, has n *different values*, depending on the division of the circumference of a circle into n equal parts, in the way lately illustrated, for the case $n = 3$, by figure 54; and only differing from ordinary algebra by the *reality* here attributed to i. In fact, *each* of these *n^{th} roots of unity* is with us a *real versor*; namely the *quotient of two radii of a circle*, which make with each other an *angle*, equal to the *n^{th} part of some whole number of circumferences.*

(1.) We propose, however, to interpret the particular symbol $i^{\frac{1}{n}}$, as always denoting the *principal value* of the n^{th} root of i; thus writing,

$$\text{IX.} \ldots i^{\frac{1}{n}} = \sqrt[n]{i};$$

whence it will follow that when this root is expressed under the form of a *couple* (228), the two constituents x and y shall both be positive, and the quotient $y : x$ shall have a smaller value than for any other couple $x + iy$ (with constituents thus positive), of which the n^{th} power equals i.

(2.) For example, although the equation

$$q^2 = (x + iy)^2 = i,$$

is satisfied by the *two* values, $\pm (1 + i) : \sqrt{2}$, we shall write *definitely*,

$$\text{X.} \ldots i^{\frac{1}{2}} = + \sqrt{i} = \frac{1 + i}{\sqrt{2}}.$$

(3.) And although the equation,

$$q^3 = (x + iy)^3 = i,$$

is satisfied by the *three* distinct and real couples, $(i \pm \sqrt{3}) : 2$, and $-i$, we shall adopt only the *one* value,

$$\text{XI.} \ldots i^{\frac{1}{3}} = \sqrt[3]{i} = \frac{i + \sqrt{3}}{2}.$$

(4.) In general, we shall thus have the expression,

$$\text{XII.} \ldots i^{\frac{1}{n}} = \cos\frac{\pi}{2n} + i \sin\frac{\pi}{2n};$$

which we shall occasionally *abridge* to the following:

$$\text{XII}' \ldots i^{\frac{1}{n}} = \operatorname{cis}\frac{\pi}{2n};$$

and this *root*, $i^{\frac{1}{n}}$, thus interpreted, denotes a *versor*, which *turns* any line on which it operates, through an angle equal to the n^{th} *part of a right angle*, in the positive direction of rotation, round the given axis of i.

234. If m and n be *any two positive whole numbers*, and q any quaternion, the definition contained in the formula 233, II., of the *whole power*, q^n, enables us to write, as in algebra, the two equations:

$$\text{I.} \ldots q^m q^n = q^{m+n}; \qquad \text{II.} \ldots (q^n)^m = q^{mn};$$

and we propose to extend the former to the case of *null* and *negative* whole *exponents*, writing therefore,

$$\text{III.} \ldots q^0 = 1; \qquad \text{IV.} \ldots q^{m-n} = q^m : q^n;$$

and in particular,

$$\text{V.} \ldots q^{-1} = 1 : q = \frac{1}{q} = \textit{reciprocal*}\ (134)\ \text{of}\ q.$$

We shall also extend the formula II., by writing,

$$\text{VI.} \ldots (q^{\frac{1}{n}})^m = q^{\frac{m}{n}},$$

whether m be positive or negative; so that this last symbol, if m and n be still *whole* numbers, whereof n may be supposed to be *positive*, has *as many distinct values* as there are *units in the denominator* of its *fractional exponent*, when *reduced to its least terms*; among which values of $q^{\frac{m}{n}}$, we shall naturally consider as the *principal* one, that which is the m^{th} *power* of the principal n^{th} *root* (233) of q.

* Compare the first Note to page 123.

(1.) For example, the symbol $q^{\frac{2}{3}}$ denotes, on this plan, the *square of any cube-root* of q; it has therefore *three* distinct values, namely, the three values of the *cube-root of the square* of the same quaternion q; but among these we regard as principal, the *square of the principal cube-root* (231) of that proposed quaternion.

(2.) Again, the symbol $q^{\frac{2}{4}}$ is interpreted, on the same plan, as denoting the *square of any fourth root* of q; but because $(1^{\frac{1}{4}})^2 = 1^{\frac{1}{2}} = \pm 1$, this *square* has only *two* distinct values, namely those of the *square root* $q^{\frac{1}{2}}$, the fractional exponent $\frac{2}{4}$ being thus reduced to its least terms; and among these the *principal* value is the *square of the principal fourth root*, which square is, at the same time, the *principal square root* (199, (1.), or 227) of the quaternion q.

(3.) The symbol $q^{-\frac{1}{2}}$ denotes, as in algebra, the reciprocal of a square-root of q; while q^{-2} denotes the reciprocal of the square, &c.

(4.) If the exponent t, in a symbol of the form q^t, be still a *scalar*, but a *surd* (or incommensurable), we may consider this *surd exponent*, t, as a *limit*, towards which a *variable fraction tends*: and the symbol itself may then be interpreted as the corresponding limit of a *fractional power* of a quaternion, which has however (in this case) *indefinitely many values*, and can therefore be of little or no *use*, until a *selection* shall have been made, of *one* value of this *surd power* as *principal*, according to a law which will be best understood by the introduction of the conception of the *amplitude* of a quaternion, to which in the next section we shall proceed.

(5.) Meanwhile (comp. 233, (4.)), we may already *definitely interpret* the symbol i^t as denoting a *versor*, which *turns* any line in the given plane, *through t right angles*, round $\mathrm{Ax}\,.\,i$, in the positive or negative direction, according as this *scalar exponent*, t, whether rational or irrational, is itself positive or negative; and thus may establish the formula,

$$\text{VII.} \ldots i^t = \cos\frac{t\pi}{2} + i\sin\frac{t\pi}{2};$$

or briefly (comp. 233, XII′.),

$$\text{VIII.} \ldots i^t = \operatorname{cis}\frac{t\pi}{2}.$$

SECTION 3.

On the Amplitudes of Quaternions in a given Plane; and on Trigonometric Expressions for such Quaternions, and for their Powers.

235. Using the binomial or *couple form* (228) for a quaternion in the plane of i (225), if we introduce two new and real scalars, r and z, whereof the former shall be supposed to be positive, and which are connected with the two former scalars x and y by the equations,

$$\text{I.} \ldots x = r\cos z, \quad y = r\sin z, \quad r > 0,$$

we shall then evidently have the formulæ (comp. 228, (5.)):

$$\text{II.} \ldots \mathrm{T}q = \mathrm{T}(x + iy) \ = r;$$

$$\text{III.} \ldots \mathrm{U}q = \mathrm{U}(x + iy) = \cos z + i\sin z;$$

which last expression may be conveniently abridged (comp. 233, XII'., and 234, VIII.) to the following:

$$\text{IV.} \ldots \mathrm{U}q = \operatorname{cis} z; \quad \text{so that} \quad \text{V.} \ldots q = r\operatorname{cis} z.$$

And the arcual or angular quantity, z, may be called the *Amplitude** of the quaternion q; this name being here preferred by us to "*Angle*," because we have already appropriated the latter name, and the corresponding symbol $\angle q$, to denote (130) an *angle of the Euclidean kind*, or at least one not exceeding, in either direction, the *limits* 0 and π; whereas the *amplitude*, z, considered as obliged only to satisfy the equations I., may have *any real and scalar value*. We shall *denote* this amplitude, at least for the present, by the *symbol*,† am . q,

* Compare the Note to Art. 130.

† The symbol V was spoken of, in 202, as completing the *system of notations* peculiar to the present Calculus; and in fact, besides the *three letters*, i, j, k, of which the laws are expressed by the *fundamental formula* (A) of Art. 183, and which were originally (namely in the year 1843, and in the two following years) the *only peculiar symbols of quaternions* (see Note to page 160), that Calculus does not *habitually* employ, with peculiar significations, any more than the *five characteristics of operation*, K, S, T, U, V, for *conjugate*, *scalar*, *tensor*, *versor*, and *vector* (or *right part*): although perhaps the mark N for *norm*, which in the present work has been adopted from the *Theory of Numbers*, will gradually come more into use than it has yet done, in connexion with quaternions also. As to the marks, $\angle$, Ax., I, R, and now am. (or am_n), for *angle*, *axis*, *index*, *reciprocal*, and *amplitude*, they are to be considered as chiefly available for the present *exposition* of the system, and as not often wanted, nor employed, in the subsequent *practice* thereof; and the same remark applies to the recent *abridgment* cis, for $\cos + i\sin$; to some notations in the present section for *powers* and *roots*, serving to express the conception of one n^{th} root, &c., as *distinguished* from another; and to the characteristic P, of what we shall call in the next section the *ponential* of a quaternion, though not requiring that notation afterwards. No apology need be made for employing the purely *geometrical signs*, $\perp$, $\|$, $|||$, for *perpendicularity*, *parallelism*, and *complanarity*: although the *last* of them was perhaps first introduced by the present writer, who has found it frequently useful.

or simply, am q; and thus shall have the following formula, of *connexion between amplitude and angle*,

$$\text{VI.} \ldots (z =) \operatorname{am} . q = 2n\pi \pm \angle q;$$

the upper or the lower *sign* being taken, according as $\operatorname{Ax} . q = \pm \operatorname{Ax} . i$; and n being *any whole number*, positive or negative or null. We may then write also (for any quaternion $q \parallel\!| i$) the general transformations following:

$$\text{VII.} \ldots \mathrm{U}q = \operatorname{cis} \operatorname{am} q; \qquad \text{VIII.} \ldots q = \mathrm{T}q . \operatorname{cis} \operatorname{am} q.$$

(1.) Writing $q = \beta : \alpha$, the amplitude am . q, or am $(\beta : \alpha)$, is thus a scalar quantity, expressing (*with its proper sign*) the *amount of rotation*, round $\operatorname{Ax} . i$, from the line α to the line β; and admitting, *in general*, of being increased or diminished by *any whole number of circumferences*, or of *entire revolutions*, when only the *directions of the two lines*, α and β, in the given plane of i, are given.

(2.) But the *particular* quaternion, or right versor, i *itself*, shall be considered as having *definitely*, for *its* amplitude, *one right angle*; so that we shall establish the particular formula,

$$\text{IX.} \ldots \operatorname{am} . i = \angle i = \frac{\pi}{2}.$$

(3.) When, for any *other* given quaternion q, the generally *arbitrary integer* n in VI. receives any one *determined value*, the corresponding value of the amplitude may be denoted by either of the two following temporary symbols,* which we here treat as equivalent to each other,

$$\operatorname{am}_n . q, \quad \text{or} \quad \angle_n q;$$

so that (with the same rule of signs as before) we may write, as a more *definite* formula than VI., the equation:

$$\text{X.} \ldots \operatorname{am}_n . q = \angle_n q = 2n\pi \pm \angle q;$$

and may say that this last quantity is the n^{th} *value of the amplitude* of q; while the *zero-value*, $\operatorname{am}_0 q$, may be called the *principal amplitude* (or the *principal value* of the amplitude).

(4.) With these notations, and with the convention, $\operatorname{am}_0(-1) = +\pi$, we may write,

$$\text{XI.} \ldots \operatorname{am}_0 q = \angle_0 q = \pm \angle q;$$

$$\text{XII.} \ldots \operatorname{am}_n a = \operatorname{am}_n 1 = \angle_n 1 = 2n\pi, \quad \text{if} \quad a > 0;$$

and

$$\text{XIII.} \ldots \operatorname{am}_n(-a) = \operatorname{am}_n(-1) = \angle_n(-1) = (2n+1)\pi,$$

if a be still a positive scalar.

* Compare the recent Note, respecting the *notations* employed.

236. From the foregoing definition of amplitude, and from the formerly established connexion of *multiplication of versors* with *composition of rotations* (207), it is obvious that (within the given *plane*, and with abstraction made of *tensors*) *multiplication and division of quaternions* answer respectively to (algebraical) *addition and subtraction of amplitudes*: so that, if the symbol am . q be interpreted in the *general* (or indefinite) sense of the equation 235, VI., we may write:

$$\text{I.} \ldots \text{am}\,(q'.\,q) = \text{am}\, q' + \text{am}\, q\,; \qquad \text{II.} \ldots \text{am}\,(q' : q) = \text{am}\, q' - \text{am}\, q\,;$$

implying hereby that, in each formula, *one* of the values of the first member is *among* the values of the second member; but not here specifying *which* value. With the same generality of signification, it follows evidently that, for a *product* of *any number* of (complanar) quaternions, and for a *whole power* of any one quaternion, we have the analogous formulæ:

$$\text{III.} \ldots \text{am}\, \Pi q = \Sigma\, \text{am}\, q\,; \qquad \text{IV.} \ldots \text{am}\,.\, q^p = p\,.\,\text{am}\, q\,;$$

where the exponent p may be any positive or negative integer, or zero.

(1.) It was proved, in 191, II., that for *any two* quaternions, the formula $\text{U}q'q = \text{U}q'.\text{U}q$ holds good; a result which, by the associative principle of multiplication (223), is easily extended to *any number* of quaternion factors (complanar or diplanar), with an analogous result for tensors: so that we may write, generally,

$$\text{V.} \ldots \text{U}\Pi q = \Pi \text{U} q\,; \qquad \text{VI.} \ldots \text{T}\Pi q = \Pi \text{T} q.$$

(2.) Confining ourselves to the first of these two equations, and combining it with III., and with 235, VII., we arrive at the important formula:

$$\text{VII.} \ldots \Pi\, \text{cis am}\, q \;(= \Pi \text{U} q = \text{U}\Pi q = \text{cis am}\, \Pi q) = \text{cis}\, \Sigma\, \text{am}\, q\,;$$

whence in particular (comp. IV.),

$$\text{VIII.} \ldots (\text{cis am}\, q)^p = \text{cis}\,(p\,.\,\text{am}\, q),$$

at least if the exponent p be still any whole number.

(3.) In these last formulæ, the amplitudes am . q, am . q', &c., may represent *any angular quantities*, z, z', &c.; we may therefore write them thus,

$$\text{IX.} \ldots \Pi\, \text{cis}\, z = \text{cis}\, \Sigma z\,; \qquad \text{X.} \ldots (\text{cis}\, z)^p = \text{cis}\, pz\,;$$

including thus, under *abridged forms*, some known and useful theorems, respecting *cosines and sines* of *sums and multiples* of arcs.

(4.) For example, if the number of factors of the form cis z be *two*, we have thus,

$$\text{IX}'\ldots \operatorname{cis} z' . \operatorname{cis} z = \operatorname{cis}(z'+z); \qquad \text{X}'\ldots (\operatorname{cis} z)^2 = \operatorname{cis} 2z;$$

whence

$$\cos(z'+z) = \text{S}(\operatorname{cis} z' . \operatorname{cis} z) = \cos z' \cos z - \sin z' \sin z;$$

$$\sin(z'+z) = i^{-1}\text{V}(\operatorname{cis} z' . \operatorname{cis} z) = \cos z' \sin z + \sin z' \cos z;$$

$$\cos 2z = (\cos z)^2 - (\sin z)^2; \quad \sin 2z = 2 \cos z \sin z;$$

with similar results for more factors than two.

(5.) Without expressly introducing the conception, or at least the notation of *amplitude*, we may derive the recent formula IX. and X., from the consideration of the *power* i^t (234), as follows. That *power of i*, with a *scalar exponent*, t, has been interpreted in 234, (5.), as a symbol satisfying an equation which may be written thus:

$$\text{XI.}\ldots i^t = \operatorname{cis} z, \quad \text{if} \quad z = \tfrac{1}{2}t\pi;$$

or geometrically as a *versor*, which *turns a line through t right angles*, where t may be *any scalar*. We see then at once, *from this interpretation*, that if t' be either the same or any *other* scalar, the formula,

$$\text{XII.}\ldots i^t . i^{t'} = i^{t+t'}, \quad \text{or} \quad \text{XIII.}\ldots \Pi . i^t = i^{\Sigma t},$$

must hold good, as in algebra. And because the number of the factors i^t is easily seen to be arbitrary in this last formula, we may write also,

$$\text{XIV.}\ldots (i^t)^p = i^{pt},$$

if p be any whole* number. But the two last formulæ may be changed by XI., to the equations IX. and X., which are therefore thus *again* obtained; although the later *forms*, namely XIII. and XIV., are perhaps somewhat *simpler*: having indeed the appearance of being mere *algebraical identities*, although we see that their *geometrical interpretations*, as given above, are important.

(6.) In connexion with the same interpretation XI. of the same useful symbol i^t, it may be noticed here that

$$\text{XV.}\ldots \text{K} . i^t = i^{-t};$$

* It will soon be seen that there is a sense, although one not quite so *definite*, in which this formula holds good, even when the exponent p is fractional, or surd; namely, that the second member is then *one of the values* of the first.

and that therefore,

$$\text{XVI.} \ldots \cos \frac{t\pi}{2} = \mathrm{S}.i^t = \tfrac{1}{2}(i^t + i^{-t});$$

$$\text{XVII.} \ldots \sin \frac{t\pi}{2} = i^{-1}\,\mathrm{V}.i^t = \tfrac{1}{2}\,i^{-1}(i^t - i^{-t}).$$

(7.) Hence, by raising the double of each member of XVI. to any positive whole power p, halving, and substituting z for $\frac{1}{2}t\pi$, we get the equation,

$$\text{XVIII.} \ldots 2^{p-1}(\cos z)^p = \tfrac{1}{2}(i^t + i^{-t})^p = \tfrac{1}{2}(i^{pt} + i^{-pt}) + \tfrac{1}{2}p\,(i^{(p-2)t} + i^{(2-p)t}) + \&c.$$

$$= \cos pz + p\cos(p-2)z + \frac{p(p-1)}{2}\cos(p-4)z + \&c.,$$

with the usual rule for halving the coefficient of $\cos 0z$, if p be an even integer; and with analogous processes for obtaining the known expansions of $2^{p-1}(\sin z)^p$, for any positive whole value, even or odd, of p; and many other known results of the same kind.

237. If p be still a whole number, we have thus the transformation,

$$\text{I.} \ldots q^p = (r \operatorname{cis} z)^p = r^p \operatorname{cis} pz = (\mathrm{T}q)^p \operatorname{cis}(p.\operatorname{am} q);$$

in which (comp. 190, 161) the two factors, of the tensor and versor kinds, may be thus written:

$$\text{II.} \ldots \mathrm{T}(q)^p = (\mathrm{T}q)^p = \mathrm{T}q^p; \qquad \text{III.} \ldots \mathrm{U}(q^p) = (\mathrm{U}q)^p = \mathrm{U}q^p;$$

and *any value* (235) of the amplitude $\operatorname{am}.q$ may be taken, since *all* will conduct to one *common* value of this *whole power* q^p. And if, for I., we substitute this slightly different formula (comp. 235, (3.)),

$$\text{IV.} \ldots (q^p)_n = \mathrm{T}q^p.\operatorname{cis}(p.\operatorname{am}_n q), \text{ with } p = \frac{m'}{n'},\ n' > 0,$$

m', n', n being whole numbers whereof the first is supposed to be *prime* to the second, so that the *exponent* p is here a *fraction in its least terms*, with a *positive denominator* n', while the factor $\mathrm{T}q^p$ is interpreted as a *positive scalar* (of which the positive or negative logarithm, in any given system, is equal to $p \times$ the logarithm of $\mathrm{T}q$), then the expression in the second member admits of n' *distinct values*, answering to different values of n; which are precisely the n' values (comp. 234) of the *fractional power* q^p, on principles already established: the *principal* value of that power corresponding to the value $n = 0$.

(1.) For *any* value of the integer n, we may say that the symbol $(q^p)_n$, defined by the formula IV., represents the n^{th} *value of the power* q^p; such values, however, *recurring* periodically, when p is, as above, a *fraction*.

(2.) Abridging $(1^p)_n$ to $1^p{}_n$, we have thus, *generally*, by 235, XII.,

$$\text{V.} \ldots \ 1^p{}_n = \operatorname{cis} 2pn\pi, \text{ if } p \text{ be any fraction,}$$

a restriction which however we shall soon remove; and in particular,

$$\text{VI.} \ldots \ \textit{Principal value of } 1^p = 1^p{}_0 = 1.$$

(3.) Thus, making successively $p = \frac{1}{2}$, $p = \frac{1}{3}$, we have

$$\text{VII.} \ldots \ 1^{\frac{1}{2}}{}_n = \operatorname{cis} n\pi, \quad 1^{\frac{1}{2}}{}_0 = +1, \quad 1^{\frac{1}{2}}{}_1 = -1, \quad 1^{\frac{1}{2}}{}_2 = +1, \ \&\text{c.};$$

$$\text{VIII.} \ldots \ 1^{\frac{1}{3}}{}_n = \operatorname{cis}\frac{2n\pi}{3}, \quad 1^{\frac{1}{3}}{}_0 = 1, \quad 1^{\frac{1}{3}}{}_1 = \frac{-1 + i\sqrt{3}}{2}, \quad 1^{\frac{1}{3}}{}_2 = \frac{-1 - i\sqrt{3}}{2}, 1^{\frac{1}{3}}{}_3 = 1, \ \&\text{c.}$$

(4.) Denoting in like manner the n^{th} value of $(-1)^p$ by the abridged symbol $(-1)^p{}_n$, we have, on the same plan (comp. 235, XIII.), for any fractional* value of p,

$$\text{IX.} \ldots \ (-1)^p{}_n = \operatorname{cis} p(2n+1)\pi; \text{ whence (comp. 232),}$$

$$\text{X.} \ldots \ (-1)^{\frac{1}{2}}{}_0 = \operatorname{cis}\frac{\pi}{2} = +i, \quad (-1)^{\frac{1}{2}}{}_1 = \operatorname{cis}\frac{3\pi}{2} = -i, \quad (-1)^{\frac{1}{2}}{}_2 = +i, \ \&\text{c.};$$

and

$$\text{XI.} \ldots \ (-1)^{\frac{1}{3}}{}_0 = \frac{1 + i\sqrt{3}}{2}, \quad (-1)^{\frac{1}{3}}{}_1 = -1, \quad (-1)^{\frac{1}{3}}{}_2 = \frac{1 - i\sqrt{3}}{2}, \ \&\text{c.},$$

these three values of $(-1)^{\frac{1}{3}}$ recurring periodically.

(5.) The formula IV. gives, generally, by V., the transformation,

$$\text{XII.} \ldots \ (q^p)_n = (q^p)_0 \operatorname{cis} 2pn\pi = 1^p{}_n (q^p)_0;$$

so that the n^{th} *value* of q^p is equal to the *principal value* of that power of q, multiplied by the *corresponding value* of the *same power of positive unity*; and it may be remarked, that if the *base a* be any *positive scalar*, the *principal p^{th} power*, $(a^p)_0$, is simply, by our definitions, the *arithmetical value* of a^p.

(6.) The n^{th} value of the p^{th} power of any *negative scalar*, $-a$, is in like manner equal to the *arithmetical p^{th} power* of the positive opposite, $+a$, multiplied by the corresponding value of the same power of *negative unity*; or in symbols,

$$\text{XIII.} \ldots \ (-a)^p{}_n = (-1)^p{}_n (a^p)_0 = (a^p)_0 \operatorname{cis} p(2n+1)\pi.$$

(7.) The formula IV., with its consequences V. VI. IX. XII. XIII., may be *extended* so as to include, *as a limit*, the case when the *exponent p* being still *scalar*, becomes *incommensurable*, or surd; and although the *number of values* of the power q^p becomes thus *unlimited* (comp. 234, (4.)), yet we can still

* As before, this restriction is only a temporary one.

consider *one* of them as the *principal value* of this (now) *surd power*: namely the value,

$$\text{XIV.} \ldots (q^p)_0 = \mathrm{T}q^p \,.\, \mathrm{cis}\,(p\,\mathrm{am}_0\, q),$$

which answers to the *principal amplitude* (235, (3.)) of the proposed quaternion q.

238. We may therefore consider the *symbol*,

$$q^p,$$

in which the *base*, q, is *any quaternion*, while the *exponent*, p, is *any scalar*, as being now fully *interpreted*; but no interpretation has been as yet assigned to this *other* symbol of the same kind,

$$q^{q'},$$

in which *both* the base q, and the exponent q', are supposed to be (generally) *quaternions*, although for the purposes of this Chapter *complanar* (225).* To do this, in a way which shall be completely *consistent* with the foregoing conventions and conclusions, or rather which shall *include* and *reproduce* them, for the *case* where the new *quaternion exponent*, q', *degenerates* (131) *into a scalar*, will be one main object of the following section: which however will also contain a theory of *logarithms of quaternions*, and of the *connexion* of both *logarithms* and *powers* with the properties of a certain function, which we shall call the *ponential* of a quaternion, and to consider which we next proceed.

SECTION 4.

On the Ponential and Logarithm of a Quaternion; and on Powers of Quaternions, with Quaternions for their Exponents.

239. If we consider the polynomial function,

$$\text{I.} \ldots \mathrm{P}\,(q, m) = 1 + q_1 + q_2 + \ldots q_m,$$

in which q is any quaternion, and m is any positive whole number, while it is supposed (for conciseness) that

$$\text{II.} \ldots q_m = \frac{q^m}{1\,.\,2\,.\,3\,.\,.\,m}\left(= \frac{q^m}{\Gamma\,(m+1)}\right),$$

then it is not difficult to prove that *however great*, but *finite* and *given*, the

* [For the general case see 316.]

tensor Tq may be, a *finite number* m can be assigned, for which the inequality

$$\text{III.} \ldots \mathrm{T}\,(\mathrm{P}\,(q, m+n) - \mathrm{P}\,(q, m)) < a, \quad \text{if } a > 0,$$

shall be satisfied, *however large* the (positive whole) number n may be, and *however small* the (positive) scalar a, provided that this last is *given.* In other words, if we write (comp. 228),

$$\text{IV.} \ldots q = x + iy, \qquad \mathrm{P}\,(q, m) = X_m + iY_m,$$

a finite value of the number m can always be assigned, such that the following inequality,

$$\text{V.} \ldots (X_{m+n} - X_m)^2 + (Y_{m+n} - Y_m)^2 < a^2,$$

shall hold good, however large the number n, and however small (but given and > 0) the scalar a may be. It follows evidently that *each of the two scalar series*, or succession of scalar functions,

$$\text{VI.} \ldots X_0 = 1, \qquad X_1 = 1 + x, \qquad X_2 = 1 + x + \frac{x^2 - y^2}{2}, \ldots \qquad X_m, \ldots$$

$$\text{VII.} \ldots Y_0 = 0, \qquad Y_1 = y, \qquad Y_2 = y + xy, \ldots \qquad Y_m, \ldots$$

converges ultimately to a fixed and finite limit, whereof the one may be called X_∞, or simply X, and the latter Y_∞, or Y, and of which each is a certain *function of the two scalars*, x and y. Writing then

$$\text{VIII.} \ldots Q = X_\infty + iY_\infty = X + iY,$$

we must consider this *quaternion* Q (namely the *limit* to which the following *series of quaternions*,

$$\text{IX.} \ldots \mathrm{P}\,(q, 0) = 1,\ \mathrm{P}\,(q, 1) = 1 + q,\ \mathrm{P}\,(q, 2) = 1 + q + \frac{q^2}{2}, \ldots \mathrm{P}\,(q, m), \ldots$$

converges ultimately) as being in like manner a certain *function*, which we shall call the *ponential function*, or simply *the Ponential of* q, in consequence of its possessing certain *exponential properties*; and which may be denoted by any one of the three symbols,

$$\mathrm{P}\,(q, \infty), \quad \text{or} \quad \mathrm{P}\,(q), \quad \text{or simply} \quad \mathrm{P}q.$$

We have therefore the equation,

$$\text{X.} \ldots \textit{Ponential of } q = Q = \mathrm{P}q = 1 + q_1 + q_2 + \ldots + q_\infty,$$

with the signification II. of the term q_m.

(1.) In connexion with the *convergence* of this *ponential series*, or with the inequality III., it may be remarked that if we write (comp. 235) $r = \mathrm{T}q$, and $r_m = \mathrm{T}q_m$, we shall have, by 212, (2.),

$$\text{XI.} \ldots \mathrm{T}(\mathrm{P}(q, m+n) - \mathrm{P}(q, m)) \leqq \mathrm{P}(r, m+n) - \mathrm{P}(r, m);$$

it is sufficient then to prove that this last difference, or the sum of the n positive terms, $r_{m+1}, \ldots r_{m+n}$, can be made $< a$. Now if we take a number $p > 2r - 1$, we shall have $r_{p+1} < \frac{1}{2}r_p$, $r_{p+2} < \frac{1}{2}r_{p+1}$, &c., so that a finite number $m > p > 2r - 1$ can be assigned, such that $r_m < a$; and then,

$$\text{XII.} \ldots \mathrm{P}(r, m+n) - \mathrm{P}(r, m) < a(2^{-1} + 2^{-2} + \ldots + 2^{-n}) < a;$$

the asserted inequality is therefore proved to exist.

(2.) In general, if an ascending series, with positive coefficients, such as

$$\text{XIII.} \ldots \mathrm{A}_0 + \mathrm{A}_1 q + \mathrm{A}_2 q^2 + \text{\&c.}, \quad \text{where} \quad \mathrm{A}_0 > 0,\ \mathrm{A}_1 > 0, \text{\&c.},$$

be *convergent* when q is changed to a *positive scalar*, it will *à fortiori* converge, when q is a *quaternion*.

240. Let q and q' be any two complanar quaternions, and let q'' be their sum, so that

$$\text{I.} \ldots q'' = q' + q, \qquad q'' \parallel\!| \, q' \parallel\!| \, q;$$

then, as in algebra, with the signification 239, II. of q_m, and with corresponding significations of q'_m and q''_m, we have

$$\text{II.} \ldots q''_m = \frac{(q' + q)^m}{1 \,.\, 2 \,.\, 3 \,..\, m} = q'_m q_0 + q'_{m-1} q_1 + q'_{m-2} q_2 + \ldots + q'_0 q_m,$$

where $q_0 = q'_0 = 1$. Hence, writing again $r = \mathrm{T}q$, $r_m = \mathrm{T}q_m$, and in like manner $r' = \mathrm{T}q'$, &c., the two differences,

$$\text{III.} \ldots \mathrm{P}(r', m) \,.\, \mathrm{P}(r, m) - \mathrm{P}(r + r', m),$$

and

$$\text{IV.} \ldots \mathrm{P}(r + r', 2m) - \mathrm{P}(r', m) \,.\, \mathrm{P}(r, m),$$

can be expanded as sums of positive terms of the form $r'_{p'} . r_p$ (one sum containing $\frac{1}{2}m(m + 1)$, and the other containing $m(m + 1)$ such terms)*; but, by 239, III., the *sum* of these two positive differences can be made less than any given small positive scalar a, since

$$\text{V.} \ldots \mathrm{P}(r + r', 2m) - \mathrm{P}(r + r', m) < a, \quad \text{if} \quad a > 0,$$

* [For the total number of terms in $\mathrm{P}(r + r', m)$ is $1 + 2 + 3 + \ldots + (m + 1) = \frac{1}{2}(m + 1)(m + 2)$. On expansion of III. the series is seen to be $\Sigma r'_{p'} . r_p$ where $p + p' > m$, and there are $(m + 1)^2 - \frac{1}{2}(m + 1)(m + 2)$ terms, all of which are positive: similarly for IV. From the expanded form of III. it is seen at once that

$$\mathrm{T}(\mathrm{P}(q', m) \,.\, \mathrm{P}(q, m) - \mathrm{P}(q + q', m)) \leqq \mathrm{P}(r', m) \,.\, \mathrm{P}(r, m) - \mathrm{P}(r + r', m).]$$

provided that the number m is taken large enough; *each* difference, therefore, separately *tends* to 0, as m tends to ∞; a tendency which must exist *à fortiori*, when the *tensors*, r, r', are replaced by the *quaternions*, q, q'. The *function* $\mathrm{P}q$ is therefore subject to the *Exponential Law*,

$$\text{VI.} \ldots \mathrm{P}(q' + q) = \mathrm{P}q' . \mathrm{P}q = \mathrm{P}q . \mathrm{P}q', \quad \text{if} \quad q' \parallel\!| \, q.$$

(1.) If we write (comp. 237, (5.)),

$$\text{VII.} \ldots \mathrm{P}1 = \varepsilon, \quad \text{then} \quad \text{VIII.} \ldots \mathrm{P}x = (\varepsilon^x)_0 = \textit{arithmetical value of } \varepsilon^x;$$

where ε is the known base of the natural system of logarithms, and x is any scalar. We shall henceforth write simply ε^x to denote this *principal* (or arithmetical) value of the x^{th} power of ε, and so shall have the simplified equation,

$$\text{VIII}' \ldots \mathrm{P}x = \varepsilon^x.$$

(2.) Already we have thus a motive for writing, *generally*,

$$\text{IX.} \ldots \mathrm{P}q = \varepsilon^q;$$

but this formula is *here* to be considered merely as a *definition* of the sense in which we *interpret* this *exponential symbol*, ε^q; namely as what we have lately called the *ponential function*, $\mathrm{P}q$, considered as the sum of the infinite but converging *series*, 239, X. It will however be soon seen to be *included* in a *more general definition* (comp. 238) of the symbol $q^{q'}$.

(3.) For any scalar x, we have by VIII. the transformation:

$$\text{X.} \ldots x = \mathrm{lP}x = \textit{natural logarithm of ponential of } x.$$

241. The exponential law (240) gives the following general *decomposition of a ponential into factors*,

$$\text{I.} \ldots \mathrm{P}q = \mathrm{P}(x + iy) = \mathrm{P}x . \mathrm{P}iy;$$

in which we have just seen that the factor $\mathrm{P}x$ is a positive scalar. The other factor, $\mathrm{P}iy$, is easily proved to be a versor, and therefore to be *the versor of* $\mathrm{P}q$, while $\mathrm{P}x$ is the *tensor* of the same ponential; because we have in general,

$$\text{II.} \ldots \mathrm{P}q . \mathrm{P}(-q) = \mathrm{P}0 = 1, \quad \text{and} \quad \text{III.} \ldots \mathrm{PK}q = \mathrm{KP}q,$$

since $\quad \text{IV.} \ldots (\mathrm{K}q)^m = \mathrm{K}(q^m) = \text{(say) } \mathrm{K}q^m$ (comp. 199, IX.);

and therefore, in particular (comp. 150, 158),

$$\text{V.} \ldots 1 : \mathrm{P}iy = \mathrm{P}(-iy) = \mathrm{KP}iy, \quad \text{or} \quad \text{VI.} \ldots \mathrm{NP}iy = 1.$$

We may therefore write (comp. 240, IX., X.),

$$\text{VII.} \ldots \mathrm{TP}q = \mathrm{PS}q = \mathrm{P}x = \varepsilon^x; \quad \text{VIII.} \ldots x = \mathrm{S}q = \mathrm{lTP}q;$$

$$\text{IX.} \ldots \mathrm{UP}q = \mathrm{PV}q = \mathrm{P}iy = \varepsilon^{iy} = \operatorname{cis} y \text{ (comp. 235, IV.)};$$

this last transformation being obtained from the two series,

$$\text{X.} \ldots \mathrm{SP}iy = 1 - \frac{y^2}{2} + \&\text{c.} = \cos y;$$

$$\text{XI.} \ldots i^{-1}\,\mathrm{VP}iy = y - \frac{y^3}{2\,.\,3} + \&\text{c.} = \sin y.$$

Hence the ponential Pq may be thus transformed:

$$\text{XII.} \ldots \mathrm{P}q = \mathrm{P}(x + iy) = \varepsilon^x \operatorname{cis} y.$$

(1.) If we had not chosen to *assume as known* the *series* for *cosine* and *sine*, nor to *select* (at first) any one *unit of angle*, such as that known one on which their validity depends, we might then have proceeded as follows. Writing

$$\text{XIII.} \ldots \mathrm{P}iy = fy + i\phi y, \quad f(-y) = +fy, \quad \phi(-y) = -\phi y,$$

we should have, by the exponential law (240),

$$\text{XIV.} \ldots f(y + y') = \mathrm{S}(\mathrm{P}iy\,.\,\mathrm{P}iy') = fy\,.fy' - \phi y\,.\,\phi y';$$

$$\text{XV.} \ldots f(y - y') = \qquad fy\,.fy' + \phi y\,.\,\phi y';$$

and then the *functional equation*, which results, namely,

$$\text{XVI.} \ldots f(y + y') + f(y - y') = 2fy\,.fy',$$

would show that

$$\text{XVII.} \ldots fy = \cos\left(\frac{y}{c} \times \textit{a right angle}\right),$$

whatever unit of angle may be adopted, provided that we determine the *constant c* by the condition,

$$\text{XVIII.} \ldots c = \textit{least positive root of the equation } fy\,(= \mathrm{SP}iy) = 0;$$

or nearly,

XVIII'. . . $c = 1{\cdot}5708$, as the study of the series* would show.

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* In fact, the value of the constant c may be obtained to *this* degree of accuracy, by simple interpolation between the two approximate values of the function f,

$$f(1{\cdot}5) = +\,0{\cdot}070737, \qquad f(1{\cdot}6) = -\,0{\cdot}029200;$$

and of course there are artifices, not necessary to be mentioned here, by which a far more accurate value can be found

(2.) A *motive* would thus arise for *representing a right angle by this numerical constant*, c; or for *so selecting* the *angular unit*, as to have the equation (π still denoting two right angles),

$$\text{XIX.} \ldots \pi = 2c = \textit{least positive root of the equation } fy = -1;$$

giving nearly,

$$\text{XIX}'. \ldots \pi = 3{\cdot}14159, \text{ as usual};$$

for thus we should reduce XVII. to the simpler form,

$$\text{XX.} \ldots fy = \cos y.$$

(3.) As to the function ϕy, since

$$\text{XXI.} \ldots (fy)^2 + (\phi y)^2 = \text{P}iy \,.\, \text{P}(-iy) = 1,$$

it is evident that $\phi y = \pm \sin y$; and it is easy to prove that the upper sign is to be taken. In fact, it can be shown (without supposing any previous knowledge of cosines or sines) that ϕc is positive, and therefore that

$$\text{XXII.} \ldots \phi c = +1, \quad \text{or} \quad \text{XXIII.} \ldots \text{P}ic = i;$$

whence

$$\text{XXIV.} \ldots \phi y = \text{S} \,.\, i^{-1}\,\text{P}iy = \text{SP}i\,(y - c) = f(y - c),$$

and

$$\text{XXV.} \ldots \text{P}iy = fy + if(y - c).$$

If then we replace c by $\frac{\pi}{2}$, we have

$$\text{XXVI.} \ldots \phi y = \cos\left(y - \frac{\pi}{2}\right) = \sin y; \quad \text{and} \quad \text{XXVII.} \ldots \text{P}iy = \text{cis}\, y, \text{ as in IX.}$$

(4.) The series X. XI. for cosine and sine might thus be *deduced*, instead of being *assumed* as known: and since we have the limiting value,

$$\text{XXVIII.} \ldots \lim_{y=0}.\ y^{-1} \sin y = \lim_{y=0}.\ y^{-1}\, i^{-1}\, \text{VP}iy = 1,$$

it follows that the *unit of angle*, which thus gives $\text{P}iy = \text{cis}\, y$, is (as usual) the angle subtended at the centre by the *arc equal to radius*; or that the number π (or $2c$) is to 1, as the *circumference* is to the *diameter* of a circle.

(5.) If any *other angular unit* had been, for any reason, chosen, then a *right angle* would of course be represented by a *different number*, and not by $1{\cdot}5708$ nearly; but we should *still* have the *transformation*,

$$\text{XXIX.} \ldots \text{P}iy = \text{cis}\left(\frac{y}{c} \times \text{a right angle}\right),$$

though *not* the same *series* as before, for $\cos y$ and $\sin y$.

242. The usual unit being retained, we see, by 241, XII., that

$$\text{I.} \ldots \mathrm{P}\,.\,2in\pi = 1, \quad \text{and} \quad \text{II.} \ldots \mathrm{P}\,(q + 2in\pi) = \mathrm{P}q,$$

if n be any whole number; it follows, then, that the *inverse ponential function*, $\mathrm{P}^{-1}q$, or what we may call the *Imponential*, of a given quaternion q, has *indefinitely many values*, which may all be represented by the formula,

$$\text{III.} \ldots \mathrm{P}_n^{-1}\, q = \mathrm{lT}q + i\,\mathrm{am}_n\, q\,;$$

and of which *each* satisfies the equation,

$$\text{IV.} \ldots \mathrm{PP}_n^{-1}\, q = q\,;$$

while the one which corresponds to $n = 0$, may be called the *Principal Imponential.* It will be found that when the *exponent* p is *any scalar*, the definition already given (237, IV., XII.) for the n^{th} *value* of the p^{th} *power* of q enables us to establish the formula,

$$\text{V.} \ldots (q^p)_n = \mathrm{P}\,(p\mathrm{P}_n^{-1}q)\,;$$

and we now propose to *extend* this last formula, by a *new definition*, to the *more general case* (238), when the *exponent* is a *quaternion* q': thus writing generally, for *any two complanar quaternions*, q and q' the *General Exponential Formula*,

$$\text{VI.} \ldots (q^{q'})_n = \mathrm{P}\,(q'\mathrm{P}_n^{-1}q)\,;$$

the *principal value* of $q^{q'}$ being still conceived to correspond to $n = 0$, or to the *principal amplitude* of q (comp. 235, (3.)).

(1.) For example,

$$\text{VII.} \ldots (\varepsilon^q)_0 = \mathrm{P}(q\mathrm{P}_0^{-1}\varepsilon) = \mathrm{P}q, \quad \text{because} \quad \mathrm{P}_0^{-1}\varepsilon = \mathrm{l}\varepsilon = 1\,;$$

the *ponential* $\mathrm{P}q$, which we agreed, in 240, (2.), to denote simply by ε^q, is therefore now seen to be in fact, by our general definition, the *principal value of that power*, or exponential.

(2.) With the same notations,

$$\text{VIII.} \ldots \varepsilon^{iy} = \mathrm{cis}\, y, \qquad \cos y = \tfrac{1}{2}\,(\varepsilon^{iy} + \varepsilon^{-iy}), \qquad \sin y = \frac{1}{2i}\,(\varepsilon^{iy} - \varepsilon^{-iy})\,;$$

these two last only differing from the usual imaginary expressions for cosine and sine, by the geometrical *reality** of the versor i.

* Compare 232, (2.), and the Notes to pp. 253, 258.

(3.) The *cosine and sine of a quaternion* (in the given plane) may now be *defined* by the equations:

$$\text{IX.} \ldots \cos q = \tfrac{1}{2}(\varepsilon^{iq} + \varepsilon^{-iq})\,; \qquad \text{X.} \ldots \sin q = \frac{1}{2i}(\varepsilon^{iq} - \varepsilon^{-iq})\,;$$

and we may write (comp. 241, IX.),

$$\text{XI.} \ldots \operatorname{cis} q = \varepsilon^{iq} = \mathrm{P}iq.$$

(4.) With this interpretation of cis q, the exponential properties, 236, IX., X., continue to hold good; and we may write,

$$\text{XII.} \ldots (q^{q'})_n = \mathrm{P}\,(q'\mathrm{lT}q) \,.\, \mathrm{P}\,(iq' \operatorname{am}_n q) = (\mathrm{T}q)_0{}^{q'} \operatorname{cis}\,(q' \operatorname{am}_n q)\,;$$

a formula which evidently includes the corresponding one, 237, IV., for the n^{th} value of the p^{th} power of q, when p is scalar.

(5.) The definitions III. and VI., combined with 235, XII., give generally,

$$\text{XIII.} \ldots 1_n{}^{q'} = (1^{q'})_n = \mathrm{P}\,.\,2in\pi q'\,; \qquad \text{XIV.} \ldots (q^{q'})_n = 1_n{}^{q'} . (q^{q'})_0\,;$$

this last equation including the formula 237, XII.

(6.) The same definitions give,

$$\text{XV.} \ldots \mathrm{P}_0{}^{-1}i = \frac{i\pi}{2}\,; \qquad \text{XVI.} \ldots (i^i)_0 = \varepsilon^{-\frac{\pi}{2}}\,;$$

which last equation agrees with a known interpretation of the symbol,

$$\sqrt{-1}^{\sqrt{-1}},$$

considered as denoting in algebra a real quantity.

(7.) The formula VI. may even be extended to the case where the *exponent* q' is a *quaternion*, which is *not in the given plane of* i, and therefore *not complanar with the base* q; thus we may write,

$$\text{XVII.} \ldots (i^j)_0 = \mathrm{P}\,(j\mathrm{P}_0{}^{-1}i) = \mathrm{P}\left(-\frac{k\pi}{2}\right) = -k\,;$$

but it would be foreign (225) to the plan of this Chapter to enter into any further details, on the subject of the interpretation of the exponential symbol $q^{q'}$, for this case of *diplanar quaternions*, though we see that there would be no difficulty in treating it, after what has been shown respecting *complanars*.

243. As regards the *general logarithm* q' *of a quaternion* q (in the given plane), we may regard it as any quaternion which satisfies the equation,

$$\text{I.} \ldots \varepsilon^{q'} = \mathrm{P}q' = q\,;$$

and in this view it is simply the *Imponential* $\mathrm{P}^{-1}q$, of which the n^{th} value is expressed by the formula 242, III. But the *principal imponential*, which answers (as above) to $n = 0$, may be said to be the *principal logarithm*, or simply *the Logarithm*, of the quaternion q, and may be denoted by the symbol,

$$lq;$$

so that we may write,

$$\text{I.} \ldots lq = P_0^{-1}q = lTq + i\,\text{am}_0\,q;$$

or still more simply,

$$\text{II.} \ldots lq = l\,(Tq \,.\, Uq) = lTq + lUq,$$

because $lTUq = l1 = 0$, and therefore,

$$\text{III.} \ldots lUq = i\,\text{am}_0\,q.$$

We have thus the two general equations,

$$\text{IV.} \ldots Slq = lTq; \qquad \text{V.} \ldots Vlq = lUq;$$

in which lTq is still the scalar and natural logarithm of the positive scalar Tq.

(1.) As examples (comp. 235, (2.), and (4.)),

$$\text{VI.} \ldots li = \tfrac{1}{2}i\pi; \qquad \text{VII.} \ldots l\,(-1) = i\pi.$$

(2.) The *general logarithm* of q may be denoted by any one of the symbols,

$$\log . q, \quad \text{or} \quad \log q, \quad \text{or} \quad (\log q)_n,$$

this last denoting the n^{th} *value*; and then we shall have,

$$\text{VIII.} \ldots (\log q)_n = lq + 2in\pi.$$

(3.) The formula,

$$\text{IX.} \ldots \log . q'q = \log q' + \log q, \quad \text{if} \quad q' \parallel\!| \; q,$$

holds good, in the sense that *every* value of the first member is *one* of the values of the second (comp. 236).

(4.) *Principal value of* $q^{q'} = \varepsilon^{q'lq}$; and *one* value of $\log . q^{q'} = q'lq$.

(5.) The *quotient* of *two* general logarithms,

$$\text{X.} \ldots (\log q')_{n'} : (\log q)_n = \frac{lq' + 2in'\pi}{lq + 2in\pi},$$

may be said to be the *general logarithm of the quaternion*, q', *to the complanar quaternion base*, q; and we see that its expression involves* *two arbitrary and independent integers*, while its *principal value* may be defined to be $lq' : lq$.

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* As the corresponding expression in algebra, according to Graves and Ohm.

SECTION 5.

On Finite* (or Polynomial) Equations of Algebraic Form, involving Complanar Quaternions; and on the Existence of n Real Quaternion Roots, of any such Equation of the n^{th} Degree.

244. We have seen (233) that an equation of the form,

$$\text{I.} \ldots q^n - Q = 0,$$

where n is any given positive integer, and Q is any† given, real, and actual quaternion (144), has always *n real, actual,* and *unequal quaternion roots, q,* complanar with Q; namely, the n distinct and real values of the symbol $Q^{\frac{1}{n}}$ (223, VIII.), determined on a plan lately laid down. This result is, however, included in a much more general *Theorem,* respecting *Quaternion Equations of Algebraic Form*; namely, that *if $q_1, q_2, \ldots q_n$ be any n given, real, and complanar quaternions,* then *the equation,*

$$\text{II.} \ldots q^n + q_1 q^{n-1} + q_2 q^{n-2} + \ldots + q_n = 0,$$

has always n real quaternion roots, $q', q'', \ldots q^{(n)}$, and no more in the given plane; of which roots it is possible however that some, or all may become *equal,* in consequence of certain *relations* existing between the n given *coefficients.*

245. As another statement of the same Theorem, if we write,

$$\text{I.} \ldots \mathrm{F}_n q \equiv q^n + q_1 q^{n-1} + \ldots + q_n,$$

the coefficients $q_1 \ldots q_n$ being as before, we may say that *every such polynomial function, $\mathrm{F}_n q$, is equal to a product of n real, complanar, and linear* (or *binomial*) *factors,* of the form $q - q'$; or that an equation of the form,

$$\text{II.} \ldots \mathrm{F}_n q \equiv (q - q')(q - q'') \ldots (q - q^{(n)}),$$

can be *proved* in all cases to *exist*: although we may not be able, with our present methods, to *assign expressions for the roots, $q', \ldots q^{(n)}$, in terms of the coefficients $q_1, \ldots q_n$.*

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* By saying *finite equations,* we merely intend to exclude here equations with *infinitely many terms,* such as $\mathrm{P}q = 1$, which has been seen (242) to have *infinitely many roots,* represented by the expression $q = 2in\pi$, where n may be any whole number.

† It is true that we have supposed $Q \parallel\!\!| \, i$ (225); but nothing hinders us, in any other case, from substituting for i the versor $\mathrm{UV}Q$, and then proceeding as before.

246. Or we may say that there is always a certain *system of n real quaternions* q', &c., $|||\, i$, which satisfies the *system of equations*, of known algebraic form,

$$\text{III.} \ldots \begin{cases} q' + q'' + \ldots + q^{(n)} = -q_1; \\ q'q'' + q'q''' + q''q''' + \ldots = +q_2; \\ q'q''q''' + \ldots = -q_3; \ \&\text{c.} \end{cases}$$

247. Or because the difference $F_n q - F_n q'$ is *divisible* by $q - q'$, as in algebra, under the supposed conditions of *complanarity* (224), it is sufficient to say that *at least one real quaternion* q' *always exists* (whether we can *assign* it or not), *which satisfies the equation,*

$$\text{IV.} \ldots F_n q' = 0,$$

with the foregoing form (245, I.) of the polynomial function F.*

248. Or finally, because the theorem is evidently true for the case $n = 1$, while the case 244, I., has been considered, and the case $q_n = 0$ is satisfied by the supposition $q = 0$, we may, without essential loss of generality, reduce the enunciation to the following:

Every equation of the form,†

$$\text{I.} \ldots q(q - q')\,(q - q'') \ldots (q - q^{(n-1)}) = Q,$$

in which q', q'', .. and Q are any n real and given quaternions in the given plane, whereof at least Q and q' may be supposed *actual* (144), *is satisfied by at least one real, actual, and complanar quaternion*, q [see 253 (1.)].

249. Supposing that the $m - 1$ last of the $n - 1$ given quaternions $q' \ldots q^{(n-1)}$ vanish, but that the $n - m$ first of them are actual, where m may be any whole number, from 1 to $n - 1$, and introducing a new real, known, complanar, and actual quaternion q_0, which satisfies the condition,

$$\text{II.} \ldots q_0^{\,m} = \frac{Q}{q'q'' \ldots q^{(n-m)}},$$

* [Thus $\frac{F_n q}{q - q'} \equiv F_{n-1} q = q^{n-1} + q'_1 q^{n-2} + q'_2 q^{n-3} + \ldots + q'_{n-1}$, which is of the form 245, I. If then every equation of this form has a root, $F_{n-1} q'' = 0$, and q'' is a second root of $F_n q$.]

† The corresponding *form*, of the *algebraical equation of the* n^{th} *degree*, was proposed by Mourey, in his very ingenious and original little work, entitled *La vraie théorie des Quantités Négatives, et des Quantités prétendues Imaginaires* (Paris, 1828). *Suggestions* also, towards the *geometrical proof* of the theorem in the text have been taken from the same work; in which, however, the *curve* here called (in 251) an *oval* is not perhaps defined with sufficient precision: the *inequality*, here numbered as 251, XII., being not employed. It is to be observed that Mourey's book contains *no hint* of the *present calculus*, being *confined*, like the *Double Algebra* of Prof. De Morgan (London, 1849), and like the earlier work of Mr. Warren (Cambridge, 1828), to questions *within the plane*: whereas the very *conception* of the *Quaternion* involves, as we have seen, a reference to *Tridimensional* SPACE.

we may write thus the recent equation I.,

$$\text{III.} \ldots fq = \left(\frac{q}{q_0}\right)^m \left(\frac{q}{q'} - 1\right)\left(\frac{q}{q''} - 1\right) \ldots \left(\frac{q}{q^{(n-m)}} - 1\right) = 1;$$

and may (by 187, 159, 235) decompose it into the two following:

$$\text{IV.} \ldots \mathrm{T}fq = 1; \quad \text{and} \quad \text{V.} \ldots \mathrm{U}fq = 1, \quad \text{or} \quad \text{VI.} \ldots \mathrm{am}fq = 2p\pi;$$

in which p is some whole number (negatives and zero included).

250. To give a more *geometrical form* to the equation, let λ be any given or assumed line $|||\, i$, and let it be supposed that $\alpha, \beta, \ldots$ and ρ, σ, or OA, OB, ... and OP, OS, are $n - m + 2$ other lines in the same planes, and that $\phi\rho$ is a known function of ρ, such that

$$\text{VII.} \ldots \alpha = q'\lambda, \qquad \beta = q''\lambda, \ldots \qquad \rho = q\lambda, \qquad \sigma = q_0\lambda,$$

and

$$\text{VIII.} \ldots \phi\rho = fq = \left(\frac{\rho}{\sigma}\right)^m \cdot \frac{\rho - \alpha}{\alpha} \cdot \frac{\rho - \beta}{\beta} \ldots = \left(\frac{\text{OP}}{\text{OS}}\right)^m \cdot \frac{\text{AP}}{\text{OA}} \cdot \frac{\text{BP}}{\text{OB}} \ldots;$$

the theorem to be proved may then be said to be, that *whatever system of real points*, O, A, B, .. and S, in a given plane, and *whatever positive whole number* m, *may be assumed*, or given, *there is always at least one real point* P, in the same plane, *which satisfies the two conditions*:

$$\text{IX.} \ldots \mathrm{T}\phi\rho = 1; \qquad \text{X.} \ldots \mathrm{am}\,\phi\rho = 2p\pi.$$

251. Whatever value $\iota \,|||\, i$ we may assume for the *versor* (or unit-vector) $\mathrm{U}\rho$, there always exists *at least one* value of the *tensor* $\mathrm{T}\rho$, which satisfies the condition IX.; because the function $\mathrm{T}\phi\rho$ vanishes with $\mathrm{T}\rho$, and becomes infinite when $\mathrm{T}\rho = \infty$, having varied continuously (although perhaps with fluctuations) in the interval. Attending then only to the *least value* (if there be more than one) of $\mathrm{T}\rho$, which thus renders $\mathrm{T}\phi\rho$ equal to unity, we can conceive a real, unambiguous, and scalar function $\psi\iota$, which shall have the two following properties:

$$\text{XI.} \ldots \mathrm{T}\phi\,(\iota\psi\iota) = 1; \qquad \text{XII.} \ldots \mathrm{T}\phi\,(x\iota\psi\iota) < 1, \text{ if } x > 0, < 1.$$

And in this way the equation, or system of equations,

$$\text{XIII.} \ldots \rho = \iota\psi\iota, \quad \text{or} \quad \text{XIV.} \ldots \mathrm{U}\rho = \iota, \quad \mathrm{T}\rho = \psi\iota,$$

may be conceived to *determine a real, finite, and plane closed curve*, which we shall call generally an *Oval*, and which shall have the two following properties: 1st, *every right line*, or *ray*, drawn *from the origin* O, in any arbitrary direction within the plane, *meets the curve once*, but *once only*; and IInd, no *one* of the $n-m$ *other given points* A, B, . . *is on the oval*, because $\phi\alpha = \phi\beta = \ldots = 0$.*

252. This being laid down, let us conceive a point P to perform *one circuit* of the oval, moving in the *positive* direction relatively to the given interior point O; so that, whatever the given direction of the line OS may be, the *amplitude* am $(\rho : \sigma)$, if supposed to vary *continuously*,† will have increased by *four right angles*, or by 2π, in the course of this *one positive circuit*; and consequently, the amplitude of the left-hand factor $(\rho : \sigma)^m$, of $\phi\rho$, will have increased, at the same time, by $2m\pi$. Then, if the point A be also *interior* to the oval, so that the line OA must be prolonged to meet that curve, the ray AP will have likewise made *one* positive revolution, and the amplitude of the factor $(\rho - \alpha) : \alpha$ will have increased by 2π. But if A be an *exterior* point, so that the *finite line* OA *intersects* the curve in a point M, and therefore *never meets it again* if prolonged, although the prolongation of the *opposite line* AO must meet it *once* in some point N, then while the *point* P performs first what we may call the positive *half-circuit* from M to N, and afterwards the *other* positive half-circuit from N to M *again*, the *ray* AP has only *oscillated* about its initial and final direction, namely that of the line AO, without ever attaining the *opposite direction*; in *this* case, therefore, the amplitude am (AP : OA), if still supposed to vary *continuously*, has only *fluctuated* in its value, and has (upon the whole) undergone *no change* at all. And since precisely similar remarks apply to the other given points, B, &c., it follows that the amplitude, am $\phi\rho$, *of the product* (VIII.) of all these factors, has (by 236) received a *total increment* $= 2(m + t)\pi$, if t be the *number* (perhaps zero) *of given internal points*, A, B, . .; while the number m is (by 249) *at least* $= 1$. Thus, while P

* [A curve traced out by a point moving so that the product of powers of its distances from fixed points is equal to a constant parameter, consists of closed curves or ovals surrounding the fixed points and enclosing all ovals corresponding to smaller parameters. If the parameter is small, each oval encloses but one fixed point, but as it increases, two ovals will combine into a curve with a "certain undulation" (254 (4.)). It is not generally true that a ray OP from one of the fixed points meets an undulatory oval only once. In this case OP will oscillate in its motion as P traces out the oval. But am . $\phi\rho = m \angle \mathrm{POS} + \Sigma(\pi - \angle \mathrm{PAO}) =$ const., defines a set of curves diverging like half-lines or rays from the fixed points, and approximating to straight lines at great distances from them. By the properties of Conjugate Functions each of these curves which originates from O cuts at right angles each oval round O and does not meet it again. Near O, am . $\phi\rho$ is nearly equal to $m \angle \mathrm{POS}$ *plus* a constant. From this it appears that IX. and X. can always be satisfied, and that as P traces out an oval round O without oscillation, am . $\phi\rho$ continually increases or diminishes without oscillation. The ovals are lines of magnetic force, and the orthogonal curves are traces of equipotential surfaces for a system of electric currents normal to the plane.]

† That is, so as not to receive any *sudden* increment, or decrement, of one or more whole circumferences (comp. 235, (1.)).

performs (as above) *one positive circuit, the amplitude* am $\phi\rho$ *has passed at least m times,* and therefore *at least once, through a value of the form* $2p\pi$; and consequently the *condition* X. has been *at least once satisfied.* But the *other* condition, IX., is satisfied *throughout,* by the supposed *construction* of the oval: there is therefore *at least one real position* P, upon that curve, for which $\phi\rho$ or $fq = 1$; so that, *for this position* of that point, the equation 249, III., and therefore also the equation 248, I., is satisfied. The *theorem* of Art. 248, and consequently also, by 247, the theorem of 244, with its transformations 245 and 246, is therefore in this manner *proved.*

253. This conclusion is so important, that it may be useful to illustrate the general reasoning, by applying it to the case of a *quadratic equation,* of the form,

$$\text{I.} \ldots fq = \frac{q}{q_0}\left(\frac{q}{q'} - 1\right) = 1;$$

or

$$\text{II.} \ldots \phi\rho = \frac{\rho}{\sigma}\left(\frac{\rho}{\epsilon} - 1\right) = \frac{\text{OP}}{\text{OS}} \cdot \frac{\text{AP}}{\text{OA}} = 1.$$

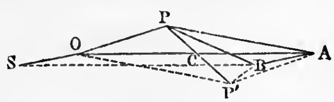

Fig. 55.

We have now to prove (comp. 250, VIII.) that a (real) point P exists, which renders the fourth proportional (226) to the three lines OA, OP, AP equal to a given line OS, or AB, if this latter be drawn = OS; or which satisfies the following condition of similarity of triangles (118),

$$\text{III.} \ldots \Delta\,\text{AOP} \propto \text{PAB};$$

which *includes* the equation of rectangles,

$$\text{IV.} \ldots \overline{\text{OP}} \cdot \overline{\text{AP}} = \overline{\text{OA}} \cdot \overline{\text{AB}}.$$

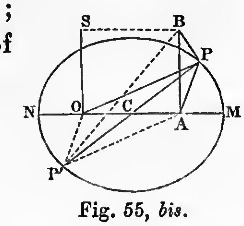

Fig. 55, *bis.*

(Compare the annexed figures, 55, and 55, *bis.*) Conceive, then, that a continuous curve* is described as a *locus* (or as *part* of the locus) of P, by means of this equality IV., with the additional condition when necessary, that O shall be *within* it; in such a manner that when (as in fig. 56) a right line from O meets the general or total *locus* in *several* points, M, M′, N′, we *reject* all but the

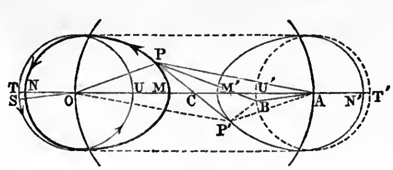

Fig. 56.

* This curve of the fourth degree is the well-known *Cassinian*; but when it breaks up, as in fig. 56, into *two separate ovals,* we *here retain,* as *the oval of the proof, only* the one round O, *rejecting* for the present that round A.

point M which is *nearest* to O, as not belonging (comp. 251, XII.) to the *oval here considered.* Then while P moves upon *that* oval, in the positive direction relatively to O, from M to N, and from N to M again, so that the ray OP performs one positive revolution, and the amplitude of the factor OP : OS increases continuously by 2π, the ray AP performs in like manner one positive revolution, or (on the whole) does not revolve at all, and the amplitude of the factor AP : OA increases by 2π or by 0, according as the point A is *interior* or *exterior* to the oval. In the one case, therefore, the amplitude am $\phi\rho$ of the *product* increases by 4π (as in fig. 55, *bis*); and in the other case, it increases by 2π (as in fig. 56); so that in *each* case, it passes *at least once* through a value of the *form* $2p\pi$, whatever its initial value may have been. Hence, *for at least one real position*, P, upon the oval, we have

$$\text{V.} \ldots \operatorname{am} \phi\rho = 2p\pi, \quad \text{and therefore} \quad \text{VI.} \ldots \mathrm{U}\phi\rho = 1;$$

but

$$\text{VII.} \ldots \mathrm{T}\phi\rho = 1,$$

throughout, by the *construction*, or by the equation of the *locus* IV.; the *geometrical condition* $\phi\rho = 1$ (II.) is therefore satisfied by *at least one real vector* ρ; and consequently the *quadratic equation* $fq = 1$ (I.) is satisfied by *at least one real quaternion root*, $q = \rho : \lambda$ (250, VII.). But the recent form I. has the same generality as the earlier form,

$$\text{VIII.} \ldots \mathrm{F}_2 q = q^2 + q_1 q + q_2 = 0 \text{ (comp. 245)},$$

where q_1 and q_2 are any two given, real, actual, and complanar quaternions; thus there is always a real quaternion q' in the given plane, which satisfies the equation,

$$\text{VIII}'. \ldots \mathrm{F}_2 q' = q'^2 + q_1 q' + q_2 = 0 \text{ (comp. 247)};$$

subtracting, therefore, and dividing by $q - q'$, as in algebra (comp. 224), we obtain the following *depressed* or *linear* equation q,

$$\text{IX.} \ldots q + q' + q_1 = 0, \quad \text{or} \quad \text{IX}'. \ldots q = q'' = -q' - q_1 \text{ (comp. 246)}.$$

The quadratic VIII. has therefore a *second real quaternion root, q'' related* in this manner to the *first*; and because the *quadratic function* $\mathrm{F}_2 q$ (comp. again 245) is thus decomposable into *two linear factors*, or can be put under the form,

$$\text{X.} \ldots \mathrm{F}_2 q = (q - q')(q - q''),$$

it *cannot vanish* for *any third real quaternion*, q; so that (comp. 244) the quadratic equation has *no more than two* such *real roots*.

(1.) The *cubic equation* may therefore be put under the *form* (comp. 248),

$$\mathrm{X}'\ldots \mathrm{F}_3 q = q^3 + q_1 q^2 + q_2 q + q_3 = q(q - q')(q - q'') + q_3 = 0;$$

it has therefore *one real root*, say $q^\backprime$, by the *general proof* (252), which has been above illustrated by the case of the *quadratic* equation; subtracting therefore (comp. 247) the equation $\mathrm{F}_3 q^\backprime = 0$, and dividing by $q - q^\backprime$, we can *depress* the cubic to a quadratic, which will have *two* new real roots, $q^{\backprime\backprime}$ and $q^{\backprime\backprime\backprime}$; and thus the *cubic function* may be put under the form,

$$\mathrm{XI}.\ldots \mathrm{F}_3 q = (q - q^\backprime)(q - q^{\backprime\backprime})(q - q^{\backprime\backprime\backprime}),$$

which cannot vanish for any *fourth* real value of q; the cubic equation X. has therefore *no more than three real quaternion roots* (comp. 244): and similarly for equations of *higher degrees.*

(2.) The existence of *two* real roots q of the *quadratic* I., or of *two real vectors*, ρ and ρ', which satisfy the equation II., might have been *geometrically anticipated*, from the recently proved increase $= 4\pi$ of amplitude $\phi\rho$, in the course of one circuit, for the case of fig. 55, *bis*, in consequence of which there must be *two real positions*, P and P′, *on the one oval* of *that* figure, of which *each* satisfies the condition of similarity III.; and for the case of fig. 56, from the consideration that the *second* (or *lighter*) *oval*, which in this case exists, although not employed above, is related to A exactly as the *first* (or *dark*) oval of the figure is related to O; so that, to the real position P on the first, there must correspond *another* real position P′, upon the second.

(3.) As regards the *law* of this *correspondence*, if the equation II. be put under the form,

$$\mathrm{XII}.\ldots \left(\frac{\rho}{a}\right)^2 - \left(\frac{\rho}{a}\right)^1 - \frac{\sigma}{a} = 0,$$

and if we now write

$$\mathrm{XIII}.\ldots \rho = qa, \quad \text{we may write} \quad \mathrm{XIV}.\ldots q_1 = -1, \quad q_2 = -\sigma : a,$$

for comparison with the form VIII.; and then the recent relation IX′. (or 246) between the two *roots* will take the form of the following relation between *vectors*,

$$\mathrm{XV}.\ldots \rho + \rho' = a; \quad \text{or} \quad \mathrm{XV}'.\ldots \mathrm{OP}' = \rho' = a - \rho = \mathrm{PA};$$

so that the point P′ completes (as in the cited figures) the parallelogram OPAP′, and the line PP′ is bisected by the middle point C of OA. Accordingly, with this position of P′, we have (comp. III.) the similarity, and (comp. II. and 226) the equation,

$$\mathrm{XVI}.\ldots \Delta\,\mathrm{AOP}' \propto \mathrm{P'AB}; \quad \mathrm{XVII}.\ldots \phi\rho' = \phi(a - \rho) = \phi\rho = 1.$$

(4.) The *other* relation between the two roots of the quadratic VIII., namely (comp. 246),

$$\text{XVIII.} \ldots q'q'' = q_2, \quad \text{gives} \quad \text{XIX.} \ldots \frac{\rho}{\alpha}\rho' = -\sigma;$$

and accordingly, the line σ, or OS, is a fourth proportional to the three lines OA, OP, and AP, or α, ρ, and $-\rho'$.

(5.) The *actual solution*, by *calculation*, of the *quadratic equation* VIII. in *complanar quaternions*, is performed *exactly as in algebra*; the *formula* being,

$$\text{XX.} \ldots q = -\tfrac{1}{2}q_1 \pm \sqrt{(\tfrac{1}{4}q_1^2 - q_2)},$$

in which, however, the *square root* is to be interpreted as a *real quaternion*, on principles already laid down.

(6.) *Cubic* and *biquadratic* equations, with quaternion coefficients of the kind considered in 244, are in like manner *resolved* by the known *formulæ* of algebra; but we have now (as has been proved) *three real* (quaternion) *roots* for the former, and *four* such real roots for the latter.

254. The following is another mode of presenting the geometrical reasonings of the foregoing Article, without expressly introducing the notation or conception of *amplitude*. The equation $\phi\rho = 1$ of 253 being written as follows,

$$\text{I.} \ldots \sigma = \chi\rho = \frac{\rho}{\alpha}(\rho - \alpha), \quad \text{or} \quad \text{II.} \ldots \mathrm{T}\sigma = \mathrm{T}\chi\rho, \quad \text{and} \quad \text{III.} \ldots \mathrm{U}\sigma = \mathrm{U}\chi\rho,$$

we may thus regard the *vector* σ as a *known function of the vector* ρ, or the *point* S as a *function of the point* P; in the sense that, while O and A are *fixed*, P and S *vary together*: although it may (and does) happen, that S may *return* to a former position without P having similarly returned. Now the essential property of the *oval* (253) may be said to be this: that it is *the locus of the points* P *nearest to* O, *for which the tensor* $\mathrm{T}\chi\rho$ *has a given value*, say b; namely the *given value of* $\mathrm{T}\sigma$, or of $\overline{\text{OS}}$, when the *point* S, like O and A, is *given*. If then we conceive the point P to *move*, as before, *along the oval*, and the point S *also* to *move*, according to the *law* expressed by the recent formula I., this *latter* point must move (by II.) *on the circumference of a given circle* (comp. again fig. 56), with the given origin O for *centre*; and the *theorem* is, that *in so moving*, S will *pass*, at least *once*, through *every position on that circle*, while P performs *one circuit* of the oval. And *this* may be proved by observing that (by III.) the *angular motion of the radius* OS is equal to the *sum of the angular motions* of the *two rays*, OP and AP; but this latter *sum* amounts to *eight right angles* for the case of fig. 55, *bis*, and to *four right angles* for the case of fig. 56;

the radius OS, and the point S, must therefore have *revolved twice* in the first case, and *once* in the second case, which proves the theorem in question.

(1.) In the first of these two cases, namely when A is an interior point, *each* of the three angular velocities is *positive* throughout, and the *mean angular velocity of the radius* OS is *double* of that of *each* of the *two rays* OP, AP. But in the second case, when A is exterior, the *mean* angular velocity of the ray AP is *zero*; and we might for a moment doubt, whether the *sometimes negative* velocity of *that* ray might not, for parts of the circuit, *exceed* the *always positive* velocity of the ray OP, and so cause the radius OS to move *backwards*, for a while. This cannot be, however; for if we conceive P to describe, like P′, a circuit of the *other* (or lighter) *oval*, in fig. 56, the point S (if still dependent on it by the law I.) would *again* traverse the whole of the same circumference as before; if then it could ever *fluctuate* in its motion, it would pass *more than twice* through some given series of real positions on that circle, during the successive description of the *two ovals* by P; and thus, within certain limiting values of the coefficients, the quadratic equation would have *more than two real roots*: a result which has been proved to be impossible.*

(2.) While S thus describes a circle round O, we may conceive the *connected point* B to describe an *equal circle* round A; and in the case at least of fig. 56, it is easy to prove *geometrically*, from the constant equality (253, IV.) of the rectangles $\overline{\mathrm{OP}}\,.\,\overline{\mathrm{AP}}$ and $\overline{\mathrm{OA}}\,.\,\overline{\mathrm{AB}}$, that these *two circles* (with T′U and T′U′ as *diameters*), and the *two ovals* (with MN and M′N′ as *axes*), have *two common tangents*, parallel to the line OA, which connects what we may call the *two given foci* (or *focal points*), O and A: the new or *third circle*, which is described on this *focal interval* OA as diameter, passing *through the four points of contact on the ovals*, as the figure may serve to exhibit.

(3.) To prove the same things *by quaternions*, we shall find it convenient to *change the origin* (18), for the sake of symmetry, to the *central point* C; and thus to denote *now* CP by ρ, and CA by α, writing also $\overline{\mathrm{CA}} = \mathrm{T}\alpha = a$, and representing still the radius of each of the two equal circles by b. We shall then have, as the *joint equation* of the system of the *two ovals*, the following:

$$\mathrm{IV.} \ldots \mathrm{T}(\rho+\alpha)\,.\,\mathrm{T}(\rho-\alpha) = 2ab;$$

or

$$\mathrm{V.} \ldots \mathrm{T}(q^2-1) = 2c, \quad \text{if} \quad q = \frac{\rho}{\alpha} \quad \text{and} \quad c = \frac{b}{a}.$$

But because we have *generally* (by 199, 204, &c.) the transformations,

$$\mathrm{VI.} \ldots \mathrm{S}\,.\,q^2 = 2\mathrm{S}q^2 - \mathrm{T}q^2 = \mathrm{T}q^2 + 2\mathrm{V}q^2 = 2\mathrm{NS}q - \mathrm{N}q = \mathrm{N}q - 2\mathrm{NV}q,$$

* [See the Note to 251, page 280.]

the square of the equation V. may (by 210, (8.)) be written under either of the two following forms:

$$\text{VII.} \ldots (\mathrm{N}q-1)^2 + 4\mathrm{NV}q = 4c^2; \quad \text{VIII.} \ldots (\mathrm{N}q+1)^2 - 4\mathrm{NS}q = 4c^2;$$

whereof the first shows that the *maximum* value of $\mathrm{TV}q$ is c, at least if $2c < 1$, as happens for this case of fig. 56; and that this maximum corresponds to the value $\mathrm{T}q = 1$, or $\mathrm{T}\rho = a$: results which, when interpreted, reproduce those of the preceding sub-article.

(4.) When $2c > 1$, it is permitted to suppose $\mathrm{S}q = 0$, $\mathrm{NV}q = \mathrm{N}q = 2c - 1$; and then we have only *one* continuous oval, as in the case of fig. 55, *bis*; but if $c < 1$, though $> \frac{1}{2}$, there exists a certain *undulation* in the form of the curve (not represented in that figure), $\mathrm{TV}q$ being a *minimum* for $\mathrm{S}q = 0$, or for $\rho \perp \alpha$, but becoming (as before) a *maximum* when $\mathrm{T}q = 1$, and *vanishing* when $\mathrm{S}q^2 = 2c + 1$, namely at the two *summits* M, N, where the oval meets the axis.

(5.) In the intermediate case, when $2c = 1$, the *Cassinian curve* IV. becomes (as is known) a *lemniscata*; of which the *quaternion equation* may, by V., be written (comp. 200, (8.)) under any one of the following forms:

$$\text{IX.} \ldots \mathrm{T}(q^2 - 1) = 1; \quad \text{or} \quad \text{X.} \ldots \mathrm{N}q^2 = 2\mathrm{S}\,.\,q^2; \quad \text{or} \quad \text{XI.} \ldots \mathrm{T}q^2 = 2\mathrm{SU}\,.\,q^2;$$

or finally,

$$\text{XII.} \ldots \mathrm{T}\rho^2 = 2\mathrm{T}\alpha^2 \cos 2\angle\frac{\rho}{\alpha};$$

which last, when written as

$$\text{XII}'. \ldots \overline{\mathrm{CP}}^2 = 2\overline{\mathrm{CA}}^2 \,.\, \cos 2\mathrm{ACP},$$

agrees evidently with known results.

(6.) This corresponds to the case when

$$\text{XIII.} \ldots \sigma = \frac{-\alpha}{4}, \quad \text{and} \quad \text{XIV.} \ldots \rho = \rho' = +\frac{\alpha}{2}, \text{ in 253, XII.},$$

that *quadratic equation* having thus its roots *equal*; and in general, for *all degrees*, cases of *equal roots* answer to some interesting *peculiarities of form* of the *ovals*, on which we cannot here delay.

(7.) It may, however, be remarked, in passing, that if we *remove the restriction* that the vector ρ, or CP, shall be *in a given plane* (225), drawn through the line which connects the *two foci*, O and A, the recent equation V. will then represent the *surface* (or *surfaces*) generated by the *revolution* of the *oval* (or ovals), or lemniscata, about that line OA as an *axis*.

255. If we look back, for a moment, on the formula of *similarity*, 253, III., we shall see that it involves not merely an *equality of rectangles*, 253, IV., but also an *equality of angles*, AOP and PAB; so that the angle OAB represents (in the figures 55) a *given difference of the base angles* AOP, PAO of the triangle OAP: but to *construct* a *triangle*, by means of *such* a given *difference*, combined with a *given base*, and a *given rectangle of sides*, is a known problem of elementary geometry. To solve it briefly, as an exercise, *by quaternions*, let the given base be the line AA′, with O for its middle point, as in the annexed figure 57; let BAA′ represent the given difference of base angles, PAA′ − AA′P; and let $\overline{\text{OA}}\,.\,\overline{\text{AB}}$ be equal to the given rectangle of sides, $\overline{\text{AP}}\,.\,\overline{\text{A'P}}$. We shall then have the similarity and equation,

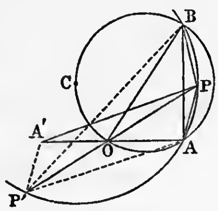

Fig. 57.

$$\text{I.}\ldots \Delta\,\text{OA'P} \propto \text{PAB}; \quad \text{II.}\ldots \frac{\rho+\alpha}{\alpha} = \frac{\beta-\alpha}{\rho-\alpha}$$

whence it follows by the simplest calculations, that

$$\text{III.}\ldots \left(\frac{\rho}{\alpha}\right)^2 = \left(\frac{\rho}{\alpha}+1\right)\left(\frac{\rho}{\alpha}-1\right)+1 = \frac{\beta-\alpha}{\alpha}+1 = \frac{\beta}{\alpha};$$

or that ρ is a mean proportional (227) between α and β. Draw, therefore, a line OP, which shall be in length a *geometric mean* between the two given lines, OA, OB, and shall also *bisect their angle* AOB; its extremity will be the required *vertex*, P, of the sought triangle AA′P: a result of the *quaternion analysis*, which *geometrical synthesis** easily confirms.

(1.) The equation III. is however satisfied also (comp. 227) by the *opposite* vector, OP′ = PO, or $\rho' = -\rho$; and because $\beta = (\rho : \alpha) \cdot \rho$, we have

$$\text{IV.}\ldots \frac{\rho+\beta}{\rho+\alpha} = \frac{\rho}{\alpha} = \frac{\beta}{\rho} = \frac{\rho'}{\alpha'}, \quad \text{or} \quad \text{IV}'\ldots \frac{\text{P'B}}{\text{P'A}} = \frac{\text{OP}}{\text{OA}} = \frac{\text{OB}}{\text{OP}} = \frac{\text{OP}'}{\text{OA}'};$$

so that the *four* following triangles are *similar* (the *two first* of them indeed being *equal*):

$$\text{V.}\ldots \Delta\,\text{A'OP'} \propto \text{AOP} \propto \text{POB} \propto \text{AP'B};$$

as geometry again would confirm.

(2.) The angles AP′B, BPA, are therefore *supplementary*, their sum being equal to the sum of the angles in the triangle OAP; whence it follows that

* In fact, the two triangles I. are similar, as required, because their angles at O and P are equal, and the sides about them are proportional.

the four points A, P, B, P′ *are concircular* :* or in other words, *the quadrilateral* APBP′ is *inscriptible in a circle*, of which (we may add) *the centre* C *is on the circle* OAB (see again fig. 57), because the angle AOB is *double* of the angle AP′B, by what has been already proved.

(3.) Quadratic equations in quaternions may also be employed in the solution of many other geometrical problems; for example, to decompose a given vector into two others, which shall have a given geometrical mean, &c.

SECTION 6.

On the $n^2 - n$ Imaginary (or Symbolical) Roots of a Quaternion Equation of the n^{th} Degree, with coefficients of the kind considered in the foregoing Section.

256. The polynomial function $F_n q$ (245), like the quaternions $q, q_1, .. q_n$ on which it depends, may always be reduced to the form of a couple (228); and thus we may establish the transformation (comp. 239),

$$\text{I.} \ldots F_n q = F_n(x + iy) = X_n + iY_n = G_n(x, y) + iH_n(x, y),$$

X_n and Y_n, or G_n and H_n, being *two* known, real, finite, and scalar *functions* of the *two sought scalars*, x and y; which functions, relatively to *them*, are each of the n^{th} dimension, but which involve also, though only in the *first* dimension, the $2n$ *given* and *real scalars*, $x_1, y_1, \ldots x_n, y_n$. And since the *one quaternion* (or *couple*) *equation*, $F_n q = 0$, is equivalent (by 228, IV.) to the *system* of the *two scalar equations*,

$$\text{II.} \ldots X_n = 0, \quad Y_n = 0, \quad \text{or} \quad \text{III.} \ldots G_n(x, y) = 0, \quad H_n(x, y) = 0,$$

we see (by what has been stated in 244, and proved in 252) that *such a system*, of *two* equations of the n^{th} dimension, *can always be satisfied by n systems* (or *pairs*) *of real scalars*, and by *not more than n*, such as,

$$\text{IV.} \ldots x', y'; \quad x'', y''; \ldots \quad x^{(n)}, y^{(n)};$$

---

* Geometrically, the construction gives at once the similarity,

$$\Delta\, \text{AOP} \propto \text{POB}, \quad \text{whence} \quad \angle\, \text{BPA} = \text{OPA} + \text{PAO} = \text{POA}';$$

and if we complete the parallelogram APA′P′, the new similarity,

$$\Delta\, \text{OA}'\text{P} \propto \text{OP}'\text{B}, \quad \text{gives} \quad \angle\, \text{AP}'\text{B} = \text{OA}'\text{P} + \text{A}'\text{PO} = \text{AOP};$$

thus the opposite angles BPA, AP′B are supplementary, and the quadrilateral APBP′ is inscriptible. It will be shown, in a shortly subsequent section [261, (6.)], that these four points, A, P, B, P′, form a *harmonic group* upon their common circle.

although it may happen that *two* or *more* of these systems shall *coincide* with (or become *equal* to) each other.

(1.) If x and y be treated as *co-ordinates* (comp. 228, (3.)), the two equations II. or III. represent a *system of two curves*, in the given plane; and then the *theorem* is, that these two curves *intersect each other* (*generally**) *in* n *real points*, and in *no more*: although two or more of these n points may happen to *coincide* with each other.

(2.) Let h denote, as a temporary abridgment, the *old* or *ordinary imaginary*, $\sqrt{-1}$, of *algebra*, considered as an *uninterpreted symbol*, and as *not* equal to *any real versor*, such as i (comp. 181, and 214, (3.)), but as following the *rules of scalars*, especially as regards the *commutative property* of multiplication (126); so that

$$\text{V.} \ldots h^2 + 1 = 0, \quad \text{and} \quad \text{VI.} \ldots hi = ih, \quad \text{but} \quad \text{VII.} \ldots h \textit{ not} = \pm i.$$

(3.) Let q denote still a *real quaternion*, or *real couple*, $x + iy$; and with the meaning just now proposed of h, let $[q]$ denote the connected but *imaginary algebraic quantity*, or *bi-scalar* (214, (7.)), $x + hy$; so that

$$\text{VIII.} \ldots q = x + iy, \quad \text{but} \quad \text{IX.} \ldots [q] = x + hy;$$

and let any *biquaternion* (214, (8.)), or (as we may *here* call it) BI-COUPLE, of the *form* $[q'] + i\,[q'']$, be said to be *complanar* with i; with the old *notation* (123) of complanarity.

(4.) Then, for the *polynomial equation in real and complanar quaternions*, $F_n q = 0$ (244, 245), we may be led to *substitute* the following *connected algebraical equation*, of the *same degree*, n, and *involving real scalars similarly*:

$$\text{X.} \ldots [F_n q] = [q]^n + [q_1]\,[q]^{n-1} + \ldots + [q_n] = 0;$$

which, after the reductions depending on the substitution V. of -1 for h^2, receives the form,

$$\text{XI.} \ldots [F_n q] = X_n + hY_n = 0;$$

where X_n and Y_n are the *same real and scalar functions* as in I.

(5.) But we have seen in II., that *these two real functions* can be made to *vanish together*, by selecting *any one of n real pairs* IV. of *scalar values*, x *and* y;

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* Cases of *equal roots* may cause points of intersection, which are *generally imaginary*, to *become real*, but *coincident* with each other, and with *former* real roots: for instance the *hyperbola*, $x^2 - y^2 = a$, is intersected in *two real* and *distinct points*, by the pair *of right lines* $xy = 0$, if the scalar $a >$ or < 0; but for the case $a = 0$, the *two pairs of lines*, $x^2 - y^2 = 0$ and $xy = 0$, may be considered to have *four coincident intersections* at the origin.

the General Algebraical Equation X., *of the* n^{th} *Degree, has therefore n Real or Imaginary Roots*,* *of the Form* $x + y\sqrt{-1}$; and it has *no more than n* such roots.

(6.) *Elimination* of y, between the two equations II. or III., conducts generally to an algebraic equation in x, of the degree n^2; which equation has therefore n^2 *algebraic roots* (5.), real *or* imaginary; namely, by what has been lately proved, n *real* and *scalar roots* $x', \ldots x^{(n)}$, with *real* and *scalar values* $y', \ldots y^{(n)}$ (comp. IV.) of y to *correspond*; and $n(n-1)$ *other roots*, with the same number of corresponding values of y, which may be thus denoted,

$$\text{XII.} \ldots [x^{(n+1)}], \ldots [x^{(n^2)}]; \qquad \text{XIII.} \ldots [y^{(n+1)}], \ldots [y^{(n^2)}];$$

and which are either themselves *imaginary* (or *bi-scalar*, 214, (7.)), or at least *correspond*, by the supposed elimination, to *imaginary* or *bi-scalar values* of y; since if $x^{(n+1)}$ and $y^{(n+1)}$, for example, could *both* be *real*, the *quaternion equation* $F_nq = 0$, would then have an $(n+1)$st *real root*, of the form, $q^{(n+1)} = x^{(n+1)} + iy^{(n+1)}$, contrary to what has been proved (252).

257. On the whole, then, it results that the equation $F_nq = 0$ in complanar quaternions, of the n^{th} degree, with real coefficients, while it admits of only *n real quaternion roots*,

$$\text{I.} \ldots q', q'', \ldots q^{(n)} \ (244, \&c.),$$

is *symbolically satisfied also* (comp. 214, (3.)) *by* $n(n-1)$ *imaginary quaternion roots*, or by $n^2 - n$ *bi-quaternions* (214, (8.)), or *bi-couples* (256, (3.)), which may be thus denoted,

$$\text{II.} \ldots [q^{(n+1)}], \ldots [q^{(n^2)}];$$

and of which the first, for example, has the *form*,

$$\text{III.} \ldots [q^{(n+1)}] = [x^{(n+1)}] + i[y^{(n+1)}] = x_{\prime}^{(n+1)} + hx_{\prime\prime}^{(n+1)} + i(y_{\prime}^{(n+1)} + hy_{\prime\prime}^{(n+1)});$$

where $x_{\prime}^{(n+1)}$, $x_{\prime\prime}^{(n+1)}$, $y_{\prime}^{(n+1)}$, and $y_{\prime\prime}^{(n+1)}$ are *four real scalars*, but h is the *imaginary of algebra* (256, (2.)).

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* This celebrated *Theorem of Algebra* has long been known, and has been proved in other ways; but it seemed necessary, or at least useful, for the purpose of the present work, to prove it anew, in connexion with *Quaternions*: or rather to establish the theorem (244, 252), to which in the present Calculus it *corresponds*. Compare the Note to page 278.

(1.) There must, for instance, be $n(n-1)$ *imaginary* n^{th} *roots of unity, in the given plane* of i (comp. 256, (3.)), besides the n *real roots* already determined (233, 237); and accordingly in the case $n = 2$, we have the *four* following *square-roots of* 1 ||| i, two real and two imaginary:

$$\text{IV.} \ldots +1, \quad -1; \qquad +hi, \quad -hi;$$

for, by 256, (2.), we have

$$\text{V.} \ldots (\pm hi)^2 = h^2i^2 = (-1)(-1) = +1.$$

And the *two imaginary roots* of the *quadratic* equation $F_2q = 0$, which *generally* exist, at least as *symbols* (214, (3.)), may be obtained by *multiplying the square-root* in the formula 253, XX. by hi; so that in the particular *case*, when that *radical vanishes*, the *four roots* of the equation become *real* and *equal*: *zero* having thus *only itself* for a *square-root*.

(2.) Again, if we write (comp. 237, (3.)),

$$\text{VI.} \ldots q = 1^{\frac{1}{3}}_{1} = \frac{-1 + i\sqrt{3}}{2}, \qquad q^2 = 1^{\frac{1}{3}}_{2} = \frac{-1 - i\sqrt{3}}{2},$$

so that 1, q, q^2 are the *three real cube-roots of positive unity*, in the given plane; and if we write also,

$$\text{VII.} \ldots \theta = [q] = \frac{-1 + h\sqrt{3}}{2}, \qquad \theta^2 = [q]^2 = \frac{-1 - h\sqrt{3}}{2},$$

so that θ and θ^2 are (as usual) the *two ordinary* (or *algebraical*) *imaginary cube-roots of unity*; then the *nine cube-roots of* 1 (||| i) are the following:

$$\text{VIII.} \ldots 1; \quad q, q^2; \quad \theta, \theta^2; \quad \theta q, \theta q^2; \quad \theta^2 q, \theta^2 q^2;$$

whereof the first is a *real scalar*; the two next are *real couples*, or *quaternions* ||| i; the two following are *imaginary scalars*, or *biscalars*; and the four that remain are *imaginary couples*, or *bi-couples*, or *biquaternions*.

(3.) The *sixteen fourth roots of unity* (||| i) are:

$$\text{IX.} \ldots \pm 1; \quad \pm i; \quad \pm h; \quad \pm hi; \quad \pm \tfrac{1}{2}(1 \pm h)(1 \pm i);$$

the three ambiguous signs in the last expression being all independent of each other.

(4.) *Imaginary roots*, of *this* sort, are sometimes *useful*, or rather *necessary*, in calculations respecting *ideal intersections*,* and *ideal contacts*, in *geometry*: although in what remains of the present Volume, we shall have little or no occasion to employ them.

(5.) We may, however, here observe, that when the *restriction* (225) on the *plane* of the quaternion q is *removed*, the *General Quaternion Equation of the n^{th} Degree* admits, by the foregoing principles, no fewer than n^4 *Roots, real or imaginary*; because, when that general equation is reduced, by 221, to the *Standard Quadrinomial Form*,

$$\text{X.} \ldots F_n q = W_n + iX_n + jY_n + kZ_n = 0,$$

it breaks up (comp. 221, VI.) into a *System of Four Scalar Equations, each* (generally) of the n^{th} *dimension*, in w, x, y, z; namely,

$$\text{XI.} \ldots W_n = 0, \quad X_n = 0, \quad Y_n = 0, \quad Z_n = 0;$$

and if x, y, z be eliminated between these four, the result is (generally) *a scalar* (or algebraical) *equation of the degree* n^4, relatively to the remaining *constituent*, w; which therefore has n^4 (algebraical) *values*, real or imaginary: and similarly for the three other constituents, x, y, z, of the sought quaternion q.

(6.) It may even happen, *when no plane is given*, that the *number of roots* (or solutions) of a *finite*† *equation in quaternions* shall become *infinite*; as has been seen to be the case for the equation $q^2 = -1$ (149, 154), *even* when we confine ourselves to what we have considered as *real* roots. If *imaginary* roots be admitted, we may write, *still more generally*, besides the *two bi-scalar values*, $\pm h$, the expression,

$$\text{XII.} \ldots (-1)^{\frac{1}{2}} = v + hv', \quad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = 0, \quad \mathrm{N}v - \mathrm{N}v' = 1;$$

v and v' being thus *any two real and right quaternions, in rectangular planes*, provided that the *norm* of the *first exceeds* that of the *second* by *unity*.

(7.) And in like manner, besides the *two real* and *scalar values*, ± 1, we have this general symbolical expression for a square root of positive unity, with merely the difference of the norms reversed:

$$\text{XIII.} \ldots 1^{\frac{1}{2}} = v + hv', \quad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = 0, \quad \mathrm{N}v' - \mathrm{N}v = 1.$$

* Comp. Art. 214, and the Notes there referred to.

† Compare the Note to page 277.

SECTION 7.

On the Reciprocal of a Vector, and on Harmonic Means of Vectors; with Remarks on the Anharmonic Quaternion of a Group of Four Points, and on Conditions of Concircularity.

258. When two vectors, α and α', are so related that

$$\text{I.} \ldots \alpha' = -\mathrm{U}\alpha : \mathrm{T}\alpha, \quad \text{and therefore} \quad \text{II.} \ldots \alpha = -\mathrm{U}\alpha' : \mathrm{T}\alpha,$$

or that

$$\text{III.} \ldots \mathrm{T}\alpha \,.\, \mathrm{T}\alpha' = 1, \quad \text{and} \quad \text{IV.} \ldots \mathrm{U}\alpha + \mathrm{U}\alpha' = 0,$$

we shall say that each of these two vectors is the *Reciprocal** of the other; and shall (at least for the present) *denote* this relation between them, by writing

$$\text{V.} \ldots \alpha' = \mathrm{R}\alpha, \quad \text{or} \quad \text{VI.} \ldots \alpha = \mathrm{R}\alpha';$$

so that for *every vector* α, and *every right quotient* v,

$$\text{VII.} \ldots \mathrm{R}\alpha = -\mathrm{U}\alpha : \mathrm{T}\alpha; \qquad \text{VIII.} \ldots \mathrm{R}^2\alpha = \mathrm{RR}\alpha = \alpha;$$

and

$$\text{IX.} \ldots \mathrm{RI}v = \mathrm{IR}v \text{ (comp. 161, (3.), and 204, XXXV'.)}.$$

259. One of the most important properties of such reciprocals is contained in the following theorem:

If any two vectors OA, OB, have OA′, OB′ for their *reciprocals*, then (comp. fig. 58) the *right line* A′B′ *is parallel to the tangent* OD, *at the origin* O, *to the circle* OAB; and the *two triangles*, OAB, OB′A′, are *inversely similar* (118). Or in symbols,

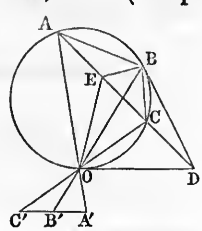

Fig. 58.

$$\text{I.} \ldots \text{if } \mathrm{OA}' = \mathrm{R}\,.\,\mathrm{OA}, \quad \text{and} \quad \mathrm{OB}' = \mathrm{R}\,.\,\mathrm{OB},$$

then

$$\Delta\, \mathrm{OAB} \propto' \mathrm{OB'A'}.$$

(1.) Of course, under the same conditions, the tangent at O to the circle OA′B′ is parallel to the line AB.

(2.) The angles BAO and OB′A′ or BOD being equal, the fourth proportional (226) to AB, AO, and OB, or to BA, OA, and OB, has the direction of OD, or the *direction opposite* to that of A′B′; and its *length* is easily proved to be the

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* Accordingly, under these conditions, we shall afterwards *denote* this *reciprocal* of a vector α by the *symbol* α^{-1}; but we postpone the use of this notation, until we shall be prepared to connect it with a general theory of *products and powers of vectors*. Compare 234, V., and the first Note to page 123. And as regards the temporary use of the characteristic R, compare the second Note to page 262.

reciprocal (or *inverse*) of the length of the same line A′B′, because the similar triangles give,

$$\text{II.} \ldots (\overline{\text{OA}} : \overline{\text{BA}}) \,.\, \overline{\text{OB}} = (\overline{\text{OB}'} : \overline{\text{A}'\text{B}'}) \,.\, \overline{\text{OB}} = 1 : \overline{\text{A}'\text{B}'},$$

it being remembered that

$$\text{III.} \ldots \overline{\text{OA}} \,.\, \overline{\text{OA}'} = \overline{\text{OB}} \,.\, \overline{\text{OB}'} = 1;$$

we may therefore write,

$$\text{IV.} \ldots (\text{OA} : \text{BA}) \,.\, \text{OB} = \text{R} \,.\, \text{A}'\text{B}', \quad \text{or} \quad \text{V.} \ldots \frac{\alpha}{\alpha - \beta}\beta = \text{R}(\text{R}\beta - \text{R}\alpha),$$

whatever two vectors α and β may be.

(3.) Changing α and β to their reciprocals, the last formula becomes,

$$\text{VI.} \ldots \text{R}(\beta - \alpha) = \frac{\text{R}\alpha}{\text{R}\alpha - \text{R}\beta} \,.\, \text{R}\beta; \quad \text{or} \quad \text{VII.} \ldots (\text{OA}' : \text{B}'\text{A}') \,.\, \text{OB}' = \text{R} \,.\, \text{AB}.$$

(4.) The inverse similarity I. gives also, generally, the relation,

$$\text{VIII.} \ldots \text{K}\frac{\beta}{\alpha} = \frac{\text{R}\alpha}{\text{R}\beta}.$$

(5.) Since, then, by 195, II., or 207, (2.),

$$\text{IX.} \ldots \text{K}\frac{\beta}{\alpha} \pm 1 = \text{K}\frac{\beta \pm \alpha}{\alpha}, \quad \text{we have} \quad \text{X.} \ldots \frac{\text{R}\alpha \pm \text{R}\beta}{\text{R}\beta} = \frac{\text{R}\alpha}{\text{R}(\beta \pm \alpha)};$$

the lower signs agreeing with VI.

(6.) In general, the *reciprocals* of *opposite* vectors are themselves *opposite*; or in symbols,

$$\text{XI.} \ldots \text{R}(-\alpha) = -\text{R}\alpha.$$

(7.) More generally,

$$\text{XII.} \ldots \text{R}x\alpha = x^{-1}\,\text{R}\alpha,$$

if x be any scalar.

(8.) Taking lower signs in X., changing α to γ, dividing, and taking conjugates, we find for *any three vectors* α, β, γ (*complanar or diplanar*) the formula:

$$\text{XIII.} \ldots \text{K}\frac{\text{R}\gamma - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{K}\left(\frac{\text{R}\gamma}{\text{R}(\beta - \gamma)} \cdot \frac{\text{R}(\beta - \alpha)}{\text{R}\alpha}\right) = \frac{\alpha}{\beta - \alpha} \cdot \frac{\gamma - \beta}{-\gamma} = \frac{\text{OA}}{\text{AB}} \cdot \frac{\text{BC}}{\text{CO}},$$

if $\alpha = \text{OA}$, $\beta = \text{OB}$, and $\gamma = \text{OC}$, as usual.

(9.) If then we *extend*, to *any four points of space*, the *notation* (25.),

$$\text{XIV.} \ldots (\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}},$$

interpreting each of these two *factor-quotients* as a *quaternion*, and *defining* that their *product* (in *this order*) is the *anharmonic quaternion function*, or simply *the*

Anharmonic, of the Group of four points A, B, C, D, or of the (*plane or gauche*) *Quadrilateral* ABCD, we shall have the following general and useful *formula of transformation*:

$$\text{XV.}\ldots (\text{OABC}) = \text{K}\,\frac{\text{R}\gamma - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{K}\,\frac{\text{B}'\text{C}'}{\text{B}'\text{A}'},$$

where OA′, OB′, OC′ are supposed to be reciprocals of OA, OB, OC.

(10.) With this notation XIV., we have *generally*, and not merely for *collinear groups* (35.), the relations:

$$\text{XVI.}\ldots (\text{ABCD}) + (\text{ACBD}) = 1; \qquad \text{XVII.}\ldots (\text{ABCD}) \,.\, (\text{ADCB}) = 1.$$

(11.) Let O, A, B, C, D be *any five points*, and OA′, . . OD′ the reciprocals of OA, . . OD; we shall then have, by XV.,

$$\text{XVIII.}\ldots \frac{\text{B}'\text{A}'}{\text{B}'\text{C}'} = \text{K}\,(\text{OCBA}), \quad \frac{\text{D}'\text{C}'}{\text{D}'\text{A}'} = \text{K}\,(\text{OADC});$$

and therefore,

$$\text{XIX.}\ldots \text{K}\,(\text{A}'\text{B}'\text{C}'\text{D}') = (\text{OADC})\,(\text{OCBA}) = -\,(\text{OADCBA}),$$

if we agree to write generally, for *any six points*, the formula,*

$$\text{XX.}\ldots (\text{ABCDEF}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DE}} \cdot \frac{\text{EF}}{\text{FA}}.$$

(12.) If then the five points O . . D be *complanar* (225), we have, by 226, and by XIV.,

$$\text{XXI.}\ldots \text{K}\,(\text{A}'\text{B}'\text{C}'\text{D}') = (\text{ABCD}), \quad \text{or} \quad \text{XXI}'.\ldots (\text{A}'\text{B}'\text{C}'\text{D}') = \text{K}\,(\text{ABCD});$$

the *anharmonic quaternion* (ABCD) being thus changed to its *conjugate*, when the *four rays* OA, . . OD are changed to their *reciprocals*.

260. Another very important consequence from the definition (258) of reciprocals of vectors, or from the recent theorem (259), may be expressed as follows:

If any three coinitial vectors, OA, OB, OC, *be chords of one common circle*, then (see again fig. 58) *their three coinitial reciprocals*, OA′, OB′, OC′, *are termino-*

* There is a convenience in calling, generally, this *product of three quotients*, (ABCDEF), the *evolutionary quaternion*, or simply *the Evolutionary*, of the *Group of Six Points*, A . . F, or (if they be not collinear) of the plane or gauche *Hexagon* ABCDEF: because the equation,

$$(\text{ABCA}'\text{B}'\text{C}') = -1,$$

expresses either Ist, that the *three pairs of points*, AA′, BB′, CC′, form a *collinear involution* (26.) of a well-known kind; or IInd, that those three *pairs*, or the three corresponding *diagonals* of the hexagon, compose a *complanar* or a *homospheric Involution*, of a *new kind suggested by quaternions* (comp. 261, (11.)).

collinear (24) : or, in other words, if the *four points* O, A, B, C be *concircular*, then the *three points* A′, B′, C′ are situated *on one right line.*

And conversely, *if three coinitial vectors*, OA′, OB′, OC′, thus terminate on one *right line*, then their three coinitial *reciprocals*, OA, OB, OC, are *chords* of one *circle*; the *tangent* to which circle, at the origin, is *parallel* to the *right line*; while the *anharmonic function* (259, (9.)), of the *inscribed quadrilateral* OABC, reduces itself to a *scalar quotient of segments* of that line (which therefore is its own conjugate, by 139): namely,

$$\text{I.} \ldots (\text{OABC}) = \text{B}'\text{C}' : \text{B}'\text{A}' = (\infty\text{A}'\text{B}'\text{C}') = (\text{O}\,.\,\text{OABC}),$$

if the symbol ∞ be used *here* to denote the *point at infinity* on the right line A′B′C′; and if, in thus employing the notation (35) for the *anharmonic of a plane pencil*, we consider the *null chord*, OO, as having the *direction** of the *tangent*, OD.

(1.) If ρ = OP be the *variable vector* of a point P upon the circle OAB, the *quaternion equation* of that circle may be thus written:

$$\text{II.} \ldots \text{R}\rho = \text{R}\beta + x(\text{R}\alpha - \text{R}\beta), \quad \text{where} \quad \text{III.} \ldots x = (\text{OABP});$$

the coefficient x being thus a *variable scalar* (comp. 99, I.), which depends on the *variable position* of the point P on the circumference.

(2.) Or we may write,

$$\text{IV.} \ldots \text{R}\rho = \frac{t\text{R}\alpha + u\text{R}\beta}{t+u},$$

as another form of the equation of the same *circle* OAB; with which may usefully be contrasted the earlier form (comp. 25.), of the equation of the *line* AB,

$$\text{V.} \ldots \rho = \frac{t\alpha + u\beta}{t+u}.$$

(3.) Or, dividing the second member of IV. by the first, and taking conjugates, we have for the circle,

$$\text{VI.} \ldots \frac{t\rho}{\alpha} + \frac{u\rho}{\beta} = t + u; \quad \text{while} \quad \text{VII.} \ldots \frac{t\alpha}{\rho} + \frac{u\beta}{\rho} = t + u,$$

for the right line.

(4.) Or we may write, by II.,

$$\text{VIII.} \ldots \text{V}\,\frac{\text{R}\rho - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = 0; \quad \text{or} \quad \text{VIII}'. \ldots \frac{\text{R}\rho - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{V}^{-1}0;$$

this latter symbol, by 204, (18.), denoting *any scalar.*

* Compare the remarks in the Note to page 140, respecting the possible determinateness of signification of the symbol U0, when the *zero* denotes a *line*, which *vanishes* according to a *law.*

(5.) Or still more briefly,

$$\text{IX.} \ldots \text{V}(\text{OABP}) = 0; \quad \text{or} \quad \text{IX}' \ldots (\text{OABP}) = \text{V}^{-1}0.$$

(6.) If the *four* points O, A, B, C be still *concircular*, and if P be *any fifth point in their plane*, while $\text{PO}_1, \ldots \text{PC}_1$ are the reciprocals of PO, .. PC, then by 259, XXI., we have the relation,

$$\text{X.} \ldots (\text{O}_1\text{A}_1\text{B}_1\text{C}_1) = \text{K}(\text{OABC}) = (\text{OABC}) = \text{V}^{-1}0;$$

the *four new points* $\text{O}_1 \ldots \text{C}_1$ are therefore *generally concircular*.

(7.) If, however, the point P be again placed *on the circle* OABC, those four new points are (by the present Article) *collinear*; being the *intersections* of the *pencil* P . OABC with a *parallel to the tangent* at P. In this case, therefore, we have the equation,

$$\text{XI.} \ldots (\text{P} \,.\, \text{OABC}) = (\text{O}_1\text{A}_1\text{B}_1\text{C}_1) = (\text{OABC});$$

so that the *constant anharmonic of the pencil* (35) is thus seen to be equal to what we have defined (259, (9.)) to be the *anharmonic of the group*.

(8.) And because the *anharmonic* of a *circular group* is a *scalar*, it is equal (by 187, (8.)) to its own *tensor*, either positively or negatively taken: we may therefore write, for any *inscribed quadrilateral* OABC, the formula,

$$\text{XII.} \ldots (\text{OABC}) = \mp \text{T}(\text{OABC}) = \mp (\overline{\text{OA}} \,.\, \overline{\text{BC}}) : (\overline{\text{AB}} \,.\, \overline{\text{CO}}),$$

$= \mp$ a *quotient of rectangles of opposite sides*; the upper or the lower *sign* being taken, according as the point B′ falls, or does not fall, *between* the points A′ and C′: that is, according as the quadrilateral OABC is an *uncrossed* or a *crossed* one.

(9.) Hence it is easy to infer that *for any circular group* O, A, B, C, we have the equation,

$$\text{XIII.} \ldots \text{U}\frac{\text{OA}}{\text{AB}} = \pm \text{U}\frac{\text{CO}}{\text{CB}};$$

the *upper* sign being taken when the *succession* OABC is a *direct* one, that is, when the *quadrilateral* OABC is *uncrossed*; and the lower sign, in the contrary case, namely, when the succession is (what may be called) *indirect*, or when the quadrilateral is *crossed*: while conversely this equation XIII. is sufficient to prove, whenever it occurs, that the anharmonic (OABC) is a negative or a positive *scalar*, and therefore by (5.) that the *group* is *circular* (if not *linear*), as above.

(10.) If A, B, C, D, E be any *five homospheric points* (or points upon the surface of *one sphere*), and if O be *any sixth point of space*, while OA′, .. OE′

are the reciprocals of OA, . . OE, then the *five new points* A′ . . E′ are *generally homospheric* (with each other); but if O happens to be *on the sphere* ABCDE, then A′ . . E′ are *complanar*, their common plane being *parallel to the tangent plane* to the given sphere at O: with resulting anharmonic relations, on which we cannot here delay.

261. An interesting *case* of the foregoing theory is that when the generally *scalar* anharmonic of a *circular* group becomes equal to *negative unity*: in which case (comp. 26), the group is said to be *harmonic*. A few remarks upon such *circular and harmonic groups* may here be briefly made: the student being left to fill up hints for himself, as what must be now to him an easy exercise of calculation.

(1.) For such a group (comp. again fig. 58), we have thus the equation,

$$\text{I.} \ldots (\text{OABC}) = -1; \quad \text{and therefore} \quad \text{II.} \ldots \text{A}'\text{B}' = \text{B}'\text{C}';$$

or

$$\text{III.} \ldots \text{R}\beta = \tfrac{1}{2}(\text{R}\alpha + \text{R}\gamma);$$

and under this condition, we shall say (comp. 216, (5.)) that the *Vector* β is the *Harmonic Mean* between the *two* vectors, α and γ.

(2.) Dividing, and taking conjugates (comp. 260, (3.), and 216, (5.)), we thus obtain the equation,

$$\text{IV.} \ldots \frac{\beta}{\alpha} + \frac{\beta}{\gamma} = 2; \quad \text{or} \quad \text{V.} \ldots \beta = \frac{2\alpha}{\gamma + \alpha}\gamma = \frac{2\gamma}{\gamma + \alpha}\alpha;$$

or

$$\text{VI.} \ldots \beta = \frac{\alpha}{\varepsilon}\gamma = \frac{\gamma}{\varepsilon}\alpha, \quad \text{if} \quad \text{VII.} \ldots \varepsilon = \tfrac{1}{2}(\gamma + \alpha);$$

ε thus denoting here the vector OE (fig. 58) of the middle point of the chord AC. We may then say that the *harmonic mean* between any two lines is (as in algebra) the *fourth proportional to their semisum, and to themselves.*

(3.) Geometrically, we have thus the similar triangles,

$$\text{VIII.} \ldots \Delta\,\text{AOB} \propto \text{EOC}; \qquad \text{VIII}'. \ldots \Delta\,\text{AOE} \propto \text{BOC};$$

whence, either because the angles OBA and OCA, or because the angles OAC and OBC are equal, we may infer (comp. 260, (5.)) that, when the equation I. is satisfied, the four points O, A, B, C, if not *collinear*, are *concircular*.

(4.) We have also the similarities,

$$\text{IX.} \ldots \Delta\,\text{OEC} \propto \text{CEB}, \quad \text{and} \quad \text{IX}'. \ldots \Delta\,\text{OEA} \propto \text{AEB};$$

or the equations,

$$\text{X.} \ldots \frac{\beta - \varepsilon}{\gamma - \varepsilon} = \frac{\gamma - \varepsilon}{-\varepsilon}, \quad \text{and} \quad \text{X}'. \ldots \frac{\beta - \varepsilon}{\alpha - \varepsilon} = \frac{\alpha - \varepsilon}{-\varepsilon};$$

in fact we have, by VI. and VII.,

$$\text{XI.} \ldots \frac{\alpha}{\varepsilon} + \frac{\gamma}{\varepsilon} = 2; \quad \text{XII.} \ldots \frac{\beta - \varepsilon}{-\varepsilon}\left(= 1 - \frac{\beta}{\alpha}\frac{\alpha}{\varepsilon} = 1 - \frac{\gamma}{\varepsilon}\frac{\alpha}{\varepsilon}\right) = \left(1 - \frac{\alpha}{\varepsilon}\right)^2.$$

(5.) Hence the line EC, in fig. 58, is the *mean proportional* (227) between the lines EO and EB; or in words, the *semisum* (OE), the *semidifference* (EC), and the *excess* (BE) *of the semisum over the harmonic mean* (OB), form (as in algebra) a *continued proportion* (227).

(6.) Conversely, if any three coinitial vectors, EO, EC, EB, form thus a continued proportion, and if we take EA = CE, then the four points OABC will compose a circular and harmonic group; for example, the points APBP′ of fig. 57 are arranged so as to form such a group.*

(7.) It is easy to prove that, for the *inscribed quadrilateral* OABC of fig. 58, the *rectangles under opposite sides* are each equal to *half* of the rectangle under the *diagonals*; which geometrical relation answers to either of the two anharmonic equations (comp. 259, (10.)):

$$\text{XIII.} \ldots (\text{OBAC}) = +2; \qquad \text{XIII}'. \ldots (\text{OCAB}) = +\tfrac{1}{2}.$$

(8.) Hence, or in other ways, it may be inferred that these diagonals, OB, AC, are *conjugate chords* of the circle to which they belong: in the sense that *each* passes *through the pole* of the *other*, and that thus the line DB is the *second tangent* from the point D, in which the chord AC prolonged intersects the tangent at O.

(9.) Under the same conditions, it is easy to prove, either by quaternions or by geometry, that we have the harmonic equations:

$$\text{XIV.} \ldots (\text{ABCO}) = (\text{BCOA}) = (\text{COAB}) = -1;$$

so that AC is the harmonic mean between AB and AO; BO is such a mean between BC and BA; and CA between CO and CB.

(10.) In any such group, *any two opposite points* (or opposite corners of the quadrilateral), as for example O and B, may be said to be *harmonically conjugate* to each other, *with respect to the two other points*, A and C; and we see that when these *two* points A and C are *given*, then to *every third point* O (whether in a given plane, or in space) there always *corresponds* a *fourth* point B, which is in this sense *conjugate* to that third point: this fourth point being always *complanar* with the three points A, C, O, and being even *concircular* with them,

* Compare the Note to 255, (2.). In fig. 58, the centre of the circle OABC is concircular with the three points O, E, B.

unless they happen to be *collinear* with each other; in which extreme (or *limiting*) *case*, the *fourth point* B is *still determined*, but is now collinear with the others (as in 26, &c.).

(11.) When, after thus *selecting two* points*, A and C, or treating them as *given* or *fixed*, we determine (10.) the harmonic *conjugates* B, B′, B″, with respect to *them*, of *any three assumed points*, O, O′, O″, then the *three pairs of points*, O, B; O′, B′; O″, B″, may be said to form an *Involution*,† either *on the right line* AC, (in which case it will only be one of an already well-known kind), or *in a plane* through that line, or even generally *in space*: and the two points A, C may in all these cases be said to be the two *Double Points* (or *Foci*) of this Involution. But the field thus opened, for geometrical investigation by Quaternions, is far too extensive to be more than *mentioned* here.

(12.) We shall therefore only at present add, that the conception of the *harmonic mean* between *two vectors* may easily be extended to *any number* of such, and need not be limited to the *plane*: since we may define that η is the harmonic mean of the n arbitrary vectors $\alpha_1, \ldots \alpha_n$, when it satisfies the equation,

$$\text{XV.} \ldots \mathrm{R}\eta = \frac{1}{n}(\mathrm{R}\alpha_1 + \ldots + \mathrm{R}\alpha_n); \quad \text{or} \quad \text{XVI.} \ldots n\mathrm{R}\eta = \Sigma\mathrm{R}\alpha.$$

(13.) Finally, as regards the *notation* $\mathrm{R}\alpha$, and the *definition* (258) of the *reciprocal of a vector*, it may be observed that if we had chosen to define reciprocal vectors as having *similar* (instead of *opposite*) *directions*, we should indeed have had the positive sign in the equation 258, VII.; but should have been obliged to write, instead of 258, IX., the much less simple formula,

$$\mathrm{RI}v = -\,\mathrm{IR}v.$$

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* There is a sense in which the geometrical process here spoken of can be applied, even when the two fixed points, or *foci*, are *imaginary*. Compare the *Géométrie Supérieure* of M. Chasles, page 136.

† Compare the Note to 259, (11.).

CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS.

SECTION 1.

On some Enunciations of the Associative Property, or Principle, of Multiplication of Diplanar Quaternions.

262. In the preceding chapter we have confined ourselves almost entirely, as had been proposed (224, 225), to the considerations of quaternions *in a given plane* (that of i); alluding only, in some instances, to possible extensions* of results so obtained. But we must now return to consider, as in the First Chapter of this Second Book, the subject of *General Quotients of Vectors*: and especially their *Associative Multiplication* (223), which has hitherto been only proved in connexion with the *Distributive Principle* (212), and with the *Laws of the Symbols, i, j, k* (183). And first we shall give a few *geometrical enunciations* of that associative principle, which shall be independent of the distributive one, and in which it will be sufficient to consider (comp. 191) the *multiplication of versors*; because the multiplication of *tensors* is *evidently* an associative operation, as corresponding simply to *arithmetical* multiplication, or to the *composition of ratios* in geometry.† We shall therefore suppose, throughout the present chapter, that q, r, s are some *three given but arbitrary versors*, in *three given and distinct planes*;‡ and our object will be to throw some

* As in 227, (3.); 242, (7.); 254, (7.); 257, (6.) and (7.); 259, (8.), (9.), (10.), (11.); 260, (10.); and 261, (11.) and (12.).

† Or, more generally, for any three pairs of magnitudes, each pair separately being homogeneous.

‡ If the factors q, r, s were *complanar*, we could always (by 120) put them under the forms,

$$q = \frac{\beta}{\alpha}, \quad r = \frac{\gamma}{\beta}, \quad s = \frac{\delta}{\gamma};$$

and then should have (comp. 183, (1.)) the *two equal ternary products*,

$$sr \,.\, q = \frac{\delta}{\beta}\frac{\beta}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\gamma}\frac{\gamma}{\alpha} = s \,.\, rq;$$

so that in *this* case (comp. 224) the associative property would be proved without any difficulty.

additional light, by new enunciations in this section, and by new demonstrations in the next, on the very important, although very simple, *Associative Formula* (223, II.), which may be written thus:

$$\text{I.} \ldots sr \,.\, q = s \,.\, rq;$$

or thus, more fully,

$$\text{II.} \ldots q'q = t, \quad \text{if} \quad q' = sr, \quad s' = rq, \quad \text{and} \quad t = ss';$$

q', s', and t being here *three new and derived versors*, in *three new and derived planes*.

263. Already we may see that this *Associative Theorem of Multiplication*, in all its forms, has an essential reference to a *System of Six Planes*, namely the planes of these *six versors*,

$$\text{IV.} \ldots q, r, s, rq, sr, srq, \quad \text{or} \quad \text{IV}'. \ldots q, r, s, s', q', t;$$

on the judicious selection and arrangement of which, the clearness and elegance of every geometrical statement or proof of the theorem must very much depend: while the *versor character* of the factors (in the only part of the theorem for which *proof* is required) suggests a reference to a *Sphere*, namely to what we have called the *unit-sphere* (128). And the *three* following *arrangements* of the six planes appear to be the most natural and simple that can be considered: namely, Ist, the arrangement in which the planes all pass *through the centre* of the sphere; IInd, that in which they all *touch* its surface; and IIIrd, that in which they are the six *faces of an inscribed solid.* We proceed to consider successively these three arrangements.

264. When the *first* arrangement (263) is adopted, it is natural to employ *arcs of great circles*, as *representatives* of the *versors*, on the plan of Art. 162. Representing thus the factor q by the arc AB, and r by the successive arc BC, we represent (167) their *product* rq, or s', by AC; or by any *equal arc* (165), such as DE, in fig. 59, may be supposed to be. Again, representing s by EF, we shall have DF as the representative of the *ternary product* $s \,.\, rq$, or ss', or t, taken in *one order of association*. To represent the *other* ternary product, $sr \,.\, q$, or $q'q$, we may first determine three new points, G, H, I, by arcual equations (165), between GH, BC, and between HI, EF, so that BC, EF intersect in H, as the arcs representing s' and s had intersected in E; and then, after thus finding an arc GI which represents sr or q', may determine three other points K, L, M,

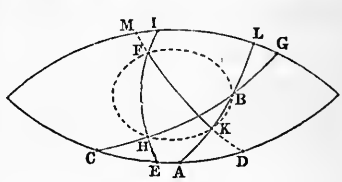

Fig. 59.

by equations between KL, AB, and between LM, GI, so that these two new arcs, KL, LM, represent q and q', and that AB, GI intersect in L; for in this way we shall have an arc, namely KM, which represents $q'q$ as required. And the *theorem* then is, that this last arc KM is *equal* to the former arc DF, *in the full sense* of Art. 165; or that *when* (as under the foregoing conditions of construction) *the five arcual equations,*

$$\text{I.} \ldots \frown \mathrm{AB} = \frown \mathrm{KL}, \quad \frown \mathrm{BC} = \frown \mathrm{GH}, \quad \frown \mathrm{EF} = \frown \mathrm{HI}, \quad \frown \mathrm{AC} = \frown \mathrm{DE}, \quad \frown \mathrm{GI} = \frown \mathrm{LM},$$

exist, then this sixth equation of the same kind is satisfied also,

$$\text{II.} \ldots \frown \mathrm{DF} = \frown \mathrm{KM}:$$

the *two points,* K and M, being both *on the same great circle* as the two previously determined points, D and F; or D and M being on the great circle through F and K: and the *two arcs,* DF and KM, of that great circle, or the two *dotted* arcs, DK, FM in the figure, being *equally long,* and *similarly directed* (165).

(1.) Or, after determining the *nine points* A . . I so as to satisfy the three middle equations I., we might determine the *three other* points K, L, M, without any other arcual *equations,* as *intersections* of the three pairs of arcs AB, DF; AB, GI; DF, GI; and then the theorem would be, that (if these three last points be suitably distinguished from their own opposites upon the sphere) the two extreme equations I., and the equation II., are satisfied.

(2.) The same geometrical theorem may also be thus enunciated: *If the first, third, and fifth sides* (KL, GH, ED) *of a spherical hexagon* KLGHED *be respectively and arcually equal* (165) *to the first, second, and third sides* (AB, BC, CA) *of a spherical triangle* ABC, *then the second, fourth, and sixth sides* (LG, HE, DK) *of the same hexagon are equal to the three successive sides* (MI, IF, FM) *of another spherical triangle* MIF.

(3.) It may be also said, that *if five successive sides* (KL, . . ED) of *one* spherical *hexagon* be respectively and arcually *equal* to the *five successive diagonals* (AB, MI, BC, IF, CA) of *another* such hexagon (AMBICF), then the *sixth side* (DK) of the *first* is equal to the *sixth diagonal* (FM) of the *second.*

(4.) Or, if we adopt the conception mentioned in 180, (3.), of an *arcual sum,* and *denote* such a sum by inserting + between the symbols of the two summands, that of the *added arc* being written to the *left-hand,* we may state the theorem, in connexion with the recent fig. 59, by the formula:

$$\text{III.} \ldots \frown \mathrm{DF} + \frown \mathrm{BA} = \frown \mathrm{EF} + \frown \mathrm{BC}, \quad \text{if} \quad \frown \mathrm{DA} = \frown \mathrm{EC};$$

where B and F may denote *any two points* upon the sphere.

(5.) We may also express* the same principle, although somewhat less simply, as follows (see again fig. 59, and compare sub-art. (2.)):

$$\text{IV.}\ldots \text{if } \frown\mathrm{ED} + \frown\mathrm{GH} + \frown\mathrm{KL} = 0, \quad \text{then} \quad \frown\mathrm{DK} + \frown\mathrm{HE} + \frown\mathrm{LG} = 0.$$

(6.) If, for a moment, we agree to write (comp. Art. 1),

$$\text{V.}\ldots \frown\mathrm{AB} = \overset{\frown}{\mathrm{B} - \mathrm{A}},$$

we may then express the recent statement IV. a little more lucidly thus:

$$\text{VI.}\ldots \text{if } \overset{\frown}{\mathrm{D} - \mathrm{E}} + \overset{\frown}{\mathrm{H} - \mathrm{G}} + \overset{\frown}{\mathrm{L} - \mathrm{K}} = 0, \quad \text{then} \quad \overset{\frown}{\mathrm{K} - \mathrm{D}} + \overset{\frown}{\mathrm{E} - \mathrm{H}} + \overset{\frown}{\mathrm{G} - \mathrm{L}} = 0.$$

(7.) Or still more simply, if $\frown$, $\frown'$, $\frown''$ be supposed to denote *any three diplanar arcs*, which are to be *added* according to the *rule* (180, (3.)) above referred to, the *theorem* may be said to be, that

$$\text{VII.}\ldots (\frown'' + \frown') + \frown = \frown'' + (\frown' + \frown);$$

or in words, that *Addition of Arcs on a Sphere is an Associative Operation.*

(8.) Conversely, if any independent demonstration be given, of the truth of any one of the foregoing statements, considered as expressing a *theorem of spherical geometry*,† a *new proof* will thereby be furnished, of the associative property of *multiplication of quaternions.*

265. In the *second* arrangement (263) of the *six planes*, instead of representing the three given versors, and their partial or total products, by *arcs*, it is natural to represent them (174, II.) by *angles* on the sphere. Conceive then that the two versors, q and r, are represented, in fig. 60, by the two spherical angles, EAB and ABE; and therefore (175) that their product, rq or s', is represented by the external vertical angle at E, of the triangle ABE. Let the second versor r be also represented by the angle FBC, and the third versor s by BCF; then the other binary product, sr or q', will be represented by the external angle at F, of the new triangle BCF. Again, to represent the *first* ternary product, $t = ss' = s \, . \, rq$, we have only to take the

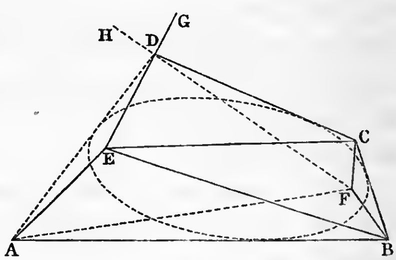

Fig. 60.

* Some of these formulæ and figures, in connexion with the associative principle, are taken, though for the most part with modifications, from the author's Sixth Lecture on Quaternions, in which that whole subject is very fully treated. Comp. the Note to page 160.

† Such a demonstration, namely a deduction of the equation II. from the five equations I., by known properties of *spherical conics*, will be briefly given in the ensuing section.

external angle at D of the triangle ECD, if D be a point determined by the two conditions, that the angle ECD shall be *equal* to BCF, and DEC *supplementary* to BEA. On the other hand, if we conceive a point D′ determined by the conditions that D′AF shall be equal to EAB, and AFD′ supplementary to CFB, then the external angle at D′, of the triangle AFD′, will represent the *second* ternary product, $q'q = sr \cdot q$, which (by the associative principle) must be *equal to the first*. Conceiving then that ED is prolonged to G, and FD′ to H, the *two spherical angles*, GDC and AD′H, must be *equal in all respects*; their *vertices* D and D′ *coinciding*, and the *rotations* (174, 177) which they represent being not only *equal in amount*, but also *similarly directed*. Or, to express the same thing otherwise, we may *enunciate* (262) *the Associative Principle* by saying, that *when the three angular equations*,

$$\text{I.} \ldots \text{ABE} = \text{FBC}, \quad \text{BCF} = \text{ECD}, \quad \text{DEC} = \pi - \text{BEA},$$

are satisfied, then these three other equations,

$$\text{II.} \ldots \text{DAF} = \text{EAB}, \quad \text{FDA} = \text{CDE}, \quad \text{AFD} = \pi - \text{CFB},$$

are satisfied also. For not only is this *theorem of spherical geometry* a *consequence* of the associative principle of *multiplication of quaternions*, but conversely any independent demonstration* of the theorem is, at the same time, a proof of the principle.

266. The *third arrangement* (263) of the six planes may be illustrated by conceiving a *gauche hexagon*, AB′CA′BC′, to be *inscribed in a sphere*, in such a manner that the intersection D of the three planes, C′AB′, B′CA′, A′BC′, is on the surface; and therefore that the *three small circles*, denoted by these three last triliteral symbols, *concur* in one point D; while the second intersection of the two other small circles, AB′C, CA′B, may be denoted by the letter D′, as in the annexed fig. 61. Let it be also for simplicity at first supposed, that (as in the figure) the *five circular successions*,

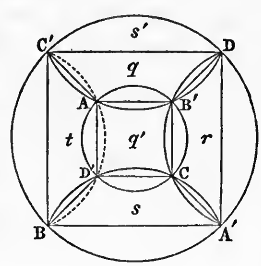

Fig. 61.

$$\text{I.} \ldots \text{C}'\text{AB}'\text{D}, \quad \text{AB}'\text{CD}', \quad \text{B}'\text{CA}'\text{D}, \quad \text{CA}'\text{BD}', \quad \text{A}'\text{BC}'\text{D},$$

are all *direct*; or that the *five inscribed quadrilaterals*, denoted by these symbols

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* Such as we shall sketch, in the following section, with the help of the known properties of the *spherical conics*. Compare the Note to the foregoing Article.

I., are all *uncrossed* ones. Then (by 260, (9.)) it is allowed to introduce *three versors*, q, r, s, each having *two expressions*, as follows:

$$\text{II.} \ldots q = \mathrm{U}\frac{\mathrm{B'D}}{\mathrm{DC'}} = + \mathrm{U}\frac{\mathrm{AB'}}{\mathrm{AC'}}; \quad r = \mathrm{U}\frac{\mathrm{DA'}}{\mathrm{B'D}} = + \mathrm{U}\frac{\mathrm{CA'}}{\mathrm{CB'}};$$

$$s = \mathrm{U}\frac{\mathrm{CD'}}{\mathrm{CA'}} = + \mathrm{U}\frac{\mathrm{BD'}}{\mathrm{A'B}};$$

although (by the cited sub-article) the last members of these three formulæ should receive the *negative* sign, if the first, third, and fourth of the successions I. were to become *indirect*, or if the corresponding quadrilaterals were *crossed* ones. We have thus (by 191) the derived expressions,

$$\text{III.} \ldots s' = rq = \mathrm{U}\frac{\mathrm{DA'}}{\mathrm{DC'}} = \mathrm{U}\frac{\mathrm{A'B}}{\mathrm{BC'}}; \quad q' = sr = \mathrm{U}\frac{\mathrm{CD'}}{\mathrm{CB'}} = \mathrm{U}\frac{\mathrm{D'A}}{\mathrm{AB'}};$$

whereof, however, the two versors in the first formula would differ in their signs, if the fifth succession I. were indirect; and those in the second formula, if the second succession were such. Hence,

$$\text{IV.} \ldots t = ss' = s \,.\, rq = \mathrm{U}\frac{\mathrm{BD'}}{\mathrm{BC'}}; \quad q'q = sr \,.\, q = \mathrm{U}\frac{\mathrm{D'A}}{\mathrm{AC'}};$$

and since, by the associative principle, these two last versors are to be *equal*, it follows that, under the supposed conditions of construction, the *four points*, B, C′, A, D′, compose a *circular* and *direct succession*; or that the *quadrilateral*, BC′AD′, is *plane, inscriptible,** and *uncrossed.*

267. It is easy, by suitable changes of sign, to adapt the recent reasoning to the case where some or all of the successions I. are indirect; and thus to *infer*, from the associative principle, this *theorem of spherical geometry*: *if* AB′CA′BC′ *be a spherical hexagon, such that the three small circles* C′AB′, B′CA′, A′BC′ *concur in one point* D, *then*, Ist, *the three other small circles*, AB′C, CA′B, BC′A, *concur in another point*, D′; and IInd, *of the six circular successions*, 266, I., and BC′AD′, *the number of those which are indirect is always even* (including *zero*). And conversely, any independent demonstration† of this geometrical theorem will be a *new proof* of the associative principle.

268. The same fertile principle of *associative multiplication* may be enunciated in other ways, without limiting the factors to be *versors*, and

* Of course, since the four points BC′AD′ are known to be *homospheric* (comp. 260, (10.)), the *inscriptibility* of the quadrilateral in a *circle* would *follow* from its being *plane*, if the latter were otherwise proved: but it is *here* deduced from the *equality* of the *two versors* IV., on the plan of 260, (9.).

† An elementary proof, by *stereographic projection*, will be proposed in the following section.

without introducing the conception of a *sphere.* Thus we may say (comp. 264, (2.)), that if O . ABCDEF (comp. 35) be any *pencil of six rays in space,* and O . A′B′C′ any *pencil of three rays,* and if the three angles AOB, COD, EOF of the first pencil be respectively equal to the angles B′OC′, C′OA′, A′OB′ of the second, then *another* pencil of *three* rays, O . A″B″C″, can be assigned, such that the three other angles BOC, DOE, FOA of the *first* pencil shall be equal to the angles B″OC″, C″OA″, A″OB″ of the *third*: *equality of angles* (with one vertex) being *here* understood (comp. 165) to *include complanarity,* and *similarity of direction of rotations.*

(1.) Again (comp. 264, (4.)), we may establish the following formula, in which the four vectors $\alpha\beta\gamma\delta$ form a complanar proportion (226), but ε and ζ are any two lines in space:

$$\text{I.} \ldots \frac{\zeta}{\gamma}\frac{\delta}{\varepsilon} = \frac{\zeta}{\alpha}\frac{\beta}{\varepsilon}, \quad \text{if} \quad \frac{\delta}{\gamma} = \frac{\beta}{\alpha};$$

for, under this last condition, we have (comp. 125),

$$\text{II.} \ldots \frac{\zeta}{\gamma}\frac{\delta}{\varepsilon} = \frac{\zeta}{\alpha}\frac{\alpha}{\gamma} \cdot \frac{\delta}{\varepsilon} = \frac{\zeta}{\alpha} \cdot \frac{\beta}{\delta}\frac{\delta}{\varepsilon}.$$

(2.) Another enunciation of the associative principle is the following:

$$\text{III.} \ldots \text{if } \frac{\delta}{\gamma}\frac{\beta}{\alpha} = \frac{\zeta}{\varepsilon}, \quad \text{then} \quad \frac{\varepsilon}{\alpha}\frac{\beta}{\gamma} = \frac{\zeta}{\delta};$$

for if we determine (120) six new vectors, $\eta\theta\iota$, and $\kappa\lambda\mu$, so that

$$\text{IV.} \ldots \left\{ \begin{array}{l} \dfrac{\theta}{\eta} = \dfrac{\delta}{\gamma}, \quad \dfrac{\eta}{\iota} = \dfrac{\beta}{\alpha}, \quad \text{whence} \quad \dfrac{\theta}{\iota} = \dfrac{\zeta}{\varepsilon}, \\ \qquad\qquad \text{and} \\ \qquad \dfrac{\lambda}{\kappa} = \dfrac{\varepsilon}{\alpha}, \quad \dfrac{\kappa}{\mu} = \dfrac{\beta}{\gamma}, \end{array} \right.$$

we shall have the transformations,

$$\text{V.} \ldots \frac{\lambda}{\zeta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\theta} = \frac{\lambda}{\varepsilon} \cdot \frac{\iota}{\eta}\frac{\eta}{\theta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\eta} \cdot \frac{\eta}{\theta} = \frac{\kappa}{\beta}\frac{\gamma}{\delta} = \frac{\mu}{\delta}, \quad \text{or} \quad \text{VI.} \ldots \frac{\lambda}{\mu} = \frac{\zeta}{\delta}.$$

(3.) Conversely, the assertion that this last equation or proportion VI. is true, whenever the twelve vectors $\alpha \ldots \mu$ are connected by the five proportions IV., is a form of enunciation of the associative principle; for it conducts (comp. IV. and V.) to the equation,

$$\text{VII.} \ldots \frac{\lambda}{\varepsilon} \cdot \frac{\iota}{\eta}\frac{\eta}{\theta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\eta} \cdot \frac{\eta}{\theta}, \quad \text{at least if} \quad \varepsilon \parallel\!| \iota, \theta;$$

but, even with this last restriction, the three factor-quotients in VII. may represent *any three quaternions.*

SECTION 2.

On some Geometrical Proofs of the Associative Property of Multiplication of Quaternions, which are independent of the Distributive* Principle.

269. We propose, in this section, to furnish *three* geometrical Demonstrations of the Associative Principle, in connexion with the three figures (59–61) which were employed in the last section for its Enunciation; and with the *three arrangements* of *six planes*, which were described in Art. 263. The two first of these proofs will suppose the knowledge of a few properties of *spherical conics* (196, (11.)); but the third will only employ the doctrine of *stereographic projection*, and will therefore be of a more strictly *elementary character*. The Principle itself is, however, of such great importance in this Calculus, that its nature and its evidence can scarcely be put in too many different points of view.

270. The only properties of a spherical conic, which we shall in this Article assume as known,† are the three following: Ist, that *through any three given points* on a given sphere, which are not on a great circle, a *conic* can be described (consisting generally of *two opposite ovals*), which shall have a *given great circle* for *one* of its *two cyclic arcs*; IInd, that if a *transversal* arc cut *both* these arcs, and the conic, the *intercepts* (suitably measured) on this transversal are *equal*; and IIIrd, that if the *vertex* of a spherical angle *move along the conic*, while its *legs* pass always *through two fixed points* thereof, those legs intercept a *constant interval*, upon *each* cyclic arc, separately taken. Admitting these three properties, we see that if, in fig. 59, we conceive a spherical conic to be described, so as to pass through the three points B, F, H,

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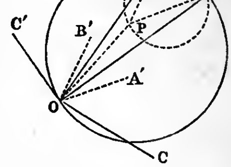
* Compare 224 and 262; and the Note to page 245.

† The reader may consult the Translation (Dublin, 1841, pp. 46, 50, 55) by the present Dean Graves, of two Memoirs by M. Chasles, on *Cones of the Second Degree, and Spherical Conics*. [If a cone have one system of cyclic sections parallel to APB, on inversion from O, the vertex of the cone, it is seen to have a second system parallel to the tangent plane at O to the sphere through the vertex and the circle APB. In the figure the circle OAB is the section of this sphere by the plane through two edges of the cone OA and OB, while OC (parallel to AB) and OC′ (tangent to OAB) are the traces of the cyclic planes. $\angle$C′OB = $\angle$BAO = $\angle$AOC proves the IInd property. Again, if OA′ and OB′ are the traces on the cyclic plane parallel to APB by the planes POA and POB respectively, $\angle$APB = $\angle$A′OB′, OA′ being parallel to PA and OB′ to PB. But as P moves along the circle APB, the angle APB is constant, and thus the IIIrd property is also proved.]



and to have the great circle DAEC for *one* cyclic arc, the second and third equations I. of 264 will prove that the arc GLIM is the *other* cyclic arc for this conic; the first equation I. proves next that the conic passes through K; and if the arcual chord FK be drawn and prolonged, the two remaining equations prove that it meets the cyclic arcs in D and M; after which, the equation II. of the same Art. 264 immediately results, at least with the arrangement* adopted in the figure.

(1.) The Ist property is easily seen to correspond to the possibility of circumscribing a circle about a given plane triangle, namely that of which the corners are the intersections of a plane parallel to the plane of the given cyclic arc, with the three radii drawn to the three given points upon the sphere: but it may be worth while, as an exercise, to prove here the IInd property *by quaternions.*

(2.) Take then the equation of a *cyclic cone*, 196, (8.), which may (by 196, XII.) be written thus:

$$\text{I.} \ldots \mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\rho}{\beta} = \mathrm{N}\frac{\rho}{\beta}; \quad \text{and let} \quad \text{II.} \ldots \mathrm{S}\frac{\rho'}{\alpha}\,\mathrm{S}\frac{\rho'}{\beta} = \mathrm{N}\frac{\rho'}{\beta},$$

ρ and ρ' being thus *two rays* (or *sides*) of the cone, which may also be considered to be the vectors of two points P and P′ of a *spherical conic*, by supposing that their lengths are each unity. Let τ and τ' be the vectors of the two points T and T′ on the two cyclic arcs, in which the arcual chord PP′ of the conic cuts them; so that

$$\text{III.} \ldots \mathrm{S}\frac{\tau}{\alpha} = 0, \quad \mathrm{S}\frac{\tau'}{\beta} = 0, \quad \text{and} \quad \text{IV.} \ldots \mathrm{T}\tau = \mathrm{T}\tau' = 1.$$

The theorem may then be stated thus: that

$$\text{V.} \ldots \text{if } \rho = x\tau + x'\tau', \quad \text{then} \quad \text{VI.} \ldots \rho' = x'\tau + x\tau';$$

or that this expression VI. satisfies II., if the equations I. III. IV. V. be satisfied.

Now, by III. V. VI., we have

$$\text{VII.} \ldots \mathrm{S}\frac{\rho}{\alpha} = x'\mathrm{S}\frac{\tau'}{\alpha} = \frac{x'}{x}\,\mathrm{S}\frac{\rho'}{\alpha}, \quad \mathrm{S}\frac{\rho}{\beta} = x\mathrm{S}\frac{\tau}{\beta} = \frac{x}{x'}\,\mathrm{S}\frac{\rho'}{\beta};$$

whence it follows that the first members of I. and II. are equal, and it only

* Modifications of that arrangement may be conceived, to which however it would be easy to adapt the reasoning.

remains to prove that their second members are equal also, or that $T\rho' = T\rho$, if $T\tau' = T\tau$.

Accordingly we have, by V. and VI.,

$$\text{VIII.} \ldots \frac{\rho' - \rho}{\rho' + \rho} = \frac{x' - x}{x' + x} \cdot \frac{\tau - \tau'}{\tau + \tau'} = S^{-1}0, \text{ by 200, (11.), and 204, (19.);}$$

and the property in question is proved.

271. To prove the associative principle, with the help of fig. 60, three other properties of a spherical conic shall be supposed known:* Ist, that for every such curve *two focal points* exist, possessing several important relations to it, one of which is, that if these *two foci* and *one tangent arc* be *given*, the conic can be constructed; IInd, that if, from any point upon the sphere, *two tangents* be drawn to the conic, and also *two arcs to the foci*, then *one focal arc* makes *with one tangent* the *same angle* as the *other* focal arc with the *other* tangent; and IIIrd, that if a spherical quadrilateral be circumscribed to such a conic (supposed here for simplicity to be a spherical *ellipse*, or the *opposite* ellipse being neglected), *opposite sides subtend supplementary angles*, at *either* of the two (interior) foci. Admitting these known properties, and supposing the arrangement to be as in fig. 60, we may conceive a conic described, which shall have E and F for its two focal points, and shall touch the arc BC; and then the two first of the equations I., in 265, will prove that it touches also the arcs AB and CD, while the third of those equations proves that it touches AD, so that ABCD is a circumscribed† quadrilateral: after which the three equations II., of the same article, are consequences of the same properties of the curve.‡

272. Finally, to prove the same important Principle in a more completely elementary way, by means of the arrangement represented in fig. 61, or to prove the theorem of spherical geometry enunciated in Art. 267, we

* The reader may again consult pages 46 and 50 of the Translation lately cited. In strictness, there are of course *four foci*, opposite two by two.

† The writer has elsewhere proposed the *notation*, EF (. .) ABCD, to denote the relation of the focal points E, F to this circumscribed quadrilateral.

‡ [The two cyclic arcs and a point determine a spherical conic. Referring to the Note on 270, describe a sphere to touch one cyclic plane at the point O. Then if OA is given, take the section APB of the sphere by a plane parallel to the second cyclic plane, and the cone is determined. Reciprocating this, the Ist property follows. The IInd property is the reciprocal of the IInd of 270, and the IIIrd is easily derived by reciprocating the IIIrd of 270, remembering that for a point P′ on the remaining arc of the circle APB, $\angle AP'B + \angle APB = \pi$.]

may assume the point D as the *pole* of a *stereographic projection*, in which the three small circles through that point shall be represented by *right lines*, but the three others by *circles*, all being in one *common plane.** And then (interchanging accents) the theorem comes to be thus stated :

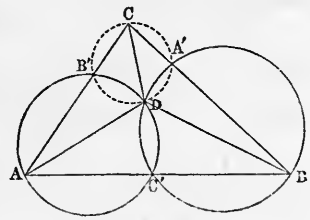


Fig. 62.

If A', B', C' *be any three points* (comp. fig. 62) *on the sides* BC, CA, AB *of any plane triangle*, or on those sides prolonged, *then*, Ist, *the three circles*,

$$\text{I.} \ldots C'AB', \; A'BC', \; B'CA',$$

will *meet in one point* D ; and IInd, *an even number* (if any) *of the six* (linear or circular) *successions*,

$$\text{II.} \ldots AB'C, \; BC'A, \; CA'B, \quad \text{and} \quad \text{II}'. \ldots C'AB'D, \; A'BC'D, \; B'CA'D,$$

will be *direct*; an *even* number therefore *also* (if any) being *indirect*. But, under this *form*,† the theorem can be proved by very elementary considerations, and still without any employment of the *distributive principle* (224, 262).

(1.) The *first part* of the theorem, as thus stated, is evident from the Third Book of Euclid ; but to prove *both parts* together, it may be useful to proceed as follows, admitting the conception (235) of *amplitudes*, or of angles as representing *rotations*, which may have *any values*, positive or negative, and are to be *added* with attention to their *signs*.

(2.) We may thus write the three *equations*,

$$\text{III.} \ldots AB'C = n\pi, \quad BC'A = n'\pi, \quad CA'B = n''\pi,$$

to express the three *collineations*, $AB'C$, &c. of fig. 62 ;‡ the *integer*, n, being *odd* or *even*, according as the point B' is on the finite line AC, or on a prolongation of that line ; or in other words, according as the first *succession* II. is *direct* or *indirect* : and similarly for the two other coefficients, n' and n''.

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* [Invert figure 61 from the point D. The sphere becomes a plane, and the circles through D right lines, the other circles remain circles.]

† The Associative Principle of Multiplication was stated nearly under this *form*, and was illustrated by the same simple *diagram*, in paragraph XXII. of a communication by the present author, which was entitled *Letters on Quaternions*, and has been printed in the First and Second Editions of the late Dr. Nichol's *Cyclopædia of the Physical Sciences* (London and Glasgow, 1857 and 1860). The same communication contained other illustrations and consequences of the same principle, which it has not been thought necessary here to reproduce ; and others may be found in the Sixth of the author's already cited *Lectures on Quaternions* (Dublin, 1853), from which (as already observed) some of the formulæ and figures of this Chapter have been taken.

‡ [$AB'C$ being the angle through which $B'A$ must be turned in the positive direction so as to coincide with $B'C$.]

(3.) Again, if OPQR be *any four points* in one plane, we may establish the formula,

$$\text{IV.} \ldots \text{POQ} + \text{QOR} = \text{POR} + 2m\pi,$$

with the same conception of addition of amplitudes; if then D be *any point in the plane* of the triangle ABC, we may write,

$$\text{V.} \ldots \text{AB}'\text{D} + \text{DB}'\text{C} = n\pi, \quad \text{BC}'\text{D} + \text{DC}'\text{A} = n'\pi, \quad \text{CA}'\text{D} + \text{DA}'\text{B} = n''\pi;$$

and therefore,

$$\text{VI.} \ldots (\text{AB}'\text{D} + \text{DC}'\text{A}) + (\text{BC}'\text{D} + \text{DA}'\text{B}) + (\text{CA}'\text{D} + \text{DB}'\text{C}) = (n + n' + n'')\,\pi.$$

(4.) Again, if any four points OPQR be not merely *complanar* but *concircular*, we have the general formula,

$$\text{VII.} \ldots \text{OPQ} + \text{QRO} = p\pi,$$

the integer p being *odd* or *even*, according as the succession OPQR is *direct* or *indirect*; if then we denote by D the *second intersection* of the first and second circles I., whereof C′ is a *first* intersection, we shall have

$$\text{VIII.} \ldots \text{AB}'\text{D} + \text{DC}'\text{A} = p\pi, \quad \text{BC}'\text{D} + \text{DA}'\text{B} = p'\pi,$$

p and p' being *odd*, when the two first successions II′. are direct, but *even* in the contrary case.

(5.) Hence, by VI., we have,

$$\text{IX.} \ldots \text{CA}'\text{D} + \text{DB}'\text{C} = p''\pi, \quad \text{where} \quad \text{X.} \ldots p + p' + p'' = n + n' + n'';$$

the *third succession* II′. is therefore *always circular*, or the *third circle* I. passes *through the intersection* D of the *two first*; and it is *direct* or *indirect*, that is to say, p'' is *odd* or *even*, according as the *number of even coefficients*, among the *five* previously considered, is itself *even* or *odd*; or in other words, according as the *number of indirect successions*, among the *five* previously considered, is *even* (including *zero*), or *odd*.

(6.) In every case, therefore, the *total number* of successions of *each kind* is *even*, and *both* parts of the theorem are proved: the importance of the *second part* of it (respecting the *even partition*, if any, of the *six successions* II. II′.) arising from the necessity of proving that we have *always*, as in algebra,

$$\text{XI.} \ldots sr \,.\, q = + s \,.\, rq, \quad \text{and } \textit{never} \quad \text{XII.} \ldots sr \,.\, q = - s \,.\, rq,$$

if q, r, s be *any three actual quaternions*.

(7.) The *associative principle* of multiplication may also be proved, without the *distributive* principle, by certain considerations of *rotations of a system*, on which we cannot enter here.

SECTION 3.

On some Additional Formulæ.

273. Before concluding the Second Book, a few additional remarks may be made, as regards some of the notations and transformations which have already occurred, or others analogous to them. And first as to *notation*, although we have reserved for the Third Book the *interpretation* of such expressions as $\beta\alpha$, or α^2, yet we have agreed, in 210, (9.), to *abridge* the frequently occurring *symbol* $(\mathrm{T}\alpha)^2$ to $\mathrm{T}\alpha^2$; and we *now* propose to abridge it still further to $\mathrm{N}\alpha$, and to *call* this *square of the tensor* (or of the *length*) *of a vector*, α, *the Norm of that Vector*: as we had (in 190, &c.), the equation $\mathrm{T}q^2 = \mathrm{N}q$, and called $\mathrm{N}q$ the *norm* of the *quaternion* q (in 145, (11.)). We shall therefore now write generally, *for any vector* α, the formula,

$$\mathrm{I}\ldots(\mathrm{T}\alpha)^2 = \mathrm{T}\alpha^2 = \mathrm{N}\alpha.$$

(1.) The equations (comp. 186, (1.) (2.) (3.) (4.)),

$$\mathrm{II}\ldots\mathrm{N}\rho = 1;\qquad \mathrm{III}\ldots\mathrm{N}\rho = \mathrm{N}\alpha;\qquad \mathrm{IV}\ldots\mathrm{N}(\rho-\alpha) = \mathrm{N}\alpha;$$

$$\mathrm{V}\ldots\mathrm{N}(\rho-\alpha) = \mathrm{N}(\beta-\alpha),$$

represent, respectively, the *unit-sphere*; the sphere through A, with O for centre; the sphere through O, with A for centre; and the sphere through B, with the same centre A.

(2.) The equations (comp. 186, (6.) (7.)),

$$\mathrm{VI}\ldots\mathrm{N}(\rho+\alpha) = \mathrm{N}(\rho-\alpha);\qquad \mathrm{VII}\ldots\mathrm{N}(\rho-\beta) = \mathrm{N}(\rho-\alpha),$$

represent, respectively, the *plane* through O, perpendicular to the line OA; and the plane which *perpendicularly bisects* the line AB.

274. As regards *transformations*, the few following may here be added, which relate partly to the *quaternion forms* (204, 216, &c.) of the *Equation of the Ellipsoid.*

(1.) Changing $\mathrm{K}(\kappa:\rho)$ to $\mathrm{R}\rho:\mathrm{R}\kappa$, by 259, VIII., in the equation 217, XVI., of the ellipsoid, and observing that the three vectors ρ, $\mathrm{R}\rho$, and $\mathrm{R}\kappa$ are complanar, while $1:\mathrm{T}\rho = \mathrm{TR}\rho$ by 258, that equation becomes, when divided by $\mathrm{TR}\rho$, and when the value 217, (5.) for t^2 is taken, and the notation 273 is employed:

$$\text{I.} \ldots \quad \left(\frac{\iota}{\mathrm{R}\rho} + \frac{\rho}{\mathrm{R}\kappa}\right) = \mathrm{N}\iota - \mathrm{N}\kappa;$$

of which the first member will soon be seen to admit of being written* as $\mathrm{T}(\iota\rho + \rho\kappa)$, and the second member as $\kappa^2 - \iota^2$.

(2.) If, in connexion with the earlier forms (204, 216) of the equation of the same surface, we introduce a *new auxiliary vector*, σ or OS, such that (comp. 216, VIII.)

$$\text{II.} \ldots \sigma = \left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right)\beta = \rho + 2\beta\mathrm{S}\frac{\rho}{\delta},$$

the equation may, by 204, (14.), be reduced to the following extremely simple form:

$$\text{III.} \ldots \mathrm{T}\sigma = \mathrm{T}\beta;$$

which expresses that the *locus* of the *new auxiliary point* S is what we have called the *mean sphere*, 216, XIV.; while the *line* PS, or $\sigma - \rho$, which *connects* any two *corresponding points*, P and S, on the ellipsoid and sphere, is seen to be *parallel to the fixed line* β; which is *one element of the homology*, mentioned in 216, (10.).

(3.) It is easy to prove that

$$\text{IV.} \ldots \mathrm{S}\frac{\sigma}{\delta} = \mathrm{S}\frac{\beta}{\alpha}\,\mathrm{S}\frac{\rho}{\delta}, \quad \text{and therefore} \quad \text{V.} \ldots \mathrm{S}\frac{\sigma'}{\delta} : \mathrm{S}\frac{\sigma}{\delta} = \mathrm{S}\frac{\rho'}{\delta} : \mathrm{S}\frac{\rho}{\delta},$$

if ρ' and σ' be the vectors of two new but corresponding points, P′ and S′, on the ellipsoid and sphere; whence it is easy to infer this *other element of the homology*, that any two *corresponding chords*, PP′ and SS′, of the two surfaces, *intersect each other on the cyclic plane* which has δ for its *cyclic normal* (comp. 216, (7.)): in fact, they intersect in the point T of which the vector is,

$$\text{VI.} \ldots \tau = \frac{x\rho + x'\rho'}{x + x'} = \frac{x\sigma + x'\sigma'}{x + x'}, \quad \text{if} \quad x = \mathrm{S}\frac{\rho'}{\delta}, \quad \text{and} \quad x' = -\mathrm{S}\frac{\rho}{\delta};$$

and this point is on the plane just mentioned (comp. 216, XI.), because

$$\text{VII.} \ldots \mathrm{S}\frac{\tau}{\delta} = 0.$$

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* Compare the Note to page 241.

(4.) Quite similar results would have followed, if we had assumed

$$\text{VIII.} \ldots \sigma = \left(-\,\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right)\beta = \rho - 2\beta\mathrm{S}\frac{\rho}{\gamma},$$

which would have given again, as in III.,

$$\text{IX.} \ldots \mathrm{T}\sigma = \mathrm{T}\beta, \quad \text{but with} \quad \text{X.} \ldots \mathrm{S}\frac{\sigma}{\gamma} = -\mathrm{S}\frac{\beta}{\alpha}\mathrm{S}\frac{\rho}{\gamma};$$

the *other cyclic plane*, with γ instead of δ for its *normal*, might therefore have been taken (as asserted in 216, (10.)), as *another plane of homology* of ellipsoid and sphere, with the *same centre of homology* as before: namely, the *point at infinity on the line* β, or on the *axis* (204, (15.)) of one of the two *circumscribed cylinders of revolution* (comp. 220, (4.)).

(5.) The same ellipsoid is, *in two other ways*, homologous to the same mean sphere, with the same two cyclic planes as *planes* of homology, but with a *new centre* of homology, which is the infinitely distant point on the axis of the *second circumscribed cylinder* (or on the line AB′ of the sub-article last cited).

(6.) Although not specially connected with the *ellipsoid*, the following general transformations may be noted here (comp. 199, XII., and 204, XXXIV′.):

$$\text{XI.} \ldots \mathrm{TV}\sqrt{q} = \sqrt{\{\tfrac{1}{2}(\mathrm{T}q - \mathrm{S}q)\}}; \quad \text{XII.} \ldots \tan\tfrac{1}{2}\angle q = (\mathrm{TV}:\mathrm{S})\sqrt{q} = \sqrt{\frac{\mathrm{T}q - \mathrm{S}q}{\mathrm{T}q + \mathrm{S}q}}.$$

(7.) The equations 204, XVI. and XXXV., give easily,

$$\text{XIII.} \ldots \mathrm{UV}q = \mathrm{UVU}q; \quad \text{XIV.} \ldots \mathrm{UIV}q = \mathrm{Ax}\,.\,q; \quad \text{XV.} \ldots \mathrm{TIV}q = \mathrm{TV}q;$$

or the more symbolical forms,

$$\text{XIII}'. \ldots \mathrm{UVU} = \mathrm{UV}; \quad \text{XIV}'. \ldots \mathrm{UIV} = \mathrm{Ax.}; \quad \text{XV}'. \ldots \mathrm{TIV} = \mathrm{TV};$$

and the identity 200, IX. becomes more evident, when we observe that

$$\text{XVI.} \ldots q - \mathrm{N}q = q(1 - \mathrm{K}q).$$

(8.) We have also generally (comp. 200, (10.) and 218, (10.)),

$$\text{XVII.} \ldots \frac{q-1}{q+1} = \frac{(q-1)(\mathrm{K}q+1)}{(q+1)(\mathrm{K}q+1)} = \frac{\mathrm{N}q - 1 + 2\mathrm{V}q}{\mathrm{N}q + 1 + 2\mathrm{S}q}.$$

(9.) The formula,*

$$\text{XVIII.} \ldots \mathrm{U}(rq + \mathrm{K}qr) = \mathrm{U}(\mathrm{S}r\,.\,\mathrm{S}q + \mathrm{V}r\,.\,\mathrm{V}q) = r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1},$$

in which q and r may be *any two quaternions*, is not perhaps of any great

* This formula was given, but in like manner without proof, in page 587 of the author's Lectures on Quaternions. [It may be expressed in terms of $p = (r^2q^2)^{\frac{1}{2}}$. Use 210, XI. and XII.]

importance in itself, but will be found to furnish a student with several useful exercises in transformation.

(10.) When it was said, in 257, (1.), that *zero* had *only itself* for a *square-root*, the meaning was (comp. 225), that *no binomial expression of the form* $x + iy$ (228) *could satisfy the equation*,

$$\text{XIX.} \ldots 0 = q^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy,$$

for any real or imaginary values of the two scalar coefficients x and y, different from zero;* for if *bi-quaternions* (214, (8.)) be admitted, and if h again denote, as in 256, (2.), the *imaginary of algebra*, then (comp. 257, (6.) and (7.)) we may write, generally, *besides the real value*, $0^{\frac{1}{2}} = 0$, the *imaginary expression*,

$$\text{XX.} \ldots 0^{\frac{1}{2}} = v + hv', \quad \text{if} \quad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = \mathrm{N}v' - \mathrm{N}v = 0;$$

v and v' being thus *any two real right quaternions, with equal norms* (or with *equal tensors*), in planes perpendicular to each other.

(11.) For example, by 256, (2.) and by the laws (183) of ijk, we have the transformations,

$$\text{XXI.} \ldots (i + hj)^2 = i^2 - j^2 + h\,(ij + ji) = 0 + h0 = 0;$$

so that the biquaternion $i + hj$ is one of the imaginary values of the symbol $0^{\frac{1}{2}}$.

(12.) In general, when *bi-quaternions* are admitted into calculation, not only the *square* of *one*, but the *product* of *two* such factors may *vanish*, without *either* of them *separately* vanishing: a circumstance which may throw some light on the existence of those *imaginary* (or *symbolical*) *roots of equations*, which were treated of in 257.

(13.) For example, although the equation

$$\text{XXII.} \ldots q^2 - 1 = (q - 1)\,(q + 1) = 0.$$

has *no real roots except* ± 1, and therefore cannot be verified by the *substitution* of *any other real scalar*, or *real quaternion*, for q, yet if we substitute for q the *bi-quaternion*† $v + hv'$, with the conditions 257, XIII., this equation XXII. is verified.

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* Compare the Note to page 289.

† This *includes* the expression $\pm\, hi$, of 257, (1.), for a *symbolical square-root* of *positive unity*. Other such roots are $\pm\, hj$, and $\pm\, hk$. [It is probable that Hamilton used the word Bi-quaternion in order to distinguish clearly the $\sqrt{-1}$ of algebra from the geometrical reals i, j, and k of the new Calculus. In his earlier writings i, j, and k are called imaginaries; and in a Paper read before the Royal Irish Academy on November 11, 1844, the scalar of a quaternion is called the "real part," and the vector, the "imaginary part." See p. 3, vol. iii., of the Proc. R.I.A.]

(14.) It will be found, however, that when *two imaginary* but *non-evanescent factors* give thus a *null product, the norm of each is zero*; provided that we agree to *extend to bi-quaternions* the formula $Nq = Sq^2 - Vq^2$ (204, XXII.); or to *define* that the *Norm of a Biquaternion* (like that of an ordinary or real quaternion) is equal to the *Square of the Scalar Part, minus the Square of the Right Part*: *each* of these two *parts* being generally *imaginary*, and the *former* being what we have called a *Bi-scalar*.

(15.) With this definition, if q and q' be any two *real quaternions*, and if h be, as above, the ordinary imaginary of *algebra*, we may establish the formula:

$$\text{XXIII.}\ldots \ N(q+hq') = (Sq + hSq')^2 - (Vq + hVq')^2;$$

or (comp. 200, VII., and 210, XX.),

$$\text{XXIV.}\ldots \ N(q+hq') = Nq - Nq' + 2hS\,.\,qKq'.$$

(16.) As regards the *norm of the sum* of any two *real* quaternions, or real vectors (273), the following transformations are occasionally useful (comp. 220, (2.)):

$$\text{XXV.}\ldots \ N(q'+q) = N(Tq'\,.\,Uq + Tq\,.\,Uq');$$

$$\text{XXVI.}\ldots \ N(\beta+\alpha) = N(T\beta\,.\,U\alpha + T\alpha\,.\,U\beta);$$

in each of which it is permitted to change the *norms* to the *tensors* of which they are the *squares*, or to write T for N.

BOOK III.

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS; AND ON SOME APPLICATIONS OF QUATERNIONS.

CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS, OR POWER OF A VECTOR, AS A QUATERNION.

SECTION 1.

On a First Method of interpreting a Product of Two Vectors as a Quaternion.

ART. 275. In the First Book of these *Elements* we interpreted, Ist, the *difference* of any two directed right lines in space (4.) ; IInd, the *sum* of two or more such lines (5–9) ; IIIrd, the *product* of one such line, multiplied by or into a positive or negative *number* (15) ; IVth, the *quotient* of such a line, divided by such a number (16), or by what we have called generally a SCALAR (17) ; and Vth, the sum of a *system* of such lines, each affected (97) with a scalar *coefficient* (99), as being in each case *itself* (generally) a *Directed Line* in Space*, or what we have called a VECTOR (1).

276. In the Second Book, the fundamental principle or pervading conception has been, that the *Quotient of two such Vectors* is, generally, a QUATERNION (112, 116). It is however to be remembered, that we have *included* under this general conception, which *usually* relates to what may be called an *Oblique Quotient*, or the quotient of two lines in space making either an *acute* or an *obtuse angle* with each other (130), the *three* following particular *cases* : Ist, the *limiting case*, when the angle becomes *null*, or when the two lines are *similarly directed*, in which case the quotient *degenerates* (131) into a *positive scalar* ; IInd, the *other* limiting case, when the angle is equal to *two right angles*, or when the lines are *oppositely directed*, and when in consequence the quotient *again* degenerates, but now into a *negative* scalar ; and IIIrd, the *intermediate* case, when the angle is *right*, or when the two lines are *perpendicular* (132), instead of being *parallel* (15), and when therefore their quotient

* The *Fourth Proportional* to any *three complanar lines* has also been since interpreted (226), as being *another line* in the *same plane*.

becomes what we have called (132) a *Right Quotient*, or a RIGHT QUATERNION: which has been seen to be a case not less important than the two former ones.

277. But *no Interpretation* has been assigned, in either of the two foregoing Books, for a PRODUCT *of two or more Vectors*; or for the SQUARE, or other POWER *of a Vector*: so that the *Symbols*,

$$\text{I.} \ldots \beta\alpha,\ \gamma\beta\alpha, \ldots \quad \text{and} \quad \text{II.} \ldots \alpha^2,\ \alpha^3, \ldots \alpha^{-1}, \ldots \alpha^t,$$

in which α, β, γ .. denote *vectors*, but t denotes a *scalar*, remain as yet entirely *uninterpreted*; and we are therefore *free* to assign, at this stage, *any meanings* to these *new symbols*, or *new combinations* of symbols, which shall *not contradict each other*, and shall appear to be consistent with *convenience* and *analogy*. And to do so will be the chief object of this First Chapter of the Third (and last) Book of these *Elements*: which is designed to be a much shorter one than either of the foregoing.

278. As a commencement of such *Interpretation* we shall here *define*, that *a vector* α *is multiplied by another vector* β, or that the *latter* vector is multiplied *into** the *former*, or that the *product* $\beta\alpha$ is obtained, *when the multiplier-line* β *is divided by the reciprocal* $\mathrm{R}\alpha$ (258) *of the multiplicand-line* α; as we had *proved* (136) that one *quaternion* is *multiplied into another*, when it is *divided by* the *reciprocal* thereof. In symbols, we shall therefore write, *as a first definition*, the formula:

$$\text{I.} \ldots \beta\alpha = \beta : \mathrm{R}\alpha; \quad \text{where} \quad \text{II.} \ldots \mathrm{R}\alpha = -\mathrm{U}\alpha : \mathrm{T}\alpha \ (258, \text{VII.}).$$

And we proceed to consider, in the following section, some of the general *consequences* of this definition, or interpretation, of a *Product of two Vectors*, as being *equal* to a certain *Quotient*, or *Quaternion*.

SECTION 2.

On some Consequences of the foregoing Interpretation.

279. The definition (278) gives the formula:

$$\text{I.} \ldots \beta\alpha = \frac{\beta}{\mathrm{R}\alpha}; \quad \text{and similarly,} \quad \text{I}' \ldots \alpha\beta = \frac{\alpha}{\mathrm{R}\beta};$$

it gives therefore, by 259, VIII., the general relation,

$$\text{II.} \ldots \beta\alpha = \mathrm{K}\alpha\beta; \quad \text{or} \quad \text{II}' \ldots \alpha\beta = \mathrm{K}\beta\alpha.$$

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* Compare the Notes to pages 147, 159.

The *Products of two Vectors*, taken in two *opposite orders*, are therefore *Conjugate Quaternions*; and the *Multiplication of Vectors*, like that of Quaternions (168), is (generally) a *Non-Commutative Operation.*

(1.) It follows from II. (by 196, comp. 223, (1.)), that

$$\text{III.} \ldots \mathrm{S}\beta\alpha = + \mathrm{S}\alpha\beta = \tfrac{1}{2}(\beta\alpha + \alpha\beta).$$

(2.) It follows also (by 204, comp. again 223, (1.)), that

$$\text{IV.} \ldots \mathrm{V}\beta\alpha = - \mathrm{V}\alpha\beta = \tfrac{1}{2}(\beta\alpha - \alpha\beta).$$

280. Again, by the same general formula 259, VIII., we have the transformations,

$$\text{I.} \ldots \frac{\beta}{\mathrm{R}(\alpha + \alpha')} = \mathrm{K}\frac{\alpha + \alpha'}{\mathrm{R}\beta} = \mathrm{K}\frac{\alpha}{\mathrm{R}\beta} + \mathrm{K}\frac{\alpha'}{\mathrm{R}\beta} = \frac{\beta}{\mathrm{R}\alpha} + \frac{\beta}{\mathrm{R}\alpha'};$$

it follows, then, from the definition (278), that

$$\text{II.} \ldots \beta(\alpha + \alpha') = \beta\alpha + \beta\alpha';$$

whence also, by taking conjugates (279), we have this other general equation,

$$\text{III.} \ldots (\alpha + \alpha')\beta = \alpha\beta + \alpha'\beta.$$

Multiplication of Vectors is, therefore, like that of *Quaternions* (212), a *Doubly Distributive Operation.*

281. As we have not yet assigned any signification for a *ternary product of vectors*, such as $\gamma\beta\alpha$, we are not yet prepared to pronounce, whether the *Associative Principle* (223) of *Multiplication of Quaternions* does or does not extend to *Vector-Multiplication.* But we can already derive several other consequences from the definition (278) of a *binary product*, $\beta\alpha$; among which, attention may be called to the *Scalar character* of a *Product of two Parallel Vectors*; and to the *Right character* of a *Product of two Perpendicular Vectors*, or of two lines at right angles with each other.

(1.) The definition (278) may be thus written,

$$\text{I.} \ldots \beta\alpha = - \mathrm{T}\beta \,.\, \mathrm{T}\alpha \,.\, \mathrm{U}(\beta : \alpha);$$

it gives, therefore,

$$\text{II.} \ldots \mathrm{T}\beta\alpha = \mathrm{T}\beta \,.\, \mathrm{T}\alpha; \quad \text{III.} \ldots \mathrm{U}\beta\alpha = - \mathrm{U}(\beta : \alpha) = \mathrm{U}\beta \,.\, \mathrm{U}\alpha;$$

the *tensor* and *versor* of the *product* of two *vectors* being thus *equal* (as for quaternions, 191) to the *product of the tensors*, and to the *product of the versors*, respectively.

(2.) Writing for abridgment (comp. 208),

$$\text{IV.}\dots a = \mathrm{T}\alpha, \quad b = \mathrm{T}\beta, \quad \gamma = \mathrm{Ax}.(\beta : \alpha), \quad x = \angle(\beta : \alpha),$$

we have thus,

$$\text{V.}\dots \mathrm{T}\beta\alpha = ba; \qquad \text{VI.}\dots \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = -ba\cos x;$$

$$\text{VII.}\dots \mathrm{SU}\beta\alpha = \mathrm{SU}\alpha\beta = -\cos x; \qquad \text{VIII.}\dots \angle\beta\alpha = \pi - x;$$

so that (comp. 198) the *angle of the product* of any two vectors is the *supplement of the angle of the quotient.*

(3.) We have next the transformations (comp. again 208),

$$\text{IX.}\dots \mathrm{TV}\beta\alpha = \mathrm{TV}\alpha\beta = ba\sin x; \qquad \text{X.}\dots \mathrm{TVU}\beta\alpha = \mathrm{TVU}\alpha\beta = \sin x;$$

$$\text{XI.}\dots \mathrm{IV}\beta\alpha = -\gamma ba\sin x; \qquad \text{XI}'\dots \mathrm{IV}\alpha\beta = +\gamma ab\sin x;$$

$$\text{XII.}\dots \mathrm{IUV}\beta\alpha = \mathrm{Ax}.\beta\alpha = -\gamma; \quad \text{XII}'\dots \mathrm{IUV}\alpha\beta = \mathrm{Ax}.\alpha\beta = +\gamma;$$

so that *the rotation round the axis of a product of two vectors, from the multiplier to the multiplicand, is positive.*

(4.) It follows also, by IX., that the *tensor of the right part* of such a *product*, $\beta\alpha$, is equal to the *parallelogram under the factors*; or to the *double of the area of the triangle* OAB, whereof those two factors α, β, or OA, OB, are two coinitial *sides*: so that if we denote here this last-mentioned *area* by the symbol

$$\Delta\,\mathrm{OAB},$$

we may write the equation,

$$\text{XIII.}\dots \mathrm{TV}\beta\alpha = \textit{parallelogram under } \alpha, \beta, = 2\Delta\,\mathrm{OAB};$$

and the *index*, $\mathrm{IV}\beta\alpha$, is a *right line perpendicular to the plane* of this *parallelogram*, of which line the *length represents* its *area*, in the sense that they bear *equal ratios* to their respective *units* (of length and of area).

(5.) Hence, by 279, IV.,

$$\text{XIV.}\dots \mathrm{T}(\beta\alpha - \alpha\beta) = 2 \times \text{parallelogram} = 4\Delta\,\mathrm{OAB}.$$

(6.) For any two vectors, α, β,

$$\text{XV.}\dots \mathrm{S}\beta\alpha = -\mathrm{N}\alpha.\mathrm{S}(\beta : \alpha); \qquad \text{XVI.}\dots \mathrm{V}\beta\alpha = -\mathrm{N}\alpha.\mathrm{V}(\beta : \alpha);$$

or briefly,*

$$\text{XVII.}\dots \beta\alpha = -\mathrm{N}\alpha.(\beta : \alpha),$$

with the signification (273) of $\mathrm{N}\alpha$, as denoting $(\mathrm{T}\alpha)^2$.

* *All* the consequences of the interpretation (278), of the product $\beta\alpha$ of two vectors, might be deduced from this formula XVII.; which, however, it would not have been so natural to have assumed for a *definition* of that symbol, as it was to assume the formula 278, I.

(7.) If the two factor-lines be *perpendicular* to each other, so that x is a *right angle*, then the *parallelogram* (4.) becomes a *rectangle*, and the *product* $\beta\alpha$ becomes a *right quaternion* (132); so that we may write,

$$\text{XVIII.} \ldots \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = 0, \quad \text{if} \quad \beta \perp \alpha, \text{ and reciprocally.}$$

(8.) Under the same condition of perpendicularity,

$$\text{XIX.} \ldots \angle\beta\alpha = \angle\alpha\beta = \frac{\pi}{2}; \quad \text{XX.} \ldots \mathrm{I}\beta\alpha = -\gamma ba; \quad \text{XXI.} \ldots \mathrm{I}\alpha\beta = +\gamma ab.$$

(9.) On the other hand, if the two factor-lines be *parallel*, the *right part* of their product vanishes, or that product reduces itself to a *scalar*, which is *negative* or *positive* according as the two vectors multiplied have *similar* or *opposite directions*; for we may establish the formula,

$$\text{XXII.} \ldots \text{if } \beta \parallel \alpha, \quad \text{then} \quad \mathrm{V}\beta\alpha = 0, \quad \mathrm{V}\alpha\beta = 0;$$

and, under the same condition of *parallelism*,

$$\text{XXIII.} \ldots \beta\alpha = \alpha\beta = \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = \mp ba,$$

the *upper* or the *lower sign* being taken, according as $x = 0$, or $= \pi$.

(10.) We may also write (by 279, (1.) and (2.)) the following *formula of perpendicularity* and *formula of parallelism*:

$$\text{XXIV.} \ldots \text{if } \beta \perp \alpha, \quad \text{then} \quad \beta\alpha = -\alpha\beta, \text{ and reciprocally;}$$

$$\text{XXV.} \ldots \text{if } \beta \parallel \alpha, \quad \text{then} \quad \beta\alpha = +\alpha\beta, \text{ with the converse.}$$

(11.) If α, β, γ be *any three unit-lines*, considered as vectors of the corners A, B, C of a *spherical triangle*, with *sides* equal to three new positive scalars, a, b, c, then because, by XVII., $\beta\alpha = -\beta : \alpha$, and $\gamma\beta = -\gamma : \beta$, the sub-articles to 208 allow us to write,

$$\text{XXVI.} \ldots \mathrm{S}(\mathrm{V}\gamma\beta \,.\, \mathrm{V}\beta\alpha) = \sin a \sin c \cos \mathrm{B};$$

$$\text{XXVII.} \ldots \mathrm{IV}(\mathrm{V}\gamma\beta \,.\, \mathrm{V}\beta\alpha) = \pm \beta \sin a \sin c \sin \mathrm{B};$$

$$\text{XXVIII.} \ldots (\mathrm{IV} : \mathrm{S})(\mathrm{V}\gamma\beta \,.\, \mathrm{V}\beta\alpha) = \pm \beta \tan \mathrm{B};$$

upper or lower *signs* being taken, in the two last formulæ, according as the *rotation* round β from α to γ, or that round B from A to C, is positive or negative.

(12.) The equation 274, I., of the *Ellipsoid*, may now be written thus:

$$\text{XXIX.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2; \quad \text{or} \quad \text{XXX.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \mathrm{N}\iota - \mathrm{N}\kappa.$$

282. Under the general head of a *product* of two *parallel vectors*, two interesting *cases* occur, which furnish two first examples of *Powers of Vectors*: namely, Ist, the case when the two factors are *equal*, which gives this remarkable result, that *the Square of a Vector is always equal to a negative Scalar*; and IInd, the case when the factors are (in the sense already defined, 258) *reciprocal* to each other, in which case it follows from the definition (278) that their *product* is equal to *Positive Unity*: so that *each* may, in this case, be considered as equal to *unity divided by the other*, or to the *Power* of that other which has *Negative Unity* for its *Exponent.*

(1.) When $\beta = \alpha$, the product $\beta\alpha$ reduces itself to what we may call the *square* of α, and may denote by α^2; and thus we may write, as a particular but important *case* of 281, XXIII., the formula (comp. 273),

$$\text{I.} \ldots \alpha^2 = -a^2 = -(\mathrm{T}\alpha)^2 = -\mathrm{N}\alpha;$$

so that the *square of any vector* α is equal to the *negative of the norm* (273) of that vector; or to the *negative of the square of the number* $\mathrm{T}\alpha$, which expresses (185) the *length* of the same vector.

(2.) More immediately, the definition (278) gives,

$$\text{II.} \ldots \alpha^2 = \alpha\alpha = \alpha : \mathrm{R}\alpha = -(\mathrm{T}\alpha)^2 = -\mathrm{N}\alpha, \text{ as before.}$$

(3.) Hence (compare the notations 161, 190, 199, 204),

$$\text{III.} \ldots \mathrm{S} \, . \, \alpha^2 = -\mathrm{N}\alpha; \quad \text{IV.} \ldots \mathrm{V} \, . \, \alpha^2 = 0;$$

and

$$\text{V.} \ldots \mathrm{T} \, . \, \alpha^2 = \mathrm{T}(\alpha^2) = +\mathrm{N}\alpha = (\mathrm{T}\alpha)^2 = \mathrm{T}\alpha^2;$$

the omission of the *parentheses*, or of the *point*, in this last symbol of a tensor,* for the square of a *vector*, as well as for the square of a *quaternion* (190), being thus justified: and in like manner we may write,

$$\text{VI.} \ldots \mathrm{U} \, . \, \alpha^2 = \mathrm{U}(\alpha^2) = -1 = (\mathrm{U}\alpha)^2 = \mathrm{U}\alpha^2;$$

the *square of an unit-vector* (129) being always equal to *negative unity*, and parentheses (or points) being *again* omitted.

(4.) The equation

$$\text{VII.} \ldots \rho^2 = \alpha^2, \quad \text{gives} \quad \text{VII}'. \ldots \mathrm{N}\rho = \mathrm{N}\alpha, \quad \text{or} \quad \text{VII}''. \ldots \mathrm{T}\rho = \mathrm{T}\alpha;$$

it represents therefore, by 186, (2.), the *sphere* with O for centre, which passes through the point A.

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* Compare the Note to page 214.

(5.) The more general equation,

$$\text{VIII.} \ldots (\rho - \alpha)^2 = (\beta - \alpha)^2, \qquad \text{(comp.* 186, (4.),)}$$

represents the sphere with A for centre, which passes through the point B.

(6.) For example, the equation,

$$\text{IX.} \ldots (\rho - \alpha)^2 = \alpha^2, \qquad \text{(comp. 186, (3.),)}$$

represents the sphere with A for centre, which passes through the origin O.

(7.) The equations (comp. 186, (6.), (7.)),

$$\text{X.} \ldots (\rho + \alpha)^2 = (\rho - \alpha)^2; \qquad \text{XI.} \ldots (\rho - \beta)^2 = (\rho - \alpha)^2,$$

represent, respectively, the *plane* through O, perpendicular to the line OA; and the plane which perpendicularly bisects the line AB.

(8.) The *distributive principle* of *vector-multiplication* (280), and the formula 279, III., enable us to establish generally (comp. 210, (9.)) the formula,

$$\text{XII.} \ldots (\beta \pm \alpha)^2 = \beta^2 \pm 2\mathrm{S}\beta\alpha + \alpha^2;$$

the recent equations IX. and X. may therefore be thus transformed:

$$\text{IX}' \ldots \rho^2 = 2\mathrm{S}\alpha\rho; \quad \text{and} \quad \text{X}' \ldots \mathrm{S}\alpha\rho = 0.$$

(9.) The equations,

$$\text{XIII.} \ldots \rho^2 + a^2 = 0; \qquad \text{XIV.} \ldots \rho^2 + 1 = 0,$$

represent the spheres with O for centre, which have a and 1 for their respective radii; so that this very simple formula, $\rho^2 + 1 = 0$, is (comp. 186, (1.)) *a form of the Equation of the Unit-Sphere* (128), and is, as such, of great importance in the present Calculus.

(10.) The equation,

$$\text{XV.} \ldots \rho^2 - 2\mathrm{S}\alpha\rho + c = 0,$$

may be transformed to the following,

$$\text{XVI.} \ldots \mathrm{N}(\rho - \alpha) = -(\rho - \alpha)^2 = c - \alpha^2 = c + \mathrm{N}\alpha;$$

or

$$\text{XVI}' \ldots \mathrm{T}(\rho - \alpha) = \sqrt{(c - \alpha^2)} = \sqrt{(c + \mathrm{N}\alpha)};$$

it represents therefore a (real or imaginary) sphere, with A for *centre*, and with this last radical (if real) for *radius*.

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* Compare also the sub-articles to 273.

(11.) This *sphere* is therefore necessarily *real*, if c be a *positive* scalar; or if this scalar constant, c, *though negative*, be (algebraically) *greater than* a^2, or than $-\mathrm{N}\alpha$: but it becomes *imaginary*, if $c + \mathrm{N}\alpha < 0$.

(12.) The *radical plane* of the *two spheres*,

$$\text{XVII.} \ldots \rho^2 - 2\mathrm{S}\alpha\rho + c = 0, \quad \rho^2 - 2\mathrm{S}\alpha'\rho + c' = 0,$$

has for equation,

$$\text{XVIII.} \ldots 2\mathrm{S}(\alpha' - \alpha)\rho = c' - c;$$

it is therefore *always real*, if the given *vectors* α, α' and the given *scalars* c, c' be such, even if one or both of the *spheres themselves* be *imaginary*.

(13.) The equation 281, XXIX., or XXX., of the *Central Ellipsoid* (or of the ellipsoid with its *centre* taken for the *origin* of vectors), may now be still further simplified,* as follows:

$$\text{XIX.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

(14.) The definition (278) gives also,

$$\text{XX.} \ldots \alpha\mathrm{R}\alpha = \alpha : \alpha = 1; \quad \text{or} \quad \text{XX}'. \ldots \mathrm{R}\alpha \,.\, \alpha = \mathrm{R}\alpha : \mathrm{R}\alpha = 1;$$

whence it is natural to write,†

$$\text{XXI.} \ldots \mathrm{R}\alpha = 1 : \alpha = \alpha^{-1},$$

if we so far anticipate here the general theory of *powers of vectors*, above alluded to (277), as to use this last symbol to denote the *quotient*, of *unity divided by the vector* α; so as to have *identically*, or for *every* vector, the equation,

$$\text{XXII.} \ldots \alpha \,.\, \alpha^{-1} = \alpha^{-1} .\, \alpha = 1.$$

(15.) It follows, by 258, VII., that

$$\text{XXIII.} \ldots \alpha^{-1} = -\mathrm{U}\alpha : \mathrm{T}\alpha; \quad \text{and} \quad \text{XXIV.} \ldots \beta\alpha = \beta : \alpha^{-1}.$$

(16.) If we had adopted the equation XXIII. *as a definition*‡ of the *symbol* α^{-1}, then the formula XXIV. might have been used, as a *formula of interpretation* for the symbol $\beta\alpha$. But we proceed to consider an entirely *different method*, of arriving at the *same* (or an *equivalent*) *Interpretation* of this latter symbol: or of a *Binary Product of Vectors*, considered as equal to a *Quaternion*.

* Compare the Note to page 241.

† Compare the Note to page 293.

‡ Compare the Note to page 324.

SECTION 3.

On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors.

283. It cannot fail to have been observed by any attentive reader of the Second Book, how close and intimate a *connexion** has been found to exist, between a *Right Quaternion* (132), and its *Index*, or *Index-Vector* (133). Thus, if v and v' denote (as in 223, (1)., &c., any two *right* quaternions, and if $\mathrm{I}v$, $\mathrm{I}v'$ denote, as usual, their *indices*, we have already seen that

$$\mathrm{I.} \ldots \mathrm{I}v' = \mathrm{I}v, \quad \text{if} \quad v' = v, \text{ and conversely } (133);$$

$$\mathrm{II.} \ldots \mathrm{I}(v' \pm v) = \mathrm{I}v' \pm \mathrm{I}v \; (206);$$

$$\mathrm{III.} \ldots \mathrm{I}v' : \mathrm{I}v = v' : v \; (193);$$

to which may be added the more recent formula,

$$\mathrm{IV.} \ldots \mathrm{RI}v = \mathrm{IR}v \; (258, \mathrm{IX.}).$$

284. It could not therefore have appeared strange, if we had proposed to establish this new formula of the same kind,

$$\mathrm{I.} \ldots \mathrm{I}v' . \mathrm{I}v = v' . v = v'v,$$

as a definition (supposing that the recent definition 278 had not occurred to us), whereby to *interpret the product of any two indices of right quaternions*, as being equal to the *product of those two quaternions themselves*. And then, *to interpret the product* $\beta\alpha$, of *any two given vectors*, taken *in a given order*, we should only have had to conceive (as we always may) that the two proposed *factors*, α and β, are the *indices* of *two right quaternions*, v and v', and to *multiply these latter*, in the *same order*. For thus we should have been led to establish the formula,

$$\mathrm{II.} \ldots \beta\alpha = v'v, \quad \text{if} \quad \alpha = \mathrm{I}v, \quad \text{and} \quad \beta = \mathrm{I}v';$$

or we should have this slightly more *symbolical* equation,

$$\mathrm{III.} \ldots \beta\alpha = \beta . \alpha = \mathrm{I}^{-1}\beta . \mathrm{I}^{-1}\alpha;$$

in which the symbols,

$$\mathrm{I}^{-1}\alpha \quad \text{and} \quad \mathrm{I}^{-1}\beta,$$

are understood to denote the two right quaternions, whereof the two lines α and β are the indices.

* Compare the Note to page 175.

(1.) To establish now the substantial *identity of these two interpretations*, 278 and 284, of a *binary product of vectors* $\beta\alpha$, notwithstanding *the difference of form* of the *definitional equations* by which they have been expressed, we have only to observe that it has been found, as a *theorem* (194), that

$$\text{IV.} \ldots v'v = \text{I}v' : \text{I}(1 : v) = \text{I}v' : \text{IR}v;$$

but the definition (258) of $\text{R}a$ gave us the lately cited equation, $\text{RI}v = \text{IR}v$; we have therefore, by the recent formula II., the equation,

$$\text{V.} \ldots \text{I}v'.\text{I}v = \text{I}v' : \text{RI}v; \quad \text{or} \quad \text{VI.} \ldots \beta . \alpha = \beta : \text{R}\alpha,$$

as in 278, I.; α and β still denoting *any two vectors*. The *two interpretations* therefore *coincide*, at least in their *results*, although they have been obtained by *different processes*, or *suggestions*, and are expressed by two different *formulæ*.

(2.) The result 279, II., respecting *conjugate products of vectors*, corresponds thus to the result 191, (2.), or to the first formula of 223, (1.).

(3.) The two formulæ of 279, (1.) and (2.), respecting the *scalar and right parts* of the product $\beta\alpha$, answer to the two other formulæ of the same sub-article, 223, (1.), respecting the corresponding parts of $v'v$.

(4.) The *doubly distributive property* (280), of *vector-multiplication*, is on this plan seen to be *included* in the corresponding but more general property (212), of *multiplication of quaternions*.

(5.) By changing $\text{IV}q$, $\text{IV}q'$, t, t', and δ, to α, β, a, b, and γ, in those formulæ of Art. 208 which are previous to its sub-articles, we should obtain, *with the recent definition* (or *interpretation*) II. of $\beta\alpha$, several of the consequences lately given (in sub-arts. to 281), as resulting from the former definition, 278, I. Thus, the equations,

VI., VII., VIII., IX., X., XI., XII., XXII., and XXIII.

of 281, correspond to, and may (with our last definition) be deduced from, the formulæ,

V., VI., VIII., XI., XII., XXII., XX., XIV., and XVI., XVIII.

of 208. (Some of the consequences from the sub-articles to 208 have been already considered, in 281, (11.).)

(6.) The *geometrical properties of the line* $\text{IV}\beta\alpha$, deduced from the *first definition* (278) of $\beta\alpha$ in 281, (3.) and (4.), (namely, the *positive rotation* round that line, from β to α; its *perpendicularity to their plane*; and the *representation* by the same line of the *parallelogram under those two factors*, regard being had to *units* of *length* and of *area*,) might also have been deduced from 223, (4.), by means of the *second definition* (284), of the *same product*, $\beta\alpha$.

SECTION 4.

On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of two Rectangular Lines, by a Third Line, Rectangular to both.

285. It has been seen, then, that the recent formula 284, II. or III., *may replace the formula* 278, I., as a *second definition* of a *product of two vectors*, which conducts to the *same consequences*, and therefore ultimately to the same *interpretation* of such a product, as the *first*. Now, in the *second* formula, we have *interpreted that product*, $\beta\alpha$, by *changing* the *two factor-lines*, α and β, to the *two right quaternions*, v and v', or $I^{-1}\alpha$ and $I^{-1}\beta$, of which they are the *indices*; and by then *defining* that the sought product $\beta\alpha$ is *equal* to the product $v'v$, of those two right quaternions. It becomes, therefore, important to inquire, at this stage, *how far such substitution*, of $I^{-1}\alpha$ for α, or of v for Iv, together with the *converse* substitution, is permitted in this Calculus, consistently with principles already established. For it is evident that if such substitutions can be shown to be generally legitimate, or allowable, we shall thereby be enabled to *enlarge* greatly the existing field of *interpretation*: and to treat, in *all* cases, *Functions of Vectors*, as being, at the same time, *Functions of Right Quaternions*.

286. We have first, by 133 (compare 283, I.), the *equality*,

$$\text{I.} \ldots I^{-1}\beta = I^{-1}\alpha, \quad \text{if} \quad \beta = \alpha.$$

In the next place, by 206 (comp. 283, II.), we have the formula of *addition or subtraction*,

$$\text{II.} \ldots I^{-1}(\beta \pm \alpha) = I^{-1}\beta \pm I^{-1}\alpha;$$

with these more general results of the same kind (comp. 207 and 99),

$$\text{III.} \ldots I^{-1}\Sigma\alpha = \Sigma I^{-1}\alpha; \qquad \text{IV.} \ldots I^{-1}\Sigma x\alpha = \Sigma x I^{-1}\alpha.$$

In the third place, by 193 (comp. 283, III.), we have, for *division*, the formula,

$$\text{V.} \ldots I^{-1}\beta : I^{-1}\alpha = \beta : \alpha;$$

while the *second definition* (284) of *multiplication of vectors*, which has been proved to be *consistent* with the *first* definition (278), has given us the analogous equation,

$$\text{VI.} \ldots I^{-1}\beta \,.\, I^{-1}\alpha = \beta \,.\, \alpha = \beta\alpha.$$

It would seem, then, that we might at once proceed to *define*, for the purpose of *interpreting any proposed Function of Vectors as a Quaternion*, that the following general *Equation* exists:

$$\text{VII.} \ldots \mathrm{I}^{-1}\alpha = \alpha; \quad \text{or} \quad \text{VIII.} \ldots \mathrm{I}v = v, \quad \text{if } \angle v = \frac{\pi}{2};$$

or still more briefly and *symbolically*, if it be understood that the *subject* of the *operation* I is always a *right* quaternion,

$$\text{IX.} \ldots \mathrm{I} = 1.$$

But, before finally adopting this conclusion, there is a *case* (or rather a *class* of cases), which it is necessary to examine, in order to be certain that *no contradiction to former results* can ever be thereby caused.

287. The *most general form of a vector function*, or of a vector regarded as a function of other vectors and of scalars, which was considered in the First Book, was the form (99, comp. 275),

$$\text{I.} \ldots \rho = \Sigma x\alpha;$$

and we have seen that *if we change, in this form*, each *vector* α to the corresponding *right quaternion* $\mathrm{I}^{-1}\alpha$, and then take the *index* of the new right quaternion which *results*, we shall thus be conducted to precisely the *same vector* ρ, as that which had been otherwise obtained before; or in symbols, that

$$\text{II.} \ldots \Sigma x\alpha = \mathrm{I}\Sigma x\mathrm{I}^{-1}\alpha \text{ (comp. 286, IV.)}.$$

But *another form of a vector-function* has been considered in the Second Book; namely, the form,

$$\text{III.} \ldots \rho = \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta}\alpha \text{ (226, III.)};$$

in which $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$ are *any odd number of complanar vectors*. And before we accept, as *general*, the *equation* VII. or VIII. or IX. of 286, we must inquire whether we are at liberty to write, under the same *conditions of complanarity*, and with the *same signification* of the *vector* ρ, the equation,

$$\text{IV.} \ldots \rho = \mathrm{I}\left(\ldots \frac{\mathrm{I}^{-1}\varepsilon}{\mathrm{I}^{-1}\delta} \cdot \frac{\mathrm{I}^{-1}\gamma}{\mathrm{I}^{-1}\beta} \cdot \mathrm{I}^{-1}\alpha\right).$$

288. To examine this, let there be at first only *three given* complanar *vectors*, $\gamma \parallel\!\parallel \alpha, \beta$; in which case there will always be (by 226) a *fourth vector* ρ, in the same plane, which will represent or construct the function $(\gamma : \beta) \cdot \alpha$; namely, the *fourth proportional* to β, γ, α. Taking then what we may call

the *Inverse Index-Functions*, or *operating* on these four vectors $\alpha, \beta, \gamma, \rho$ by the *characteristic* I^{-1}, we obtain *four collinear and right quaternions* (209), which may be denoted by v, v', v'', v'''; and we shall have the equation,

$$\text{V.} \ldots v''' : v = (\rho : \alpha = \gamma : \beta =)\, v'' : v';$$

or

$$\text{VI.} \ldots v''' = (v'' : v') \,.\, v;$$

which proves what was required. Or, more symbolically,

$$\text{VII.} \ldots \frac{I^{-1}\rho}{I^{-1}\alpha} = \frac{\rho}{\alpha} = \frac{\gamma}{\beta} = \frac{I^{-1}\gamma}{I^{-1}\beta};$$

$$\text{VIII.} \ldots \frac{\gamma}{\beta} \,.\, \alpha = \rho = I\,(I^{-1}\rho) = I\left(\frac{I^{-1}\gamma}{I^{-1}\beta} \,.\, I^{-1}\alpha\right).$$

And it is so easy to extend this reasoning to the case of any greater odd number of given vectors in one plane, that we may now consider the recent formula IV. as proved.

289. We shall therefore *adopt*, as *general*, the *symbolical equations* VII. VIII. IX. of 286; and shall thus be enabled, in a shortly subsequent section, to *interpret ternary* (and other) *products of vectors*, as well as *powers* and other *Functions of Vectors*, as being *generally Quaternions*; although they may, in particular *cases*, *degenerate* (131) into *scalars*, or may become *right* quaternions (132): in which latter event they may, in virtue of the same principle, be *represented by*, and *equated* to, *their own indices* (133), and so be treated as *vectors*. In symbols, we shall write *generally*, for *any set of vectors* $\alpha, \beta, \gamma, \ldots$ and *any function* f, the equation,

$$\text{I.} \ldots f(\alpha, \beta, \gamma, \ldots) = f(I^{-1}\alpha,\ I^{-1}\beta,\ I^{-1}\gamma, \ldots) = q,$$

q being some *quaternion*; while in the particular *case* when this quaternion is *right*, or when

$$q = v = S^{-1}\,0 = I^{-1}\rho,$$

we shall write *also*, and usually *by preference* (for *that* case), the formula,

$$\text{II.} \ldots f(\alpha, \beta, \gamma, \ldots) = If(I^{-1}\alpha,\ I^{-1}\beta,\ I^{-1}\gamma, \ldots) = \rho,$$

ρ being a *vector*.

290. For example, instead of saying (as in 281) that the *Product* of any *two Rectangular Vectors* is a *Right Quaternion*, with certain properties of its *Index*, already pointed out (284, (6.)), we may *now* say that *such a product* is *equal* to that *index*. And hence will follow the important consequence, that

the Product of any two Rectangular Lines in Space is equal to (or may be *constructed by*) *a Third Line, rectangular to both*; the *Rotation round this Product-Line, from the Multiplier-Line to the Multiplicand-Line*, being *Positive*: and the *Length of the Product* being equal to the *Product of the Lengths of the Factors*, or *representing* (with a suitable reference to *units*) the *Area of the Rectangle* under them. And generally we may now, for all purposes of *calculation* and *expression*, *identify** a *Right Quaternion with its own Index.*

SECTION 5.

On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the conception of an Unit-Line as a Right Versor.

291. An immediate consequence of the symbolical equation 286, IX., is that we may now *suppress the Characteristic* I, *of the Index of a Right Quaternion*, in all the formulæ into which it has entered; and so may *simplify the Notation*. Thus, instead of writing,

$$\mathrm{Ax}\,.\,q = \mathrm{IUV}q, \quad \text{or} \quad \mathrm{Ax}. = \mathrm{IUV}, \quad \text{as in 204, (23.),}$$

or

$$\mathrm{Ax}\,.\,q = \mathrm{UIV}q, \quad \text{or} \quad \mathrm{Ax}. = \mathrm{UIV}, \quad \text{as in 274, (7.),}$$

we may *now* write simply†,

$$\text{I.}\ldots \mathrm{Ax}\,.\,q = \mathrm{UV}q; \quad \text{or} \quad \text{II.}\ldots \mathrm{Ax}. = \mathrm{UV}.$$

The Characteristic Ax., *of the Operation of taking the Axis of a Quaternion* (132, (6.)), *may therefore henceforth be replaced* whenever we may think fit to *dispense with it*, by this *combination of two other characteristics*, U and V, which are of greater and more *general* utility, and indeed *cannot*‡ be dispensed with, in the *practice* of the present Calculus.

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* Compare the Notes to pages 121, 137, 175, 193, 203.

† Compare the first Note to page 120, and the third Note to page 203.

‡ Of course, any one who chooses may *invent new symbols*, to *denote the same operations on quaternions*, as those which are denoted in these *Elements*, and in the elsewhere cited *Lectures*, by the letters U and V; but, under *some form*, such symbols *must* be *used*: and it appears to have been hitherto thought expedient, by other writers, not hastily to innovate on notations which have been already employed in several published researches, and have been found to answer their purpose. As to the *type* used for these, and for the analogous characteristics K, S, T, that must evidently be a mere affair of taste and convenience: and in fact they have all been printed as small italic capitals, in some examination-papers by the author.

292. We are now enabled also to *diminish*, to some extent, the *number of technical terms*, which have been employed in the foregoing Book. Thus, whereas we defined, in 202, that the right quaternion $\mathrm{V}q$ was the *Right Part* of the Quaternion q, or of the *sum* $\mathrm{S}q + \mathrm{V}q$, we may now, by 290, *identify* that *part* with its own *index-vector* $\mathrm{IV}q$, and so may be led to *call* it *the vector part*, or simply *the* VECTOR,* of that Quaternion q, without henceforth *speaking* of the *right part*: although the *plan of exposition*, adopted in the Second Book, required that we should do so for some time. And thus an *enunciation*, which was put forward at an early stage of the present work, namely, at the end of the First Chapter of the First Book, or the assertion (17) that

"*Scalar plus Vector equals Quaternion,*"

becomes entirely intelligible, and acquires a perfectly *definite* signification. For we are in this manner led to conceive a *Number* (positive or negative) as being *added to a Line*,† *when it is added* (according to rules already established) *to that right quotient* (132), of which the *line* is the *Index*. In symbols, we are thus led to establish the formula,

$$\text{I.} \ldots q = a + \alpha, \quad \text{when} \quad \text{II.} \ldots q = a + \mathrm{I}^{-1}\alpha\,;$$

whatever scalar, and *whatever vector*, may be denoted by a and α. And because *either* of these *two parts*, or *summands*, may *vanish* separately, we are entitled to say, that *both Scalars and Vectors*, or *Numbers and Lines*, are *included in the Conception of a Quaternion*, as now enlarged or modified.

293. Again, the same *symbolical identification* of $\mathrm{I}v$ with v (286, VIII.) leads to the forming of a *new conception of an Unit-Line*, or *Unit-Vector* (129), as being also a *Right Versor* (153); or an *Operator*, of which the *effect* is to *turn a line*, in a plane perpendicular to itself, through a *positive quadrant of rotation*: and thereby to oblige the *Operand-Line* to take a *new direction, at right angles* to its *old* direction, but *without* any *change of length*. And then the remarks (154) on the equation $q^2 = -1$, where q was a *right versor* in the *former sense* (which is still a permitted one) of its being a *right radial quotient*

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* Compare the Note to page 193.

† On account of this possibility of conceiving a quaternion to be the *sum* of a number and a line, it was at one time suggested by the present author, that a *Quaternion* might *also* be called a *Grammarithm*, by a combination of the two Greek words γραμμή and ἀριθμός, which signify respectively a *Line* and a *Number*.

(147), or the *quotient of two equally long but mutually rectangular lines*, become immediately applicable to the *interpretation of the equation*,

$$\rho^2 = -1, \quad \text{or} \quad \rho^2 + 1 = 0 \text{ (282, XIV.)};$$

where ρ is still an *unit-vector*.

(1.) Thus (comp. fig. 41, p. 132), if α be any line perpendicular to *such* a vector ρ, we have the equations,

$$\text{I.} \ldots \rho\alpha = \beta; \quad \text{II.} \ldots \rho^2\alpha = \rho\beta = \alpha' = -\alpha;$$

β being *another* line perpendicular to ρ, which is, at the same time, at right angles to α, and of the same length with it; and from which a *third line* α', or $-\alpha$, *opposite* to the line α, but still *equally long*, is formed by a *repetition* of the *operation*, denoted by (what we may here call) the *characteristic* ρ; or having that *unit-vector* ρ for the *operator*, or *instrument employed*, as a sort of *handle*, or *axis** *of rotation*.

(2.) More generally (comp. 290), if α, β, γ be any three lines at right angles to each other, and if the *length* of γ be numerically equal to the *product of the lengths* of α and β, then (by what precedes) the *line* γ *represents*, or *constructs*, or is *equal* to, the *product of the two other lines*, at least if a certain *order* of the *factors* (comp. 279) be observed: so that we may write the equation (comp. 281, XXI.),

$$\text{III.} \ldots \alpha\beta = \gamma, \quad \text{if} \quad \text{IV.} \ldots \beta \perp \alpha, \ \gamma \perp \alpha, \ \gamma \perp \beta, \quad \text{and} \quad \text{V.} \ldots \text{T}\alpha \,.\, \text{T}\beta = \text{T}\gamma,$$

provided that the *rotation round* α, *from* β *to* γ, or that round γ from α to β, &c., has the direction taken as the *positive* one.

(3.) In this more general case, we may *still* conceive that the *multiplier-line* α has *operated* on the *multiplicand-line* β, so as to *produce* (or *generate*) the *product-line* γ; but *not now* by an operation of *version alone*, since the *tensor* of β is (generally) *multiplied* by that of α, in order to form, by V., the *tensor of the product* γ.

(4.) And if (comp. fig. 41, *bis*, in which α was first changed to β, and then to α') we *repeat this compound operation*, of *tension and version combined* (comp. 189), or if we *multiply again by* α, we obtain a *fourth line* β', *in the plane* of β, γ, but with a *direction opposite* to that of β, and with a *length generally different*: namely the line,

$$\text{VI.} \ldots \alpha\gamma = \alpha\alpha\beta = \alpha^2\beta = \beta' = -a^2\beta, \quad \text{if} \quad a = \text{T}\alpha.$$

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* Compare the second Note to page 137.

(5.) The *operator* a^2, or aa, is therefore *equivalent*, in its *effect on* β, to the *negative scalar*, $-a^2$, or $-(\mathrm{T}a)^2$, or $-\mathrm{N}a$, considered as a *coefficient*, or as a (scalar) *multiplier* (15): whence the *equation*,

$$\text{VII.} \ldots a^2 = -\mathrm{N}a \ (282, \text{I.}),$$

may be *again deduced*, but now with a *new interpretation*, which is, however, as we see, completely *consistent*, in all its *consequences*, with the one first proposed (282).

SECTION 6.

On the Interpretation of a Product of Three or more Vectors, as a Quaternion.

294. There is now no difficulty in interpreting a *ternary product of vectors* (comp. 277, I.), or a product of *more vectors than three*, taken always in some *given order*; namely, as the result (289, I.) of the *substitution* of the corresponding *right quaternions* in that product: which *result* is *generally* what we have lately called (276) an *Oblique Quotient*, or a Quaternion with either an *acute* or an *obtuse angle* (130); but may *degenerate* (131) into a *scalar*, or may become *itself* a *right* quaternion (132), and so be *constructed* (289, II.) by a *new vector*. It follows (comp. 281), that *Multiplication of Vectors*, like that of *Quaternions* (223), in which indeed we now see that it is *included*, is an *Associative Operation*: or that we may write generally (comp. 223, II.), for *any three vectors*, a, β, γ, the *Formula*,

$$\text{I.} \ldots \gamma\beta \,.\, a = \gamma \,.\, \beta a.$$

(1.) The formulæ 223, III. and IV., are now replaced by the following:

$$\text{II.} \ldots \mathrm{V}.\gamma\mathrm{V}\beta a = a\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma a;$$

$$\text{III.} \ldots \mathrm{V}\gamma\beta a = a\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma a + \gamma\mathrm{S}a\beta;*$$

in which $\mathrm{V}\gamma\beta a$ is written, for simplicity, instead of $\mathrm{V}(\gamma\beta a)$, or $\mathrm{V}.\gamma\beta a$; and with which, as with the earlier equations referred to, a student of this Calculus will find it useful to render himself *very familiar*.

* [On account of the importance of these formulæ, it is worth while to notice that, using the principles of the present Book,

$$\mathrm{V}\gamma\beta a = \tfrac{1}{2}(1 - \mathrm{K})\,\gamma\beta a = \tfrac{1}{2}(\gamma\beta a + a\beta\gamma) = \tfrac{1}{2}\gamma(\beta a + a\beta) - \tfrac{1}{2}(\gamma a + a\gamma)\,\beta$$
$$+ \tfrac{1}{2}a(\gamma\beta + \beta\gamma) = \gamma\mathrm{S}a\beta - \beta\mathrm{S}\gamma a + a\mathrm{S}\beta\gamma.]$$

(2.) Another useful form of the equation II. is the following:

$$\text{IV.} \ldots \mathrm{V}(\mathrm{V}\alpha\beta \,.\, \gamma) = \alpha \mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma\alpha.$$

(3.) The equations IX. X. XIV. of 223 enable us now to write, for *any three vectors*, the formula:

$$\begin{aligned} \text{V.} \ldots \mathrm{S}\gamma\beta\alpha = -\mathrm{S}\alpha\beta\gamma = \mathrm{S}\alpha\gamma\beta = -\mathrm{S}\beta\gamma\alpha = \mathrm{S}\beta\alpha\gamma = -\mathrm{S}\gamma\alpha\beta \\ = \pm \textit{volume of parallelepiped under } \alpha, \beta, \gamma, \\ = \pm 6 \times \textit{volume of pyramid } \mathrm{OABC}; \end{aligned}$$

upper or lower *signs* being taken, according as the *rotation* round α from β to γ is positive or negative: or in other words, the *scalar* $\mathrm{S}\gamma\beta\alpha$, of the *ternary product of vectors* $\gamma\beta\alpha$, being *positive* in the first case, but *negative* in the second.

(4.) The *condition of complanarity* of *three vectors*, α, β, γ, is therefore expressed by the equation (comp. 223, XI.):

$$\text{VI.} \ldots \mathrm{S}\gamma\beta\alpha = 0; \quad \text{or} \quad \text{VI}'. \ldots \mathrm{S}\alpha\beta\gamma = 0; \ \&\text{c.}$$

(5.) If α, β, γ be *any three vectors*, complanar or diplanar, the expression,

$$\text{VII.} \ldots \delta = \alpha\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma\alpha,$$

gives $$\text{VIII.} \ldots \mathrm{S}\gamma\delta = 0, \quad \text{and} \quad \text{IX.} \ldots \mathrm{S}\alpha\beta\delta = 0;$$

it represents therefore (comp. II. and IV.) a *fourth vector* δ, which is *perpendicular* to γ, but *complanar* with α and β: or in symbols,

$$\text{X.} \quad \delta \perp \gamma, \quad \text{and} \quad \text{XI.} \ldots \delta \,|||\, \alpha, \beta.$$

(Compare the notations 123, 129.)

(6.) For *any four vectors*, we have by II. and IV. the transformations,

$$\text{XII.} \ldots \mathrm{V}(\mathrm{V}\alpha\beta \,.\, \mathrm{V}\gamma\delta) = \delta\mathrm{S}\alpha\beta\gamma - \gamma\mathrm{S}\alpha\beta\delta;$$

$$\text{XIII.} \ldots \mathrm{V}(\mathrm{V}\alpha\beta \,.\, \mathrm{V}\gamma\delta) = \alpha\mathrm{S}\beta\gamma\delta - \beta\mathrm{S}\alpha\gamma\delta;$$

and each of these three equivalent expressions represents a *fifth vector* ε, which is at once complanar with α, β, and with γ, δ; or a *line* OE, which is in the *intersection of the two planes*, OAB and OCD.

(7.) Comparing them, we see that any *arbitrary vector* ρ may be expressed as a *linear function* of any *three given diplanar vectors*, α, β, γ, by the formula:

$$\text{XIV.} \ldots \rho\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma\rho + \beta\mathrm{S}\gamma\alpha\rho + \gamma\mathrm{S}\alpha\beta\rho;$$

which is found to be one of extensive utility.

(8.) Another very useful formula, of the same kind, is the following:

$$\text{XV.} \ldots \rho S\alpha\beta\gamma = V\beta\gamma \,.\, S\alpha\rho + V\gamma\alpha \,.\, S\beta\rho + V\alpha\beta \,.\, S\gamma\rho\,;$$

in the second member of which, the points may be omitted.*

(9.) One mode of proving the correctness of this last formula XV., is to *operate* on both members of it, by the *three symbols*, or *characteristics of operation*,

$$\text{XVI.} \ldots S\,.\,\alpha, \quad S\,.\,\beta, \quad S\,.\,\gamma\,;$$

the common results on both sides being respectively the three scalar products,

$$\text{XVII.} \ldots S\alpha\rho \,.\, S\alpha\beta\gamma, \quad S\beta\rho \,.\, S\alpha\beta\gamma, \quad S\gamma\rho \,.\, S\alpha\beta\gamma\,;$$

where again the points may be omitted.

(10.) We here employ the principle, that *if the three vectors* α, β, γ *be actual and diplanar*, then *no actual vector* λ *can satisfy at once the three scalar equations*,

$$\text{XVIII.} \ldots S\alpha\lambda = 0, \quad S\beta\lambda = 0, \quad S\gamma\lambda = 0\,;$$

because it *cannot be perpendicular at once to those three diplanar vectors.*

(11.) If, then, in any investigation with quaternions, we meet a system of this form XVIII., we can at once infer that

$$\text{XIX.} \ldots \lambda = 0, \quad \text{if} \quad \text{XX.} \ldots S\alpha\beta\gamma \gtrless 0\,;$$

while, conversely, if λ *be* an *actual* vector, then α, β, γ must be *complanar* vectors, or $S\alpha\beta\gamma = 0$, as in VI'.

(12.) Hence also, under the same condition XX., the three scalar equations,

$$\text{XXI.} \ldots S\alpha\lambda = S\alpha\mu, \quad S\beta\lambda = S\beta\mu, \quad S\gamma\lambda = S\gamma\mu,$$

give

$$\text{XXII.} \ldots \lambda = \mu.$$

(13.) *Operating* (comp. (9.)) on the equation XV. by the symbol, or *characteristic*, $S\,.\,\delta$, in which δ is any new vector, we find a result which may be written thus (with or without the points):

$$\text{XXIII.} \ldots 0 = S\alpha\rho \,.\, S\beta\gamma\delta - S\beta\rho \,.\, S\gamma\delta\alpha + S\gamma\rho \,.\, S\delta\alpha\beta - S\delta\rho \,.\, S\alpha\beta\gamma\,;$$

where α, β, γ, δ, ρ may denote *any five vectors.*

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* [Another method of proving XIV. is to assume $\rho = x\alpha + y\beta + z\gamma$. Operating by $S\,.V\beta\gamma$, $S\beta\gamma\rho = xS\alpha\beta\gamma$; and similar expressions may be found for y and z. To prove XV. assume $\rho = x'V\beta\gamma + y'V\gamma\alpha + z'V\alpha\beta$, and operate in turn by $S\,.\,\alpha$, $S\,.\,\beta$, and $S\,.\,\gamma$].

(14.) In drawing this last inference, we assume that the equation XV. holds good, *even* when the three vectors, α, β, γ are *complanar*: which in fact must be true, *as a limit*, since the equation has been proved, by (9.) and (12.), to be valid, if γ be *ever so little out of the plane* of α and β.

(15.) We have therefore this new formula:

$$\text{XXIV.} \ldots \mathrm{V}\beta\gamma\mathrm{S}\alpha\rho + \mathrm{V}\gamma\alpha\mathrm{S}\beta\rho + \mathrm{V}\alpha\beta\mathrm{S}\gamma\rho = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0;$$

in which ρ may denote *any fourth vector*, whether *in*, or *out of*, the *common plane* of α, β, γ.

(16.) If ρ be *perpendicular to that plane*, the last formula is *evidently* true, each term of the first member vanishing separately, by 281, (7.); and if we change ρ to a vector δ *in the plane* of α, β, γ, we are conducted to the following equation, *as an interpretation* of the same formula XXIV., which expresses a known theorem of plane trigonometry, including several others under it:

$$\text{XXV.} \ldots \sin \textsc{boc} \,.\, \cos \textsc{aod} + \sin \textsc{coa} \,.\, \cos \textsc{bod} + \sin \textsc{aob} \,.\, \cos \textsc{cod} = 0,$$

for *any four complanar and co-initial lines*, OA, OB, OC, OD.

(17.) By passing from OD to a line perpendicular thereto, but in their common plane, we have this other known* equation:

$$\text{XXVI.} \ldots \sin \textsc{boc} \sin \textsc{aod} + \sin \textsc{coa} \sin \textsc{bod} + \sin \textsc{aob} \sin \textsc{cod} = 0;$$

which, like the former, admits of many transformations, but is only mentioned here as offering itself naturally to our notice, when we seek to *interpret the formula* XXIV. obtained as above by quaternions.

(18.) Operating on that formula by $\mathrm{S}\,.\,\delta$, and changing ρ to ε, we have this new equation:

$$\text{XXVII.} \ldots 0 = \mathrm{S}\alpha\varepsilon\mathrm{S}\beta\gamma\delta + \mathrm{S}\beta\varepsilon\mathrm{S}\gamma\alpha\delta + \mathrm{S}\gamma\varepsilon\mathrm{S}\alpha\beta\delta, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0;$$

which might indeed have been at once deduced from XXIII.

(19.) The equation XIV., as well as XV., must hold good at the *limit*, when α, β, γ are *complanar*; hence

$$\text{XXVIII.} \ldots \alpha\mathrm{S}\beta\gamma\rho + \beta\mathrm{S}\gamma\alpha\rho + \gamma\mathrm{S}\alpha\beta\rho = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0.$$

* Compare page 20 of the *Géométrie Supérieure* of M. Chasles.

(20.) This last formula is evidently true, by (4.), if ρ be *in* the common plane of the three other vectors; and if we suppose it to be *perpendicular* to that plane, so that

$$\text{XXIX.} \ldots \rho \parallel \mathrm{V}\beta\gamma \parallel \mathrm{V}\gamma\alpha \parallel \mathrm{V}\alpha\beta,$$

and therefore, by 281, (9.), since $\mathrm{S}(\mathrm{S}\beta\gamma \cdot \rho) = 0$.

$$\text{XXX.} \ldots \mathrm{S}\beta\gamma\rho = \mathrm{S}(\mathrm{V}\beta\gamma \cdot \rho) = \mathrm{V}\beta\gamma \cdot \rho, \ \&\text{c.},$$

we may *divide each term by* ρ, and so obtain this other formula,

$$\text{XXXI.} \ldots \alpha\mathrm{V}\beta\gamma + \beta\mathrm{V}\gamma\alpha + \gamma\mathrm{V}\alpha\beta = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0.$$

(21.) In general, the *vector* (292) of this last expression *vanishes* by II.; the expression is therefore equal to its own *scalar*, and we may write,

$$\text{XXXII.} \ldots \alpha\mathrm{V}\beta\gamma + \beta\mathrm{V}\gamma\alpha + \gamma\mathrm{V}\alpha\beta = 3\mathrm{S}\alpha\beta\gamma,$$

whatever three vectors may be denoted by α, β, γ.

(22.) For the *case of complanarity*, if we suppose that the three vectors are *equally long*, we have the proportion,

$$\text{XXXIII.} \ldots \mathrm{V}\beta\gamma : \mathrm{V}\gamma\alpha : \mathrm{V}\alpha\beta = \sin \text{BOC} : \sin \text{COA} : \sin \text{AOB};$$

and the formula XXXI. becomes thus,

$$\text{XXXIV.} \ldots \text{OA} \cdot \sin \text{BOC} + \text{OB} \cdot \sin \text{COA} + \text{OC} \cdot \sin \text{AOB} = 0;$$

where OA, OB, OC are *any three radii of one circle*, and the equation is interpreted as in Articles 10, 11, &c.

(23.) The equation XXIII. might have been deduced from XIV., instead of XV., by first operating with $\mathrm{S} \cdot \delta$, and then interchanging δ and ρ.

(24.) A *vector* ρ may in general be considered (221) as *depending on three scalars* (the *co-ordinates* of its *term*); it cannot then be *determined* by *fewer than three scalar equations*; nor can it be *eliminated* between *fewer than four*.

(25.) As an example of such *determination* of a vector, let α, β, γ be again any *three given* and *diplanar vectors*; and let the three given *equations* be,

$$\text{XXXV.} \ldots \mathrm{S}\alpha\rho = a, \quad \mathrm{S}\beta\rho = b, \quad \mathrm{S}\gamma\rho = c;$$

in which a, b, c are supposed to denote *three given scalars*. Then the *sought vector* ρ has for its expression, by XV.,

$$\text{XXXVI.} \ldots \rho = e^{-1}(a\mathrm{V}\beta\gamma + b\mathrm{V}\gamma\alpha + c\mathrm{V}\alpha\beta), \quad \text{if} \quad \text{XXXVII.} \ldots e = \mathrm{S}\alpha\beta\gamma.$$

(26.) As another example, let the three equations be,

$$\text{XXXVIII.} \ldots S\beta\gamma\rho = a', \quad S\gamma\alpha\rho = b', \quad S\alpha\beta\rho = c';$$

then, with the same signification of the scalar e, we have, by XIV.,

$$\text{XXXIX.} \ldots \rho = e^{-1}(a'\alpha + b'\beta + c'\gamma).$$

(27.) As an example of *elimination of a vector*, let there be the *four* scalar equations,

$$\text{XL.} \ldots S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\delta\rho = d;$$

then, by XXIII., we have this *resulting equation, into which* ρ *does not enter*, but only the *four vectors*, $\alpha \ldots \delta$, and the *four scalars*, $a \ldots d$:

$$\text{XLI.} \ldots a \,.\, S\beta\gamma\delta - b \,.\, S\gamma\delta\alpha + c \,.\, S\delta\alpha\beta - d \,.\, S\alpha\beta\gamma = 0.$$

(28.) This last equation may therefore be considered as the *condition of concurrence of the four planes*, represented by the *four scalar equations* XL., in one *common point*; for, although it has not been expressly stated before, it follows evidently from the *definition* 278 of a *binary product of vectors*, combined with 196, (5.), that *every scalar equation of the linear form* (comp. 282, XVIII.),

$$\text{XLII.} \ldots S\alpha\rho = a, \quad \text{or} \quad S\rho\alpha = a,$$

in which $\alpha = \text{OA}$, and $\rho = \text{OP}$, as usual, *represents a plane locus of the point* P; the *vector of the foot* S, of the *perpendicular on that plane from the origin*, being

$$\text{XLIII.} \ldots \text{OS} = \sigma = a\mathrm{R}\alpha = a\alpha^{-1} \ (282, \text{XXI.}).$$

(29.) If we conceive a *pyramidal volume* (68) as having an *algebraical* (or *scalar*) *character*, so as to be capable of bearing either a *positive* or a *negative ratio* to the volume of a *given pyramid*, with a *given order of its points*, we may then *omit the ambiguous sign*, in the last expression (3.) for the *scalar of a ternary product of vectors*: and so may write, generally, OABC denoting such a *volume*, the formula,

$$\text{XLIV.} \ldots S\alpha\beta\gamma = 6 \,.\, \text{OABC},$$

= a *positive* or a *negative scalar*, according as the *rotation* round OA from OB to OC is *negative* or *positive*.

(30.) More generally, changing O to D, and OA or α to $\alpha - \delta$, &c., we have thus the formula:

$$\text{XLV.} \ldots 6 \,.\, \text{DABC} = S(\alpha-\delta)(\beta-\delta)(\gamma-\delta) = S\alpha\beta\gamma - S\beta\gamma\delta + S\gamma\delta\alpha - S\delta\alpha\beta;$$

in which it may be observed that the expression is *changed to its own opposite*, or *negative*, or is *multiplied by* -1, *when any two of the four vectors*, $\alpha, \beta, \gamma, \delta$, or when *any two of the four points*, A, B, C, D, *change places with each other*; and therefore is *restored* to its former value, by a *second* such *binary interchange*.

(31.) Denoting then the *new origin* of $\alpha, \beta, \gamma, \delta$ by E, we have first, by XLIV., XLV., the equation,

$$\text{XLVI.} \ldots \text{DABC} = \text{EABC} - \text{EBCD} + \text{ECDA} - \text{EDAB};$$

and may then write the result (comp. 68) under the more *symmetric form* (because $-$ EBCD = BECD = &c.):

$$\text{XLVII.} \ldots \text{BCDE} + \text{CDEA} + \text{DEAB} + \text{EABC} + \text{ABCD} = 0;$$

in which A, B, C, D, E may denote *any five points of space*.

(32.) And an analogous formula (69, III.) of the First Book, for *any six points* OABCDE, namely the equation (comp. 65, 70),

$$\text{XLVIII.} \ldots \text{OA}.\text{BCDE} + \text{OB}.\text{CDEA} + \text{OC}.\text{DEAB} + \text{OD}.\text{EABC} + \text{OE}.\text{ABCD} = 0,$$

in which the *additions* are performed according to the rules of *vectors*, the *volumes* being treated as scalar *coefficients*, is easily recovered from the foregoing principles and results. In fact, by XLVII., this last formula may be written as

$$\text{XLIX.} \ldots \text{ED}.\text{EABC} = \text{EA}.\text{EBCD} + \text{EB}.\text{ECAD} + \text{EC}.\text{EABD};$$

or, substituting $\alpha, \beta, \gamma, \delta$ for EA, EB, EC, ED, as

$$\text{L.} \ldots \delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta;$$

which is only another form of XIV., and ought to be *familiar* to the student.

(33.) The formula 69, II. may be deduced from XXXI. by observing that, when the *three vectors* α, β, γ are *complanar*, we have the proportion,

$$\text{LI.} \ldots V\beta\gamma : V\gamma\alpha : V\alpha\beta : V(\beta\gamma + \gamma\alpha + \alpha\beta) = \text{OBC} : \text{OCA} : \text{OAB} : \text{ABC},$$

if *signs* (or *algebraic* or *scalar ratios*) *of areas* be attended to (28, 63); and the formula 69, I., for the case of *three collinear points* A, B, C, may now be written as follows:

$$\text{LII.} \ldots \alpha(\beta-\gamma) + \beta(\gamma-\alpha) + \gamma(\alpha-\beta) = 2V(\beta\gamma + \gamma\alpha + \alpha\beta)$$
$$= 2V(\beta-\alpha)(\gamma-\alpha) = 0,$$

if the *three coinitial vectors* α, β, γ be *termino-collinear* (24).

(34.) The case when *four coinitial vectors* $\alpha, \beta, \gamma, \delta$ are *termino-complanar* (64), or when they terminate in *four complanar points* A, B, C, D, is expressed by equating to zero the second or the third member of the formula XLV.*

(35.) Finally, for *ternary products of vectors* in general, we have the formula:

$$\text{LIII.} \ldots \alpha^2\beta^2\gamma^2 + (\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\alpha\beta\gamma)^2 = (\alpha\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma\alpha + \gamma\mathrm{S}\alpha\beta)^2$$
$$= \alpha^2(\mathrm{S}\beta\gamma)^2 + \beta^2(\mathrm{S}\gamma\alpha)^2 + \gamma^2(\mathrm{S}\alpha\beta)^2 - 2\mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha\mathrm{S}\alpha\beta.\dagger$$

295. The *identity* (290) of a *right quaternion* with its *index*, and the *conception* (293) of an *unit-line* as a *right versor*, allow us now to treat the three important versors, i, j, k, as *constructed by*, and even as (in our present view) *identical with*, their own *axes*; or with the *three lines* OI, OJ, OK of 181, considered as being each a certain *instrument*, or *operator*, or *agent in a right rotation* (293, (1.)), which *causes any line*, in a plane perpendicular to itself, *to turn in that plane*, through a *positive quadrant*, without any change of its *length*. With this conception, or construction, the *Laws of the Symbols ijk* are *still included in the Fundamental Formula* of 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1; \qquad \text{(A)}$$

and if we now, in conformity with the same conception, *transfer* the *Standard Trinomial Form* (221) from *Right Quaternions* to *Vectors*, so as to write generally an expression of the form,

$$\text{I.} \ldots \rho = ix + jy + kz, \quad \text{or} \quad \text{I}'. \ldots \alpha = ia + jb + kc, \ \&\text{c.},$$

where xyz and abc are *scalars* (namely, *rectangular co-ordinates*), we can *recover* many of the foregoing results with ease: and can, if we think fit, *connect* them with *co-ordinates*.

(1.) As to the *laws* (182), included in the *Fundamental Formula* A, the law $i^2 = -1$, &c., may be interpreted on the plan of 293, (1.), as representing the *reversal* which results from *two successive quadrantal rotations*.

(2.) The *two contrasted laws*, or formulæ,

$$ij = +k, \qquad ji = -k, \qquad \text{(182, II. and III.)}$$

may now be interpreted as expressing, that although a *positive rotation through a right angle, round the line i as an axis, brings a revolving line from the position j to the position k*, or $+k$, yet, on the contrary, a *positive quadrantal rotation round the line j*, as a *new axis*, brings a *new revolving line* from a *new initial*

* [And the equation of the plane ABC is $\mathrm{S}\rho\mathrm{V}(\beta\gamma + \gamma\alpha + \alpha\beta) = \mathrm{S}\alpha\beta\gamma$.]

† [Since $\mathrm{K}\alpha\beta\gamma = -\gamma\beta\alpha$.]

position, i, to a *new final position*, denoted by $-k$, or *opposite** to the *old* final position, $+k$.

(3.) Finally, the law $ijk = -1$ (183) may be interpreted by conceiving, that we *operate on a line* α, which has at first the direction of $+j$, *by the three lines*, k, j, i, *in succession*; which gives *three new* but equally long *lines*, β, γ, δ, in the directions of $-i$, $+k$, $-j$, and so conducts at last to a line $-\alpha$, which has a *direction opposite* to *the initial one*.

(4.) The foregoing laws of ijk, which are all (as has been said) *included* (184) in the Formula A, when combined with the recent expression I. for ρ, give (comp. 222, (1.)) for the *square* of that *vector* the value:

$$\text{II.} \ldots \rho^2 = (ix + jy + kz)^2 = -(x^2 + y^2 + z^2);$$

this *square of the line* ρ is therefore equal to the *negative of the square of its length* $T\rho$ (185), or to the *negative of its norm* $N\rho$ (273), which agrees with the former result† 282, (1.) or (2.).

(5.) The *condition of perpendicularity* of the *two lines* ρ and α, when they are represented by the two trinomials I. and I'., may be expressed (281, XVIII.) by the formula,

$$\text{III.} \ldots 0 = S\alpha\rho = -(ax + by + cz);$$

which agrees with a well-known theorem of rectangular co-ordinates.

(6.) The *condition of complanarity* of *three* lines, ρ, ρ', ρ'', represented by the trinomial forms,

$$\text{IV.} \ldots \rho = ix + jy + kz, \quad \rho' = ix' + \&c., \quad \rho'' = ix'' + \&c.$$

is (by 294, VI.) expressed by the formula (comp. 223, XIII.),

$$\text{V.} \ldots 0 = S\rho''\rho'\rho = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);$$

agreeing again with known results.

(7.) When the three lines ρ, ρ', ρ'', or OP, OP', OP'', are *not in one plane*, the recent expression for $S\rho''\rho'\rho$ gives, by 294, (3.), the *volume of the parallelepiped*

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* In the *Lectures*, the *three rectangular unit-lines*, i, j, k, were supposed (in order to fix the conceptions, and with a reference to northern latitudes) to be directed, respectively, towards the *south*, the *west*, and the *zenith*; and then the *contrast* of the two formulæ, $ij = +k$, $ji = -k$, came to be illustrated by conceiving, that we at *one* time *turn a moveable line*, which is at first directed *westward*, round an *axis* (or *handle*) directed towards the *south*, with a *right-handed* (or *screwing*) *motion*, through a *right angle*, which causes the line to take an *upward position*, as its *final* one; and that at *another* time we operate, in a precisely similar manner, on a line directed at first *southward*, with an *axis* directed to the *west*, which obliges this new line to take finally a *downward* (instead of, as before, an *upward*) *direction*.

† Compare also 222, IV.

(comp. 223, (9.)) of which they are *edges*; and this *volume*, thus expressed, is a *positive* or a *negative scalar*, according as the *rotation* round ρ from ρ' to ρ'' is itself *positive* or *negative*: that is, according as it has the *same direction* as that round $+x$ from $+y$ to $+z$ (or round i from j to k), or the direction *opposite* thereto.

(8.) It may be noticed here (comp. 223, (13.)), that if α, β, γ be *any three vectors*, then (by 294, III. and V.) we have:

$$\text{VI.} \ldots \mathrm{S}\alpha\beta\gamma = -\,\mathrm{S}\gamma\beta\alpha = \tfrac{1}{2}(\alpha\beta\gamma - \gamma\beta\alpha);$$

$$\text{VII.} \ldots \mathrm{V}\alpha\beta\gamma = +\,\mathrm{V}\gamma\beta\alpha = \tfrac{1}{2}(\alpha\beta\gamma + \gamma\beta\alpha).$$

(9.) More generally (comp. 223, (12.)), since a *vector*, considered as representing a *right quaternion* (290), is always (by 144) the *opposite of its own conjugate*, so that we have the important formula,*

$$\text{VIII.} \ldots \mathrm{K}\alpha = -\,\alpha, \quad \text{and therefore} \quad \text{IX.} \ldots \mathrm{K}\Pi\alpha = \pm\,\Pi'\alpha,$$

we may write for *any number of vectors*, the transformations,

$$\text{X.} \ldots \mathrm{S}\Pi\alpha = \pm\,\mathrm{S}\Pi'\alpha = \tfrac{1}{2}(\Pi\alpha \pm \Pi'\alpha),$$

$$\text{XI.} \ldots \mathrm{V}\Pi\alpha = \mp\,\mathrm{V}\Pi'\alpha = \tfrac{1}{2}(\Pi\alpha \mp \Pi'\alpha),$$

upper or *lower* signs being taken, according as that number is *even* or *odd*: it being understood that

$$\text{XII.} \ldots \Pi'\alpha = \ldots \gamma\beta\alpha, \quad \text{if} \quad \Pi\alpha = \alpha\beta\gamma \ldots$$

(10.) The relations of *rectangularity*,

$$\text{XIII.} \ldots \mathrm{Ax.}\, i \perp \mathrm{Ax.}\, j; \quad \mathrm{Ax.}\, j \perp \mathrm{Ax.}\, k; \quad \mathrm{Ax.}\, k \perp \mathrm{Ax.}\, i,$$

which result at once from the definitions (181), may now be written more briefly, as follows:

$$\text{XIV.} \ldots i \perp j; \quad j \perp k; \quad k \perp i;$$

and similarly in other cases, where the *axes*, or the *planes*, of any two right quaternions are at *right angles* to each other.

(11.) But, with the notations of the Second Book, we might *also* have written, by 123, 181, such formulæ of *complanarity* as the following, $\mathrm{Ax.}\, j \parallel\!| \, i$, to express (comp. 225) that the *axis* of j was a *line* in the *plane* of i; and it might cause some confusion, if we were now to *abridge* that formula to $j \parallel\!| \, i$.

* If, in like manner, we interpret, on our present plan, the symbols $\mathrm{U}\alpha$, $\mathrm{T}\alpha$, $\mathrm{N}\alpha$ as equivalent to $\mathrm{UI}^{-1}\alpha$, $\mathrm{TI}^{-1}\alpha$, $\mathrm{NI}^{-1}\alpha$, we are reconducted (compare the Notes to page 137) to the same signification of those symbols as before (155, 185, 273); and it is evident that on the same plan we have now,

$$\mathrm{S}\alpha = 0, \qquad \mathrm{V}\alpha = \alpha.$$

In general, it seems convenient that we should not henceforth employ the *sign* $\parallel\!|$, except as connecting either *symbols of three lines*, considered still as *complanar*; or else *symbols of three right quaternions*, considered as being *collinear* (209), because their *indices* (or *axes*) are *complanar*: or finally, *any two complanar quaternions* (123).

(12.) On the other hand, no inconvenience will result, if we now insert the *sign of parallelism*, between the symbols of two right quaternions which are, in the former sense (123), *complanar*; for example, we may write, on our present plan,

$$\text{XV.} \ldots xi \parallel i, \qquad yj \parallel j, \qquad zk \parallel k,$$

if xyz be any three *scalars*.

296. There are a few particular but remarkable *cases*, of *ternary* and other *products of vectors*, which it may be well to mention here, and of which some may be worth a student's while to remember: especially as regards the *products of successive sides of closed polygons, inscribed in circles*, or in *spheres*.

(1.) If A, B, C, D be any four *concircular points*, we know, by the sub-articles to 260, that their *anharmonic function* (ABCD), as defined in 259, (9.), is *scalar*; being also *positive* or *negative*, according to a law of *arrangement* of those four points, which has been already stated.

(2.) But, by that definition, and by the *scalar* (though *negative*) character of the *square of a vector* (282), we have generally, for *any plane or gauche quadrilateral* ABCD, the formula:

$$\text{I.} \ldots e^2\,(\text{ABCD}) = \text{AB} \,.\, \text{BC} \,.\, \text{CD} \,.\, \text{DA} = \textit{the continued product of the four sides};$$

in which the coefficient e^2 is a *positive scalar*, namely the product of two negative or of two positive squares, as follows:

$$\text{II.} \ldots e^2 = \text{BC}^2 \,.\, \text{DA}^2 = \overline{\text{BC}}^2 \,.\, \overline{\text{DA}}^2 > 0.$$

(3.) If then ABCD be a *plane* and *inscribed* quadrilateral, we have, by 260, (8.), the formula,

$$\text{III.} \ldots \text{AB} \,.\, \text{BC} \,.\, \text{CD} \,.\, \text{DA} = \textit{a positive or negative scalar},$$

according as this *quadrilateral in a circle* is a *crossed* or an *uncrossed* one.

(4.). The *product* $\alpha\beta\gamma$ of *any three complanar vectors* is a *vector*, because its *scalar part* $S\alpha\beta\gamma$ *vanishes*, by 294, (3.) and (4.); and if the factors be three *successive sides* AB, BC, CD of a quadrilateral thus *inscribed* in a circle, their product has either the *direction* of the *fourth successive side*, DA, or else the *opposite* direction, or in symbols,

$$\text{IV.} \ldots \text{AB} \,.\, \text{BC} \,.\, \text{CD} : \text{DA} > \text{or} < 0,$$

according as the *quadrilateral* ABCD is an *uncrossed* or a *crossed* one.

(5.) By conceiving the *fourth point* D to *approach*, continuously and indefinitely, *to the first point* A, we find that the *product of the three successive sides of any plane triangle*, ABC, is given by an equation of the form:

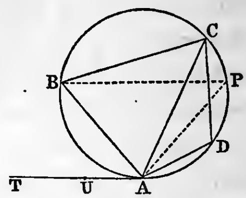

Fig. 63.

$$\text{V.} \ldots \text{AB} \cdot \text{BC} \cdot \text{CA} = \text{AT};^*$$

AT being a *line* (comp. fig. 63) *which touches the circumscribed circle*, or (more fully) which *touches the segment* ABC of that circle, at the point A; or *represents the initial direction of motion, along the circumference, from* A *through* B *to* C: while the *length* of this *tangential product line*, AT, is *equal* to, or *represents*, with the usual reference to an *unit* of length, the *product of the lengths of the three sides*, of the same inscribed triangle ABC.

(6.) Conversely, if this theorem respecting the product of the sides of an *inscribed triangle* be supposed to have been *otherwise proved*, and if it be *remembered*, then since it will give in like manner the equation,

$$\text{VI.} \ldots \text{AC} \cdot \text{CD} \cdot \text{DA} = \text{AU},$$

if D be *any fourth point, concircular with* A, B, C, while AU is, as in the annexed figures 63, a *tangent to the new segment* ACD, we can *recover* easily the theorem (3.), respecting the *product of the sides of an inscribed quadrilateral;* and thence can *return* to the corresponding theorem (260, (8.)), respecting the *anharmonic function* of any such figure ABCD: for we shall thus have, by V. and VI., the equation,

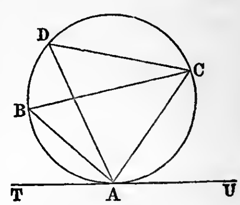

Fig. 63 *bis*.

$$\text{VII.} \ldots \text{AB} \cdot \text{BC} \cdot \text{CD} \cdot \text{DA} = (\text{AT} \cdot \text{AU}) : (\text{CA} \cdot \text{AC}),$$

in which the *divisor* CA.AC or N.AC, or $\overline{\text{AC}}^2$, is *always positive* (282, (1.)), but the *dividend* AT.AU is *negative* (281, (9.)) for the case of an *uncrossed* quadrilateral (fig. 63), being on the contrary *positive* for the other case of a *crossed* one (fig. 63, *bis*).

(7.) If P be any point on the circle through a given point A, which touches at a given origin O a given line OT = τ, as represented in fig. 64, we shall then have by (5.) an equation of the form,

$$\text{VIII.} \ldots \text{OA} \cdot \text{AP} \cdot \text{PO} = x \cdot \text{OT},$$

* [Or directly by Euclid $\text{U}\frac{\text{AT}}{\text{AB}} = \text{U}\frac{\text{CA}}{\text{CB}}$, or $\text{U}\frac{\text{AT}}{\text{CA}} = \text{U}\frac{\text{AB}}{\text{CB}}$.]

in which x is some scalar coefficient, which varies with the position of P. Making then $OA = \alpha$, and $OP = \rho$, as usual, we shall have

$$\text{IX.} \ldots \alpha(\rho - \alpha)\rho = -x\tau,$$

or

$$\text{IX}'. \ldots \rho^{-1} - \alpha^{-1} = x\tau : \alpha^2\rho^2,$$

or

$$\text{IX}''. \ldots \text{V}\tau\rho^{-1} = \text{V}\tau\alpha^{-1};$$

and any one of these may be considered as *a form of the equation of the circle*, determined by the given conditions.

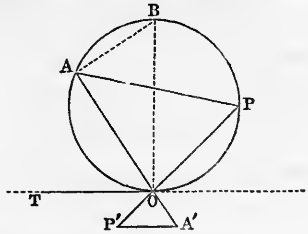

Fig. 64.

(8.) Geometrically, the last formula IX″. expresses, that *the line* $\rho^{-1} - \alpha^{-1}$, or $\text{R}\rho - \text{R}\alpha$, or A′P′ (see again fig. 64), if $OA' = \alpha^{-1} = \text{R}\alpha = \text{R} \,.\, OA$, and $OP' = \rho^{-1} = \text{R} \,.\, OP$, is *parallel to the given tangent* τ *at* O; which agrees with fig. 58, and with Art. 260.

(9.) If B be the point *opposite* to O upon the circle, then the *diameter* OB, or β, as being $\perp \tau$, so that $\tau\beta^{-1}$ is a vector, is given by the formula,

$$\text{X.} \ldots \tau\beta^{-1} = \text{V}\tau\alpha^{-1}; \quad \text{or} \quad \text{X}'. \ldots \beta = -\tau : \text{V}\tau\alpha^{-1};$$

in which the tangent τ admits, as it ought to do, of being multiplied by any scalar, without the value of β being changed.

(10.) As another verification, the last formula gives,

$$\text{XI.} \ldots \overline{OB} = \text{T}\beta = \text{T}\alpha : \text{TVU}\tau\alpha^{-1} = \overline{OA} : \sin AOT.$$

(11.) If a quadrilateral OABC be *not inscriptible* in a circle, then whether it be *plane* or *gauche*, we can always circumscribe (as in fig. 65) *two circles*, OAB and OBC, about the *two triangles*, formed by drawing the *diagonal* OB; and then, on the plan of (6.), we can draw *two tangents* OT, OU, to the *two segments* OAB, OBC, so as to represent the *two ternary products*,

$$OA \,.\, AB \,.\, BO, \quad \text{and} \quad OB \,.\, BC \,.\, CO;$$

after which we shall have the *quaternary product*,

$$\text{XII.} \ldots OA \,.\, AB \,.\, BC \,.\, CO = OT \,.\, OU : \overline{OB}^2;$$

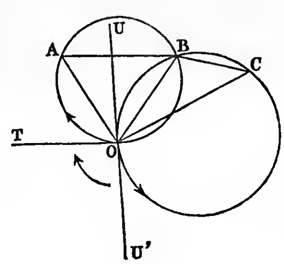

Fig. 65.

where the *divisor*, $\overline{OB}^2$, or BO . OB, or N . OB, is a *positive scalar*, but the *dividend* OT . OU, and therefore also the *quotient* in the *second* member, or the *product* in the *first* member, is a *quaternion*.

(12.) The *axis* of this quaternion is *perpendicular to the plane* TOU of the *two tangents*; and therefore to the *plane itself* of the quadrilateral OABC, if that be a *plane figure*; but if it be *gauche*, then the axis is *normal to the circumscribed sphere at the point* O: being also in all cases such, that the *rotation* round it, from OT to OU, is *positive*.

(13.) The *angle* of the same quaternion is the *supplement* of the angle TOU between the two tangents above mentioned; it is therefore *equal* to the angle U′OT, if OU′ touch the *new segment* OCB, or proceed in a new and *opposite direction* from O (see again fig. 65); it may therefore be said to be the *angle between the two arcs*, OAB and OCB, *along which a point should move*, in order to go *from* O, on the two circumferences, *to* the *opposite corner* B of the *quadrilateral* OABC, *through* the *two other corners*, A and C, respectively: or the angle between the arcs OCB, OAB.

(14.) These results, respecting the *axis* and *angle* of the *product of the four successive sides*, of any quadrilateral OABC, or ABCD, apply without any modification to the *anharmonic quaternion* (259, (9.)) of the same quadrilateral; and although, for the *case* of a quadrilateral *in a circle*, the *axis* becomes indeterminate, because the quaternary product and the anharmonic function degenerate together into *scalars*, or because the figure may then be conceived to be *inscribed in indefinitely many spheres*, yet the *angle* may still be determined by the *same rule* as in the *general* case: this angle being $= \pi$, for the inscribed and *uncrossed* quadrilateral (fig. 63); but $= 0$, for the inscribed and *crossed* one (fig. 63, *bis*).

(15.) For the *gauche* quadrilateral OABC, which may always be conceived to be inscribed *in a determined sphere*, we may say, by (13.), that the *angle of the quaternion product*, $\angle$ (OA . AB . BC . CO), is equal to the *angle of the lunule*, bounded (generally) by the two *arcs of small circles* OAB, OCB; with the *same construction* for the equal *angle of the anharmonic*,

$$\angle\,(\mathrm{OABC}), \quad \text{or} \quad \angle\,(\mathrm{OA} : \mathrm{AB} \,.\, \mathrm{BC} : \mathrm{CO}).$$

(16.) It is evident that the general principle 223, (10.), of the permissibility of *cyclical permutation* of quaternion factors *under the sign* S, must hold good for the *case* when those quaternions *degenerate* (294) into *vectors*; and it is still more obvious, that *every* permutation of factors is allowed, under the sign T: whence *cyclical* permutation is *again* allowed, *under this other sign* SU; and consequently also (comp. 196, XVI.) under the sign $\angle$.

(17.) Hence generally, for *any four vectors*, we have the three equations,

$$\text{XIII.} \ldots S\alpha\beta\gamma\delta = S\beta\gamma\delta\alpha; \qquad \text{XIV.} \ldots SU\alpha\beta\gamma\delta = SU\beta\gamma\delta\alpha;$$

$$\text{XV.} \ldots \angle\, \alpha\beta\gamma\delta = \angle\, \beta\gamma\delta\alpha;$$

and in particular, for the *successive sides* of any plane or gauche *quadrilateral* ABCD, we have the *four equal angles*,

$$\text{XVI.} \ldots \angle(\text{AB . BC . CD . DA}) = \angle(\text{BC . CD . DA . AB}) = \&\text{c.};$$

with the corresponding equality of the *angles of the four anharmonics*,

$$\text{XVII.} \ldots \angle(\text{ABCD}) = \angle(\text{BCDA}) = \angle(\text{CDAB}) = \angle(\text{DABC});$$

or of those of the four *reciprocal* anharmonics (259, XVII.),

$$\text{XVII}'. \ldots \angle(\text{ADCB}) = \angle(\text{BADC}) = \angle(\text{CBAD}) = \angle(\text{DCBA}).$$

(18.) Interpreting now, by (13.) and (15.), these last equations, we derive from them the following *theorem*, for the *plane*, or for *space*:—

Let ABCD be *any four points*, connected by *four circles*, each passing through *three* of the points: then, not only is the *angle at* A, between the *arcs* ABC, ADC, *equal* to the *angle at* C, between CDA and CBA, but also it is equal (comp. fig. 66) to the *angle at* B, between the *two other arcs* BCD and BAD, and to the *angle at* D, between the arcs DAB, DCB.

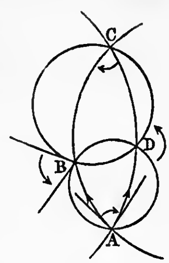

Fig. 66.

(19.) Again, let ABCDE be any *pentagon*, *inscribed in a sphere*; and conceive that the *two diagonals* AC, AD are drawn. We shall then have three equations, of the forms,

$$\text{XVIII.} \ldots \text{AB . BC . CA} = \text{AT}; \quad \text{AC . CD . DA} = \text{AU}; \quad \text{AD . DE . EA} = \text{AV};$$

where AT, AU, AV are three tangents to the sphere at A, so that their product is a fourth tangent at that point. But the equations XVIII. give

$$\text{XIX.} \ldots \text{AB . BC . CD . DE . EA} = (\text{AT . AU . AV}) : (\overline{\text{AC}}^2 . \overline{\text{AD}}^2)$$

$$= \text{AW} = \textit{a new vector, which touches the sphere at } \text{A}.$$

We have therefore this *Theorem*, which includes several others under it:—

"*The product of the five successive sides of any (generally gauche) pentagon inscribed in a sphere, is equal to a tangential vector, drawn from the point at which the pentagon begins and ends.*"

(20.) Let then P be a point on the sphere which passes through O, and through three given points A, B, C; we shall have the equation,

$$\text{XX.} \ldots 0 = \mathrm{S}\,(\text{OA} \,.\, \text{AB} \,.\, \text{BC} \,.\, \text{CP} \,.\, \text{PO}) = \mathrm{S}\alpha\,(\beta - \alpha)\,(\gamma - \beta)\,(\rho - \gamma)\,(-\rho)$$
$$= \alpha^2 \mathrm{S}\beta\gamma\rho + \beta^2 \mathrm{S}\gamma\alpha\rho + \gamma^2 \mathrm{S}\alpha\beta\rho - \rho^2 \mathrm{S}\alpha\beta\gamma.$$

(21.) Comparing with 294, XIV., we see that the *condition* for the four co-initial vectors α, β, γ, ρ thus *terminating on one spheric surface*, which passes *through their common origin* O, may be thus expressed:

$$\text{XXI.} \ldots \text{if } \rho = x\alpha + y\beta + z\gamma, \quad \text{then} \quad \rho^2 = x\alpha^2 + y\beta^2 + z\gamma^2.$$

(22.) If then we *project* (comp. 62) the variable point P into points A', B', C' on the *three given chords* OA, OB, OC, by *three planes* through that point P, respectively *parallel* to the planes BOC, COA, AOB, we shall have the equation:

$$\text{XXII.} \ldots \text{OP}^2 = \text{OA} \,.\, \text{OA}' + \text{OB} \,.\, \text{OB}' + \text{OC} \,.\, \text{OC}'.$$

(23.) That the equation XX. does in fact represent a *spheric locus* for the point P, is evident from its mere *form* (comp. 282, (10.)); and that this sphere passes *through the four given points*, O, A, B, C, may be proved by observing that the equation is satisfied, when we change ρ to any one of the four vectors, 0, α, β, γ.

(24.) Introducing an *auxiliary vector*, OD or δ, determined by the equation,

$$\text{XXIII.} \ldots \delta \mathrm{S}\alpha\beta\gamma = \alpha^2 \mathrm{V}\beta\gamma + \beta^2 \mathrm{V}\gamma\alpha + \gamma^2 \mathrm{V}\alpha\beta,$$

or by the system of the three scalar equations (comp. 294, (25.)),

$$\text{XXIV.} \ldots \alpha^2 = \mathrm{S}\delta\alpha, \quad \beta^2 = \mathrm{S}\delta\beta, \quad \gamma^2 = \mathrm{S}\delta\gamma,$$

or $$\text{XXIV}'. \ldots \mathrm{S}\delta\alpha^{-1} = \mathrm{S}\delta\beta^{-1} = \mathrm{S}\delta\gamma^{-1} = 1,$$

the equation XX. of the sphere becomes simply,

$$\text{XXV.} \ldots \rho^2 = \mathrm{S}\delta\rho, \quad \text{or} \quad \text{XXV}'. \ldots \mathrm{S}\delta\rho^{-1} = 1;$$

so that D is the point of the sphere *opposite* to O, and δ is a *diameter* (comp. 282, IX'.; and 196, (6.)).

(25.) The formula XXIII., which determines this diameter, may be written in this other way:

$$\text{XXVI.} \ldots \delta \mathrm{S}\alpha\beta\gamma = \mathrm{V}\alpha\,(\beta - \alpha)\,(\gamma - \beta)\,\gamma;$$

or $$\text{XXVI}'. \ldots 6 \,.\, \text{OABC} \,.\, \text{OD} = -\,\mathrm{V}\,(\text{OA} \,.\, \text{AB} \,.\, \text{BC} \,.\, \text{CO});$$

where the symbol OABC, considered as a *coefficient*, is interpreted as in 294, XLIV.; namely, as denoting the *volume* of the *pyramid* OABC, which is here an *inscribed* one.

(26.) This result of calculation, so far as it regards the *direction* of the *axis of the quaternion* OA . AB . BC . CO, agrees with, and may be used to confirm, the theorem (12.), respecting the *product of the successive sides of a gauche quadrilateral*, OABC; including the *rule of rotation*, which *distinguishes* that axis from its *opposite*.

(27.) The formula XXIII. for the diameter δ may also be thus written:

$$\text{XXVII.} \ldots \delta . S\alpha^{-1}\beta^{-1}\gamma^{-1} = V(\beta^{-1}\gamma^{-1} + \gamma^{-1}\alpha^{-1} + \alpha^{-1}\beta^{-1})$$
$$= V(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1});$$

and the equation XX. of the sphere may be transformed to the following:

$$\text{XXVIII.} \ldots 0 = S(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1})(\rho^{-1} - \alpha^{-1});$$

which expresses (by 294, (34.), comp. 260, (10.)), that the *four reciprocal vectors*,

$$\text{XXIX.} \ldots \text{OA}' = \alpha' = \alpha^{-1}, \quad \text{OB}' = \beta' = \beta^{-1}, \quad \text{OC}' = \gamma' = \gamma^{-1}, \quad \text{OP}' = \rho' = \rho^{-1},$$

are *termino-complanar* (64); the plane A′B′C′P′, in which they all terminate, being *parallel to the tangent plane to the sphere at* O: because the perpendicular let fall on this plane from O is

$$\text{XXX.} \ldots \delta' = \delta^{-1},$$

as appears from the three scalar equations,

$$\text{XXXI.} \ldots S\alpha'\delta = S\beta'\delta = S\gamma'\delta = 1.$$

(28.) In general, if D be the *foot of the perpendicular from* O, *on the plane* ABC, then

$$\text{XXXII.} \ldots \delta = S\alpha\beta\gamma : V(\beta\gamma + \gamma\alpha + \alpha\beta);$$

because this expression satisfies, and may be deduced from, the three equations,

$$\text{XXXIII.} \ldots S\alpha\delta^{-1} = S\beta\delta^{-1} = S\gamma\delta^{-1} = 1.$$

As a verification, the formula shows that the *length* $T\delta$, of this perpendicular, or *altitude*, OD, is equal to *the sextuple volume of the pyramid* OABC, *divided by the double area of the triangular base* ABC. (Compare 281, (4.), and 294, (3.), (33.).)

(29.) The equation XX., of the *sphere* OABC, might have been obtained by the *elimination of the vector* δ, between the *four scalar equations* XXIV. and XXV., on the plan of 294, (27.).

(30.) And another form of equation of the same sphere, answering to the development of XXVIII., may be obtained by the analogous elimination of the same vector δ, between the four other equations, XXIV′. and XXV′.

(31.) The *product* of *any even number* of *complanar vectors* is *generally a quaternion* with an *axis perpendicular to their plane*; but the product of the *successive sides of a hexagon* ABCDEF, or any other *even-sided figure, inscribed in a circle*, is a *scalar*: because by drawing *diagonals* AC, AD, AE from the *first* (or *last*) *point* A of the polygon, we find as in (6.) that it differs only by a scalar coefficient, or divisor, from the product of an *even number of tangents*, at the first point.

(32.) On the other hand, the *product* of *any odd number of complanar vectors* is *always a line*, in the same plane; and in particular (comp. (19.)), the product of the *successive sides* of a *pentagon*, or *heptagon*, &c., inscribed *in a circle*, is equal to a *tangential vector*, drawn from the *first point* of that *inscribed* and *odd-sided polygon*: because it differs only by a scalar coefficient from the product of an *odd number* of such *tangents*.

(33.) The *product* of *any number* of *lines in space* is *generally* a *quaternion* (289); and if they be the *successive sides* of a *hexagon*, or other *even-sided polygon*, inscribed *in a sphere*, the *axis* of this quaternion (comp. (12.)) is *normal to that sphere, at the initial* (or final) *point* of the polygon.

(34.) But the product of the successive sides of a *heptagon*, or other *odd-sided polygon in a sphere*, is equal (comp. (19.)) to a *vector*, which *touches the sphere* at the initial or final point; because it bears a scalar ratio to the product of an *odd number* of vectors, in the *tangent plane* at that point.*

(35.) The equation XX., or its transformation XXVIII., may be called the condition or *equation of homosphericity* (comp. 260, (10.)) of the *five points* O, A, B, C, P; and the analogous equation for the five points ABCDE, with vectors $\alpha\beta\gamma\delta\varepsilon$ from any arbitrary origin O, may be written thus:

$$\text{XXXIV.} \ldots \quad 0 = S(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \varepsilon)(\varepsilon - \alpha);$$

or thus

$$\text{XXXV.} \ldots \quad 0 = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\varepsilon^2,\dagger$$

* [The inscription of polygons in a sphere is treated very fully in the "Lectures." If $\rho_1, \rho_2, \ldots \rho_n$ are the vectors from the centre to the vertices, and if $\iota_1 = \rho_2 - \rho_1$, $\iota_2 = \rho_3 - \rho_2$, &c. denote the vector sides, then by 213 (5.) $\rho_2 = -\iota_1 \rho_1 \iota_1^{-1}$, $\rho_3 = -\iota_2 \rho_2 \iota_2^{-1} = \iota_2 \iota_1 \rho_1 \iota_1^{-1} \iota_2^{-1}$ and $\rho_{n+1} = \rho_1 = (-)^n q \rho_1 q^{-1}$, where $q = \iota_n \iota_{n-1} \ldots \iota_2 \iota_1$. Hence $\rho_1 q = (-)^n q \rho_1$; or when n is even $\rho_1 \mathrm{V}q = \mathrm{V}q \,.\, \rho_1$ or $\mathrm{V}q \parallel \rho_1$; but when n is odd the quaternion equation $\rho_1 \mathrm{S}q + \mathrm{S}\rho_1 \mathrm{V}q = 0$ affords the conditions $\mathrm{S}q = 0$ and $\mathrm{S}\rho_1 \mathrm{V}q = 0$, or q is a vector at right angles to ρ_1. See Lecture VI., Art. 336.]

† [On change of origin XXI. may be written in the form

$$a(\alpha - \epsilon) + b(\beta - \epsilon) + c(\gamma - \epsilon) + d(\delta - \epsilon) = 0, \quad a(\alpha - \epsilon)^2 + b(\beta - \epsilon)^2 + c(\gamma - \epsilon)^2 + d(\delta - \epsilon)^2 = 0.$$

Introducing e defined by XXXIX., XXXV. and XXXVII. follow. Eliminating a, b, c, d, and e from five equations connecting the squares of the mutual distances between the points, analogous to that here given, a determinant relation is at once found.]

six times the second member of this last formula being found to be equal to the second member of the one preceding it, if

$$\text{XXXVI}\ldots\ a = \text{BCDE},\quad b = \text{CDEA},\quad c = \text{DEAB},\quad d = \text{EABC},\quad e = \text{ABCD},$$

or more fully,

$$\text{XXXVII}\ldots\ 6a = S(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) = S(\gamma\delta\varepsilon - \delta\varepsilon\beta + \varepsilon\beta\gamma - \beta\gamma\delta),\ \&c.;$$

so that, by 294, XLVIII. and XLVII., we have also (comp. 65, 70) the equation,

$$\text{XXXVIII}\ldots\ 0 = a\alpha + b\beta + c\gamma + d\delta + e\varepsilon,$$

with the relation between the coefficients,

$$\text{XXXIX}\ldots\ 0 = a + b + c + d + e,$$

which allows (as above) the *origin* of vectors to be *arbitrary*.

(36.) The equation or condition XXXV. may be obtained as the result of an *elimination* (294, (27.)), of a *vector* κ, and of a *scalar* g, *between five scalar equations* of the *form* 282, (10.), namely the five following,

$$\text{XL}\ldots\ \alpha^2 - 2S\kappa\alpha + g = 0,\quad \beta^2 - 2S\kappa\beta + g = 0,\ \ldots\quad \varepsilon^2 - 2S\kappa\varepsilon + g = 0;$$

κ being the *vector of the centre* K of the *sphere* ABCD, of which the equation may be written as

$$\text{XLI}\ldots\ \rho^2 - 2S\kappa\rho + g = 0,$$

g being some scalar constant; and *on which*, by the *condition* referred to, the *fifth point* E is situated.

(37.) By treating this fifth point, or its vector ε, as *arbitrary*, we recover the condition or *equation of concircularity* (3.), of the four points A, B, C, D; or the formula,

$$\text{XLII}\ldots\ 0 = V(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha).$$

(38.) The *equation of the circle* ABC, and the *equation of the sphere* ABCD, may in general be written thus:

$$\text{XLIII}\ldots\ 0 = V(\alpha-\beta)(\beta-\gamma)(\gamma-\rho)(\rho-\alpha);$$

$$\text{XLIV}\ldots\ 0 = S(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\rho)(\rho-\alpha);$$

ρ being as usual the vector of a *variable point* P, on the one or the other *locus*.

(39.) The equations of the *tangent to the circle* ABC, and of the *tangent plane to the sphere* ABCD, at the point A, are respectively,

$$\text{XLV}\ldots\ 0 = V(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\rho-\alpha),$$

and

$$\text{XLVI}\ldots\ 0 = S(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha)(\rho-\alpha).$$

(40.) Accordingly, whether we combine the two equations XLIII. and XLV., or XLIV. and XLVI., we find in each case the equation,

$$\text{XLVII.} \ldots (\rho - \alpha)^2 = 0, \quad \text{giving} \quad \rho = \alpha, \quad \text{or} \quad \mathrm{P} = \mathrm{A} \ (20);$$

it being supposed that the *three* points A, B, C are *not collinear*, and that the *four points*, A, B, C, D are *not complanar*.

(41.) If the *centre* of the *sphere*, ABCD be taken for the *origin* O, so that

$$\text{XLVIII.} \ldots \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = -r^2, \quad \text{or} \quad \text{XLIX.} \ldots \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = \mathrm{T}\delta = r,$$

the positive scalar r denoting the *radius*, then after some reductions we obtain the transformation,

$$\text{L.} \ldots \mathrm{V}(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha) = 2\alpha \mathrm{S}(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha).$$

(42.) Hence, generally, if K be, as in (36.), the centre of the sphere, we have the equation (comp. XXVI′.),

$$\text{LI.} \ldots \mathrm{V}(\mathrm{AB} \,.\, \mathrm{BC} \,.\, \mathrm{CD} \,.\, \mathrm{DA}) = 12\mathrm{KA} \,.\, \mathrm{ABCD}.$$

(43.) We may therefore enunciate this *theorem*:

"*The vector part of the product of four successive sides, of a gauche quadrilateral inscribed in a sphere, is equal to the diameter drawn to the initial point of the polygon, multiplied by the sextuple volume of the pyramid, which its four points determine.*"

(44.) In effecting the *reductions* (41.), the following *general formulæ* of transformation have been employed, which may be useful on other occasions:

$$\text{LII.} \ldots \alpha q + q\alpha = 2(\alpha \mathrm{S}q + \mathrm{S}q\alpha); \quad \text{LII}'. \ldots \alpha q \alpha = \alpha^2 \mathrm{K}q + 2\alpha \mathrm{S}q\alpha;$$

where α may be *any vector*, and q may be *any quaternion*.

SECTION 7.

On the Fourth Proportional to Three Diplanar Vectors.

297. In general, when any *four quaternions*, q, q', q'', q''', satisfy the *equation of quotients*,

$$\text{I.} \ldots q''' : q'' = q' : q,$$

or the equivalent formula,

$$\text{II.} \ldots q''' = (q' : q) \,.\, q'' = q'q^{-1}q'',$$

we shall say that they form a *Proportion*; and that the *fourth*, namely q''', is the *Fourth Proportional* to the *first*, *second*, and *third* quaternions, namely to

q, q', and q'', taken in this given *order*. This definition will *include* (by 288) the one which was assigned in 226, for the fourth proportional to *three coplanar vectors*, α, β, γ, namely that *fourth vector in the same plane*, $\delta = \beta\alpha^{-1}\gamma$, which has been already considered; and it will enable us to *interpret* (comp. 289) the *symbol*

$$\text{III.}\ldots\ \beta\alpha^{-1}\gamma, \text{ when } \gamma \textit{ not } ||| \ \alpha, \beta,$$

as denoting *not* indeed a *Vector*, in this new case, *but* at least a *Quaternion*, which may be called (on the present general plan) *the Fourth Proportional to these three Diplanar Vectors*, α, β, γ. *Such fourth proportionals* possess some interesting properties, especially with reference to their *vector parts*, which it will be useful briefly to consider, and to illustrate by showing their connexion with *spherical trigonometry*, and generally with *spherical geometry*.

(1.) Let α, β, γ be (as in 208, (1), &c.) the vectors of the corners of a *triangle* ABC on the *unit-sphere*, whereof the sides are a, b, c; and let us write,

$$\text{IV.}\ldots \begin{cases} l = \cos a = \mathrm{S}\gamma\beta^{-1} = -\mathrm{S}\beta\gamma, \\ m = \cos b = \mathrm{S}\alpha\gamma^{-1} = -\mathrm{S}\gamma\alpha, \\ n = \cos c = \mathrm{S}\beta\alpha^{-1} = -\mathrm{S}\alpha\beta; \end{cases}$$

where it is understood that

$$\text{V.}\ldots\ \alpha^2 = \beta^2 = \gamma^2 = -1, \quad \text{or} \quad \text{VI.}\ldots\ \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = 1;$$

it being also at first supposed, for the sake of fixing the conceptions, that each of these three cosines, l, m, n, is greater than zero, or that each side of the triangle ABC is less than a quadrant.

(2.) Then, introducing three new vectors, δ, ε, ζ, defined by the equations,

$$\text{VII.}\ldots \begin{cases} \delta = \mathrm{V}\beta\alpha^{-1}\gamma = \mathrm{V}\gamma\alpha^{-1}\beta = m\beta + n\gamma - l\alpha, \\ \varepsilon = \mathrm{V}\gamma\beta^{-1}\alpha = \mathrm{V}\alpha\beta^{-1}\gamma = n\gamma + l\alpha - m\beta, \\ \zeta = \mathrm{V}\alpha\gamma^{-1}\beta = \mathrm{V}\beta\gamma^{-1}\alpha = l\alpha + m\beta - n\gamma, \end{cases}$$

we find that these *three derived vectors* have all one *common length*, say r, because they have one *common norm*; namely,

$$\text{VIII.}\ldots\ \mathrm{N}\delta = \mathrm{N}\varepsilon = \mathrm{N}\zeta = l^2 + m^2 + n^2 - 2lmn = r^2;$$

so that $$\text{IX.}\ldots\ \mathrm{T}\delta = \mathrm{T}\varepsilon = \mathrm{T}\zeta = r = \sqrt{(l^2 + m^2 + n^2 - 2lmn)}.$$

(3.) This common length, r, is *less than unity*; for if we write,

$$\text{X.}\ldots\ \mathrm{S}\alpha\beta\gamma = \mathrm{S}\beta\alpha^{-1}\gamma = e,$$

we shall have the relation,

$$\text{XI.}\ldots\ e^2 + r^2 = \mathrm{N}\beta\alpha^{-1}\gamma = 1;$$

and the scalar e is different from zero, because the vectors α, β, γ are diplanar.

(4.) *Dividing* the three lines δ, ε, ζ by their *length*, r, we change them to their *versors* (155, 156); and so obtain a *new triangle*, DEF, *on the unit-sphere*, of which the corners are determined by the *three new unit-vectors*,

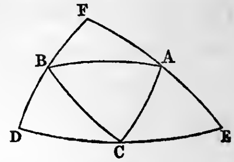

Fig. 67.

$$\text{XII.}\ldots \text{OD} = \text{U}\delta = r^{-1}\delta; \quad \text{OE} = \text{U}\varepsilon = r^{-1}\varepsilon;$$
$$\text{OF} = \text{U}\zeta = r^{-1}\zeta.$$

(5.) The *sides* opposite to D, E, F, in this *new* or *derived* triangle, are *bisected*, as in fig. 67, by the *corners* A, B, C of the *old* or *given* triangle; because we have the three equations,

$$\text{XIII.}\ldots \varepsilon + \zeta = 2l\alpha; \quad \zeta + \delta = 2m\beta; \quad \delta + \varepsilon = 2n\gamma.$$

(6.) Denoting the *halves* of the *new sides* by a', b', c' (so that the *arc* EF $= 2a'$, &c.), the equations XIII. show also, by IV. and IX., that

$$\text{XIV.}\ldots \cos a = r\cos a', \quad \cos b = r\cos b', \quad \cos c = r\cos c';$$

the *cosines of the half-sides* of the *new* (or *bisected*) *triangle*, DEF, are therefore *proportional to the cosines of the sides* of the *old* (or *bisecting*) *triangle* ABC.

(7.) The equations (IV.) give, by 279, (1.),

$$\text{XV.}\ldots 2l = -(\beta\gamma + \gamma\beta), \quad 2m = -(\gamma\alpha + \alpha\gamma), \quad 2n = -(\alpha\beta + \beta\alpha);$$

we have therefore, by VII., the three following equations between quaternions,

$$\text{XVI.}\ldots \alpha\varepsilon = \zeta\alpha, \quad \beta\zeta = \delta\beta, \quad \gamma\delta = \varepsilon\gamma;$$

which may also be thus written,

$$\text{XVI}'.\ldots \varepsilon\alpha = \alpha\zeta, \quad \zeta\beta = \beta\delta, \quad \delta\gamma = \gamma\varepsilon,$$

and express in a new way the relations of *bisection* (5.).

(8.) We have therefore the equations between vectors,

$$\text{XVII.}\ldots \varepsilon = \alpha\zeta\alpha^{-1}, \quad \zeta = \beta\delta\beta^{-1}, \quad \delta = \gamma\varepsilon\gamma^{-1};$$

or

$$\text{XVII}'.\ldots \zeta = \alpha\varepsilon\alpha^{-1}, \quad \delta = \beta\zeta\beta^{-1}, \quad \varepsilon = \gamma\delta\gamma^{-1}.$$

(9.) Hence also, by V., or because α, β, γ are *unit-vectors*,

$$\text{XVIII.}\ldots \varepsilon = -\alpha\zeta\alpha, \quad \zeta = -\beta\delta\beta, \quad \delta = -\gamma\varepsilon\gamma;$$

or

$$\text{XVIII}'.\ldots \zeta = -\alpha\varepsilon\alpha, \quad \delta = -\beta\zeta\beta, \quad \varepsilon = -\gamma\delta\gamma.$$

(10.) In general, *whatever the length of the vector* α *may be*, the first equation XVII. expresses that *the line* ε *is* (comp. 138) *the reflexion of the line* ζ, *with respect to that vector* α; because it may be put (comp. 279) under the form

$$\text{XIX.}\ldots \zeta\alpha^{-1} = \alpha^{-1}\varepsilon = \text{K}\varepsilon\alpha^{-1}, \quad \text{or} \quad \text{XIX}'.\ldots \varepsilon\alpha^{-1} = \text{K}\zeta\alpha^{-1}.$$

(11.) Another mode of arriving at the same *interpretation* of the equation $\varepsilon = a\zeta a^{-1}$, is to conceive ζ decomposed into two summand vectors, ζ' and ζ'', one parallel and the other perpendicular to a, in such a manner that

$$\text{XX.} \ldots \zeta = \zeta' + \zeta'', \quad \zeta' \parallel a, \quad \zeta'' \perp a;$$

for then we shall have, by 281, (10.), the transformations,

$$\text{XXI.} \ldots \varepsilon = a\zeta' a^{-1} + a\zeta'' a^{-1} = \zeta' a a^{-1} - \xi'' a a^{-1} = \zeta' - \zeta'';$$

the *parallel part* of ζ being thus *preserved*, but the *perpendicular part* being *reversed*, by the *operation* $a\,(\quad)\,a^{-1}$.

(12.) Or we may *return* from $\varepsilon = a\zeta a^{-1}$ to the form $\varepsilon a = a\zeta$, that is, to the first equation XVI′.; and then this equation between quaternions will show, as suggested in (7.), that whatever may be the *length* of a, we must have,

$$\text{XXII.} \ldots \mathrm{T}\varepsilon = \mathrm{T}\zeta, \quad \text{Ax.* } \varepsilon a = \text{Ax. } a\zeta, \quad \angle\, \varepsilon a = \angle\, a\zeta;$$

so that the *two lines* ε, ζ are *equally long*, and the *rotation* from ε to a is *equal* to that from a to ζ; these two rotations being *similarly directed*, and in one *common plane*.

(13.) We may also write the equations XVII. XVII′. under the forms,

$$\text{XXIII.} \ldots \varepsilon = a^{-1}\zeta a, \text{ \&c.}; \qquad \text{XXIII}'. \ldots \zeta = a^{-1}\varepsilon a, \text{ \&c.}$$

(14.) Substituting this last expression for ζ in the second equation XVII′., we derive this new equation,

$$\text{XXIV.} \ldots \delta = \beta a^{-1}\varepsilon a \beta^{-1}; \quad \text{or} \quad \text{XXIV}'. \ldots \varepsilon = a\beta^{-1}\delta\beta a^{-1};$$

that is, more briefly,

$$\text{XXV.} \ldots \delta = q\varepsilon q^{-1}, \text{ and } \text{XXV}'. \ldots \varepsilon = q^{-1}\delta q, \text{ if } \text{XXVI.} \ldots q = \beta a^{-1}.$$

(15.) An expression of this *form*, namely one with such a symbol as

$$\text{XXVII.} \ldots q\,(\quad)\,q^{-1}$$

for an *operator*, occurred before, in 179, (1.), and in 191 (5.); and was seen to indicate a *conical rotation of the axis of the operand quaternion* (of which the *symbol* is to be conceived as being written *within the parentheses*) *round the axis of* q, *through an angle* $= 2\angle\, q$, without any change of the *angle*, or of the *tensor*, of that *operand*; so that a *vector* must *remain* a *vector*, after any *operation* of

* It was remarked in 291, that this *characteristic* Ax. *can be dispensed with*, because it admits of being *replaced* by UV; but there may still be a convenience in employing it occasionally.

this sort, as being *still* a *right-angled quaternion* (290); or (comp. 223, (10.)) because

$$\text{XXVIII.} \ldots \ \mathrm{S}q\rho q^{-1} = \mathrm{S}q^{-1}q\rho = \mathrm{S}\rho = 0.$$

(16.) If then we conceive two *opposite points*, P′ and P, to be determined on the unit-sphere, by the conditions of being respectively the *positive poles* of the two *opposite arcs*, AB and BA, so that

$$\text{XXIX.} \ldots \ \mathrm{OP}' = \mathrm{Ax}.\,\beta\alpha^{-1} = \mathrm{Ax}.\,q, \quad \text{and} \quad \mathrm{OP} = \mathrm{P}'\mathrm{O} = \mathrm{Ax}.\,\alpha\beta^{-1} = \mathrm{Ax}.\,q^{-1},$$

we can infer from XXIV. that *the line* OD *may be derived from the line* OE, *by a conical rotation round the line* OP′ *as an axis, through an angle equal to the double of the angle* AOB (if O be still the *centre* of the *sphere*).

(17.) And in like manner we can infer from XXIV′., that the line OE admits of being derived from OD, by an *equal* but *opposite conical rotation*, round the line OP as a *new positive axis*, through an angle equal to twice the angle BOA.

(18.) To illustrate these and other connected results, the annexed figure 68 is drawn; in which P represents, as above, the positive pole of the arc BA, and arcs are drawn from it to D, E, F, meeting the great circle through A and B in the points R, S, T. (The other letters in the figure are not, for the moment, required, but their significations will soon be explained.)

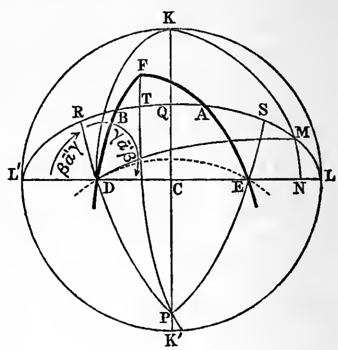

Fig. 68.

(19.) This being understood, we see, first, that because the arcs EF and FD are *bisected* (5.) at A and B, *the three arcual perpendiculars*, ES, FT, DR, let fall from E, F, D, on the great circle through A and B, are *equally long*; and that therefore the point P is the *interior pole of the small circle* DEF′, if F′ be the point diametrically *opposite to* F: so that a *conical rotation round this pole* P, or round the *axis* OP, would in fact bring the point D, or the line OD, *to the position* E, or OE, which is *one part* of the *theorem* (17.).

(20.) Again, the *quantity* of this *conical rotation*, is evidently measured by the *arc* RS of the *great circle* with P for *pole*; but the *bisections* above mentioned give (comp. 165) the two *arcual equations*,

$$\text{XXX.} \ldots \ \frown\mathrm{RB} = \frown\mathrm{BT}, \quad \frown\mathrm{TA} = \frown\mathrm{AS}; \quad \text{whence} \quad \text{XXXI.} \ldots \ \frown\mathrm{RS} = 2\frown\mathrm{BA},$$

and the *other part* of the same theorem (17.) is proved.

(21.) The *point* F may be said to be the *reflexion, on the sphere, of the point* D, *with respect to the point* B, which *bisects the interval* between them ; and thus we may say that *two successive reflexions of an arbitrary point upon a sphere* (as here from D to F, and then from F to E), *with respect to two given points* (B and A) *of a given great circle*, are jointly *equivalent to one conical rotation, round the pole* (P) *of that great circle* ; or to the description of an *arc of a small circle*, round that *pole*, or *parallel* to that great circle : and that the *angular quantity* (DPE) of this *rotation* is *double* of that represented by the *arc* (BA) connecting the *two given points* ; or is the *double of the angle* (BPA), which that *given arc subtends*, at the *same pole* (P).

(22.) There is, as we see, no difficulty in *geometrically proving* this *theorem of rotation* : but it is remarkable *how simply quaternions express it* : namely by the formula,

$$\text{XXXII.} \ldots \alpha \,.\, \beta^{-1}\rho\beta \,.\, \alpha^{-1} = \alpha\beta^{-1} \,.\, \rho \,.\, \beta\alpha^{-1},$$

in which α, β, ρ may denote *any three vectors* ; and which, as we see by the *points*, involves essentially the *associative principle of multiplication*.

(23.) Instead of conceiving that the point D, or the line OD, has been *reflected* into the position F, or OF, *with respect to* the point B, or to the line OB, with a similar *successive reflexion* from F to E, we may conceive that a point has moved *along a small semicircle*, with B for *pole*, from D to F, as indicated in fig. 69, and then along *another* small semicircle, with A for pole, from F to E ; and we see that the *result*, or *effect*, of these *two successive and semicircular motions* is *equivalent* to a motion *along an arc* DE of a *third small circle*, which is *parallel* (as before) *to the great circle through* B and A, and has a *projection* RS *thereon*, which (still as before) is *double of the given arc* BA.

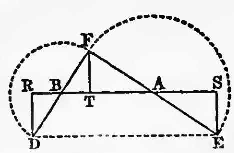

Fig. 69.

(24.) And instead of thus conceiving *two successive arcual motions of a point* D *upon a sphere*, or *two successive conical rotations of a radius* OD, considered as *compounding* themselves into *one resultant motion* of that *point*, or *rotation* of that *radius*, we may conceive an analogous *composition* of two successive *rotations of a solid body* (or *rigid system*), round *axes* passing *through a point* O, which is *fixed in space* (and *in the body*) : and so obtain a *theorem* respecting *such* rotation, which easily suggests itself from what precedes, and on which we may perhaps return.

(25.) But to draw some additional consequences from the equations VII., &c., and from the recent fig. 68, especially as regards the *Construction of the*

Fourth Proportional to three diplanar vectors, let us first remark, generally, that when we have (as in 62) a *linear equation,* of the form

$$a\alpha + b\beta + c\gamma + d\delta = 0,$$

connecting *four co-initial vectors* $\alpha \,.\,.\, \delta$, whereof *no three are complanar,* then this *fifth vector,*

$$\varepsilon = a\alpha + b\beta = - c\gamma - d\delta,$$

is evidently *complanar* (22) with α, β, and *also* with γ, δ (comp. 294, (6.)); it is therefore part of the indefinite *line of intersection* of the *plane* AOB, COD, of these *two pairs* of vectors.

(26.) And if we *divide* this fifth vector ε by the two (generally unequal) *scalars,*

$$a + b, \quad \text{and} \quad - c - d,$$

the two (generally unequal) *vectors,*

$$(a\alpha + b\beta) : (a + b), \quad \text{and} \quad (c\gamma + d\delta) : (c + d),$$

which are obtained as the *quotients* of these two divisions, are (comp. 25, 64) the vectors of two (generally distinct) *points of intersection,* of *lines* with *planes,* namely the two following:

$$\text{AB} \cdot \text{OCD}, \quad \text{and} \quad \text{CD} \cdot \text{OAB}.$$

(27.) When the *two lines,* AB and CD, happen to *intersect each other,* the two last-mentioned *points coincide;* and thus we recover, in a new way, the *condition* (63), for the *complanarity* of the *four points* O, A, B, C, or for the *termino-complanarity* of the *four vectors* α, β, γ, δ; namely the equation

$$a + b + c + d = 0,$$

which may be compared with 294, XLV., and L.

(28.) Resuming now the recent equations VII., and introducing the new vector,

$$\text{XXXIII.} \ldots \lambda = l\alpha - m\beta = \tfrac{1}{2}(\varepsilon - \delta),$$

which gives,

$$\text{XXXIV.} \ldots \mathrm{S}\gamma\lambda = 0, \quad \text{and} \quad \text{XXXV.} \ldots \mathrm{T}\lambda = \sqrt{(r^2 - n^2)} = r \sin c'.$$

we see that the two arcs BA, DE, prolonged, meet in a point L (comp. fig. 68), for which OL = Uλ, and which is *distant by a quadrant from* C: a result which may be confirmed by elementary considerations, because (by a well-known theorem respecting *transversal arcs*) the *common bisector* BA of the two sides, DE and EF, must meet the *third side* in a point L, for which

$$\sin \text{DL} = \sin \text{EL}.$$

(29.) To prove *by quaternions* this last *equality of sines*, and to assign their common value, we have only to observe that by XXXIII.,

$$\text{XXXVI.} \ldots \ \mathrm{V}\delta\lambda = \mathrm{V}\epsilon\lambda = \tfrac{1}{2}\mathrm{V}\delta\epsilon;$$

in which, $\mathrm{T}\delta\lambda = \mathrm{T}\epsilon\lambda = r^2 \sin c'$, and $\mathrm{TV}\delta\epsilon = r^2 \sin 2c'$;

the *sines* in question are therefore (by 204, XIX.),

$$\text{XXXVI}'. \ldots \ \mathrm{TVU}\delta\lambda = \mathrm{TVU}\epsilon\lambda = \tfrac{1}{2}r^2 \sin 2c' : r^2 \sin c' = \cos c'.$$

(30.) On similar principles, we may interpret the two *vector-equations*,

$$\text{XXXVII.} \ldots \ \mathrm{V}\beta\lambda = l\mathrm{V}\beta\alpha, \quad \mathrm{V}\alpha\lambda = m\mathrm{V}\beta\alpha,$$

in which $\text{XXXVIII.} \ldots \ \mathrm{T}\lambda : \mathrm{TV}\beta\alpha = r \sin c' : \sin c = \tan c' : \tan c,$

an equivalent to the *trigonometric* equations,

$$\text{XXXIX.} \ldots \frac{\tan \mathrm{CD}}{\tan \mathrm{AB}} = \frac{\cos \mathrm{BC}}{\sin \mathrm{BL}} = \frac{\cos \mathrm{AC}}{\sin \mathrm{AL}}.$$

(31.) Accordingly, if we let fall the perpendicular CQ on AB (see again fig. 68), so that Q bisects RS, and if we determine two new points M, N by the arcual equations,

$$\text{XL.} \ldots \frown \mathrm{LM} = \frown \mathrm{AB} = \frown \mathrm{QR}, \qquad \frown \mathrm{LN} = \frown \mathrm{CD},$$

the arcs MR, ND will be *quadrants*; and because the angle at R is *right* by construction (18.), M is the *pole* of DR, and DM is a quadrant; whence D is the pole of MN and the angle LNM is right: conceiving then that the arcs CA and CB are drawn, we have three triangles [BCQ, ACQ, and LMN], right-angled at Q and N, which show, by elementary principles, that the three trigonometric quotients in XXXIX. have in fact a common value, namely cos CQ, or cos L.

(32.) To prove this *last* result by *quaternions*, and *without* employing the auxiliary points M, N, Q, R, we have the transformations,

$$\text{XLI.} \ldots \cos \mathrm{L} = \mathrm{SU}\frac{\mathrm{V}\beta\alpha}{\mathrm{V}\delta\epsilon} = \mathrm{SU}\frac{\mathrm{V}\beta\alpha}{\gamma\lambda} = \mathrm{T}\frac{\lambda}{\mathrm{V}\beta\alpha} . \mathrm{S}\frac{\beta\alpha}{\gamma\lambda} = \mathrm{T}\frac{\lambda}{\mathrm{V}\beta\alpha};$$

because

$$\text{XLII.} \ldots \ \delta = n\gamma - \lambda, \quad \epsilon = n\gamma + \lambda, \quad \mathrm{V}\delta\epsilon = 2n\gamma\lambda, \quad \mathrm{UV}\delta\epsilon = \mathrm{U}\gamma\lambda,$$

and

$$\text{XLIII.} \ldots \ \mathrm{S}\frac{\beta\alpha}{\gamma\lambda} = \frac{\mathrm{S}\beta\alpha\gamma\lambda}{(\gamma\lambda)^2} = -\mathrm{S}\beta\alpha^{-1}\gamma\lambda^{-1} = -\mathrm{S}\delta\lambda^{-1} = 1;$$

it being remembered that $\lambda \perp \gamma$, whence

$$\mathrm{V}\gamma\lambda = \gamma\lambda = -\lambda\gamma, \quad (\gamma\lambda)^2 = -\gamma^2\lambda^2 = \lambda^2, \quad \mathrm{S}\gamma\lambda^{-1} = 0.$$

(33.) At the same time we see that if P be (as before) the positive pole of BA, and if K, K′ be the negative and positive poles of DE, while L′ is the negative (as L is the positive) pole of CQ, whereby all the letters in fig. 68 have their significations determined, we may write,

$$\text{XLIV.}\ldots \text{OP} = \mathrm{UV}\beta\alpha\,; \quad \text{OK}' = \gamma\mathrm{U}\lambda\,; \quad \text{OK} = -\gamma\mathrm{U}\lambda\,; \quad \text{OL}' = -\mathrm{U}\lambda\,;$$

while $\text{OL} = +\,\mathrm{U}\lambda$, as before.

(34.) Writing also,

$$\text{XLV.}\ldots \kappa = -\gamma\lambda, \quad \text{or} \quad \lambda = \gamma\kappa, \quad \text{and} \quad \mu = \beta\alpha^{-1}\lambda,$$

so that $\text{XLV}'.\ldots \text{OK} = \mathrm{U}\kappa, \quad \text{and} \quad \text{OM} = \mathrm{U}\mu,$

we have $\text{XLVI.}\ldots \beta\alpha^{-1}\,.\,\gamma = \mu\lambda^{-1}\,.\,\lambda\kappa^{-1} = \mu\kappa^{-1}\,;$

this *fourth proportional, to the three equally long but diplanar vectors*, α, β, γ, is therefore a *versor*, of which the *representative arc* (162) is KM, and the *representative angle* (174) is KDM, or L′DR, or EDP; and we may write for this versor, or quaternion, the expression:

$$\text{XLVII.}\ldots \beta\alpha^{-1}\gamma = \cos \text{L}'\text{DR} + \text{OD}\,.\,\sin \text{L}'\text{DR}.^*$$

(35.) The *double* of this representative *angle* is the *sum* of the *two base-angles* of the *isosceles triangle* DPE; and because the two other triangles, EPF′, F′PD, are *also* isosceles (19.), the *lune* FF′ shows that this sum is what *remains*, when we *subtract the vertical angle* F, of the triangle DEF, from the *sum of the supplements* of the two base-angles D and E of that triangle; or when we subtract the *sum of the three angles* of the same triangle *from four right angles*. We have therefore this very simple expression for the *Angle of the Fourth Proportional*:

$$\text{XLVIII.}\ldots \angle\,\beta\alpha^{-1}\gamma = \text{L}'\text{DR} = \pi - \tfrac{1}{2}(\text{D} + \text{E} + \text{F}).$$

(36.) Or, if we introduce the *area*, or the *spherical excess*, say Σ, of the triangle DEF, writing thus

$$\text{XLIX.}\ldots \Sigma = \text{D} + \text{E} + \text{F} - \pi,$$

we have these other expressions:

$$\text{L.}\ldots \angle\,\beta\alpha^{-1}\gamma = \tfrac{1}{2}\pi - \tfrac{1}{2}\Sigma\,; \quad \text{LI.}\ldots \beta\alpha^{-1}\gamma = \sin\tfrac{1}{2}\Sigma + r^{-1}\delta\cos\tfrac{1}{2}\Sigma\,;$$

because $\text{OD} = \mathrm{U}\delta = r^{-1}\delta$, by XII.

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* [Since $\beta\alpha^{-1}\gamma\,.\,\gamma\,.\,(\beta\alpha^{-1}\gamma)^{-1} = \beta\alpha^{-1}\,.\,\gamma\,.\,(\beta\alpha^{-1})^{-1} = \gamma'$ suppose, C is brought to a point C′ by a conical rotation round OD or round OP′ where P′ is the opposite of P (XXIX.). Hence C and C′ are the points of intersection of small circles whose poles are D and P′, and C′ is the reflexion of C to the great circle PD. This shows that the angle of the quaternion $\beta\alpha^{-1}\gamma$ is CDP.]

(37.) Having thus expressed $\beta\alpha^{-1}\gamma$, we require no new appeal to the figure, in order to express this *other* fourth proportional, $\gamma\alpha^{-1}\beta$, which is the *negative* of its *conjugate*, or has an *opposite scalar*, but an *equal vector part* (comp. 204, (1.), and 295, (9.)): the geometrical difference being merely this, that because the rotation round α from β to γ has been supposed to be *negative*, the rotation round α from γ to β must be, on the contrary, *positive*.

(38.) We may thus write, at once,

$$\text{LII.} \ldots \gamma\alpha^{-1}\beta = -\text{K}\beta\alpha^{-1}\gamma = -\sin\tfrac{1}{2}\Sigma + r^{-1}\delta\cos\tfrac{1}{2}\Sigma;$$

and we have, for the *angle* of this *new* fourth proportional, to the *same three vectors* α, β, γ, of which the *second* and *third* have merely *changed places* with each other, the formula:

$$\text{LIII.} \ldots \angle\gamma\alpha^{-1}\beta = \text{RDL} = \tfrac{1}{2}(\text{D} + \text{E} + \text{F}) = \tfrac{1}{2}\pi + \tfrac{1}{2}\Sigma.$$

(39.) But the *common vector part* of these *two* fourth proportionals is δ, by VII.; we have therefore, by XI.,

$$\text{LIV.} \ldots r = \cos\tfrac{1}{2}\Sigma; \quad e = \pm\sin\tfrac{1}{2}\Sigma;$$

the upper sign being taken, when the rotation round α from β to γ is negative, as above supposed.

(40.) It follows by (6.) that when the *sides* $2a'$, $2b'$, $2c'$, of a spherical triangle DEF, of which the *area* is Σ, are *bisected* by the *corners* A, B, C of *another* spherical triangle, of which the sides* are a, b, c, then

$$\text{LV.} \ldots \cos a : \cos a' = \cos b : \cos b' = \cos c : \cos c' = \cos\tfrac{1}{2}\Sigma.$$

(41.) It follows also, from what has been recently shown, that the *angle* RDK, or MDN, or the *arc* MN in fig. 68, *represents the semi-area* of the *bisected triangle* DEF; whence, by the right-angled triangle LMN, we can infer that the *sine* of this *semi-area* is equal to the *sine of a side* of the *bisecting triangle* ABC, multiplied into the *sine of the perpendicular*, let fall upon that side from the opposite corner of the latter triangle; because we have

$$\text{LVI.} \ldots \sin\tfrac{1}{2}\Sigma = \sin\text{MN} = \sin\text{LM} \,.\, \sin\text{L} = \sin\text{AB} \,.\, \sin\text{CQ}.$$

(42.) The same conclusion can be drawn immediately, by quaternions, from the expression,

$$\text{LVII.} \ldots \sin\tfrac{1}{2}\Sigma = e = \text{S}\alpha\beta\gamma = \text{S}(\text{V}\beta\alpha \,.\, \gamma^{-1}) = \text{TV}\beta\alpha \,.\, \text{SU}(\text{V}\beta\alpha : \gamma);$$

in which one factor is the sine of AB, and the other factor is the cosine of CP, or the sine of CQ.

* These *sides abc*, of the *bisecting triangle* ABC, have been hitherto supposed for simplicity (1.) to be *each less than a quadrant*, but it will be found that the *formula* LV. holds good, *without any such restriction*.

(43.) Under the same conditions, since

$$\text{LVIII.} \ldots a = \mathrm{U}(\varepsilon + \zeta) = \tfrac{1}{2}l^{-1}(\varepsilon + \zeta), \ \&\text{c.},$$

we may write also,

$$\text{LIX.} \ldots \sin \tfrac{1}{2}\Sigma = \mathrm{SU}(\varepsilon + \zeta)(\zeta + \delta)(\delta + \varepsilon) = \mathrm{S}\delta\varepsilon\zeta : 4lmn;$$

in which, by IV. and XIII.,

$$\text{LX.} \ldots 4lmn = -\mathrm{S}(\delta + \varepsilon)(\varepsilon + \zeta) = r^2 - \mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon).$$

(44.) Hence also, by LIV.,

$$\text{LXI.} \ldots \cos \tfrac{1}{2}\Sigma = r = (r^3 - r\mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon)) : 4lmn;$$

$$\text{LXII.} \ldots \tan \tfrac{1}{2}\Sigma = \frac{e}{r} = \frac{\mathrm{S}\delta\varepsilon\zeta}{r^3 - r\mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon)} = \frac{\mathrm{SU}\delta\varepsilon\zeta}{1 - \mathrm{SU}\varepsilon\zeta - \mathrm{SU}\zeta\delta - \mathrm{SU}\delta\varepsilon};$$

and under *this last form*, we have a *general expression for the tangent of half the spherical opening at* O, *of any triangular pyramid* ODEF, *whatever the lengths* $\mathrm{T}\delta$, $\mathrm{T}\varepsilon$, $\mathrm{T}\zeta$ *of the edges at* O *may be*.

(45.) As a verification, we have

$$\text{LXIII.} \ldots (4lmn)^2 = -\tfrac{1}{4}(\varepsilon + \zeta)^2(\zeta + \delta)^2(\delta + \varepsilon)^2 = 2(r^2 - \mathrm{S}\varepsilon\zeta)(r^2 - \mathrm{S}\zeta\delta)(r^2 - \mathrm{S}\delta\varepsilon);$$

but the elimination of $\frac{1}{2}\Sigma$ between LIX. LXI. gives

$$\text{LXIV.} \ldots (4lmn)^2 = (\mathrm{S}\delta\varepsilon\zeta)^2 + (r^3 - r(\mathrm{S}\varepsilon\zeta + \mathrm{S}\zeta\delta + \mathrm{S}\delta\varepsilon))^2;$$

we ought then to find that

$$\text{LXV.} \ldots (\mathrm{S}\delta\varepsilon\zeta)^2 = r^6 - r^2\{(\mathrm{S}\varepsilon\zeta)^2 + (\mathrm{S}\zeta\delta)^2 + (\mathrm{S}\delta\varepsilon)^2\} - 2\mathrm{S}\varepsilon\zeta\mathrm{S}\zeta\delta\mathrm{S}\delta\varepsilon,$$

if $\delta^2 = \varepsilon^2 = \zeta^2 = -r^2$; and in fact this equality results immediately from the general formula 294, LIII.

(46.) Under the same condition, respecting the equal lengths of δ, ε, ζ, we have also the formula,

$$\text{LXVI.} \ldots -\mathrm{V}(\delta + \varepsilon)(\varepsilon + \zeta)(\zeta + \delta) = 2\delta(r^2 - \mathrm{S}\varepsilon\zeta - \mathrm{S}\zeta\delta - \mathrm{S}\delta\varepsilon) = 8lmn\delta;$$

whence other verifications may be derived.

(47.) If σ denote the *area* of the bisecting triangle* ABC, the general principle LXII. enables us to infer that

$$\text{LXVII.} \ldots \tan\frac{\sigma}{2} = \frac{\mathrm{S}\alpha\beta\gamma}{1 - \mathrm{S}\beta\gamma - \mathrm{S}\gamma\alpha - \mathrm{S}\alpha\beta} = \frac{e}{1 + l + m + n}$$

$$= \frac{\sin c \sin p}{1 + \cos a + \cos b + \cos c},$$

* The reader will observe that the more usual symbol Σ, for this area of ABC, is *here* employed (6.) to denote the area of the *exscribed* triangle DEF.

if p denote the perpendicular CQ from C on AB, so that

$$e = \sin c \sin p = \sin b \sin c \sin A = \&c. \text{ (comp. 210, (21.))}.$$

(48.) But, by (IX.) and (XI.),

$$\text{LXVIII.} \ldots e^2 + (1 + l + m + n)^2 = 2(1 + l)(1 + m)(1 + n)$$
$$= \left(4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}\right)^2;$$

hence the cosine and sine of the *new* semi-area are,

$$\text{LXIX.} \ldots \cos \frac{\sigma}{2} = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}};$$

$$\text{LXX.} \ldots \sin \frac{\sigma}{2} = \frac{\sin \frac{a}{2} \sin \frac{b}{2} \sin \mathrm{c}}{\cos \frac{c}{2}} = \&c.$$

(49.) Returning to the *bisected triangle*, DEF, the last formula gives,

$$\text{LXXI.} \ldots \sin \tfrac{1}{2}\Sigma = \frac{\sin a' \sin b' \sin \mathrm{F}}{\cos c'} = \sin p' \sin c \sec c',$$

if p' denote the perpendicular from F on the bisecting arc AB, or FT in fig. 68; but $\cos \frac{1}{2}\Sigma = \cos c \sec c'$, by LV.; hence

$$\text{LXXII.} \ldots \tan \tfrac{1}{2}\Sigma = \sin p' \tan c = \sin \mathrm{FT} \,.\, \tan \mathrm{AB}.$$

Accordingly, in fig. 68, we have, by spherical trigonometry,

$$\sin \mathrm{FT} = \sin \mathrm{ES} = \sin \mathrm{LE} \sin \mathrm{L} = \cos \mathrm{LN} \sin \mathrm{MN} \operatorname{cosec} \mathrm{LM} = \tan \mathrm{MN} \cot \mathrm{AB}.$$

(50.) The *arc* MN, which thus represents in *quantity* the semiarea of DEF, has its *pole* at the point D, and may be considered as the *representative arc* (162) of a certain *new quaternion* Q, or of its *versor*, of which the *axis* is the *radius* OD, or $\mathrm{U}\delta$; and this new quaternion may be thus expressed:

$$\text{LXXIII.} \ldots Q = \delta\gamma\alpha\beta = -\delta^2 + \delta \mathrm{S}\alpha\beta\gamma = r^2 + e\delta;$$

its tensor and versor being, respectively,

$$\text{LXXIV.} \ldots \mathrm{T}Q = r = \cos \tfrac{1}{2}\Sigma; \quad \text{LXXV.} \ldots \mathrm{U}Q = \cos \tfrac{1}{2}\Sigma + \mathrm{OD} \,.\, \sin \tfrac{1}{2}\Sigma.$$

(51.) An important transformation of this last *versor* may be obtained as follows:

$$\text{LXXVI.} \ldots \mathrm{U}Q = \mathrm{U}(\delta\gamma^{-1} \,.\, \alpha\zeta^{-1} \,.\, \zeta\beta^{-1}) = (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}};$$

so that

$$\text{LXXVII.} \ldots \tfrac{1}{2}\Sigma = \angle\, Q = \angle\, \delta\gamma\alpha\beta = \angle\, (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}};$$

these *powers* of quaternions, with *exponents* each = $\frac{1}{2}$, being interpreted as *square roots* (199, (1.)), or as equivalent to the symbols $\sqrt{(\delta\epsilon^{-1})}$, &c.

(52.) The *conjugate* (or *reciprocal*) *versor*, $\mathrm{U}Q^{-1}$, which has NM for its *representative arc*, may be deduced from $\mathrm{U}Q$ by simply interchanging β and γ, or ϵ and ζ; the corresponding *quaternion* is,

$$\text{LXXVIII.} \ldots Q' = \mathrm{K}Q = \delta\beta\alpha\gamma = r^2 - e\delta ;$$

and we have

$$\text{LXXIX.} \ldots \mathrm{U}Q' = \cos \tfrac{1}{2}\Sigma - \mathrm{OD} \,.\, \sin \tfrac{1}{2}\Sigma = (\delta\zeta^{-1})^{\frac{1}{2}} (\zeta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\delta^{-1})^{\frac{1}{2}} ;$$

the rotation round D, from E to F, being still supposed to be negative.

(53.) Let H be any other point upon the sphere, and let OH = η; also let Σ' be the area of the *new spherical triangle*, DFH; then the same reasoning shows that

$$\text{LXXX.} \ldots \cos \tfrac{1}{2}\Sigma' + \mathrm{OD} \,.\, \sin \tfrac{1}{2}\Sigma' = (\delta\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}},$$

if the rotation round D from F to H be negative; and therefore, by *multiplication* of the two *co-axal versors*, LXXVI. and LXXX., we have by LXXV., the analogous formula:

$$\text{LXXXI.} \ldots \cos \tfrac{1}{2}(\Sigma + \Sigma') + \mathrm{OD} \,.\, \sin \tfrac{1}{2}(\Sigma + \Sigma') = (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}} ;$$

where $\Sigma + \Sigma'$ denotes the *area of the spherical quadrilateral*, DEFH.

(54.) It is easy to *extend* this result to the *area* of *any spherical polygon*, or to the *spherical opening* (44.) of *any pyramid*; and we may even conceive an extension of it, as a *limit*, to the *area of any closed curve upon the sphere*, considered as decomposed into an *indefinite number of indefinitely small triangles*, with some *common vertex*, such as the point D, on the spheric surface, and with *indefinitely small arcs* EF, FH, .. *of the curve*, for their respective *bases*: or to the *spherical opening of any cone*, expressed thus as the *Angle of a Quaternion*, which is the *limit** *of the product of indefinitely many factors, each equal to the square-root of a quaternion, which differs indefinitely little from unity.*

---

* *This Limit* is closely *analogous* to a *definite integral*, of the ordinary kind; or rather, we may say that it *is* a *Definite Integral*, but one of a *new kind*, which could not easily have been introduced without Quaternions. In fact, if we did not employ the *non-commutative property* (168) of *quaternion multiplication*, the *Products* here considered would evidently become each *equal to unity*: so that they would furnish no expressions for spherical or other *areas*, and in short, it would be useless to speak of them. On the contrary, when that property or principle of multiplication is introduced, these *expressions of product-form* are found, as above, to have extremely useful significations in *spherical geometry*; and it will be seen that they suggest and embody a remarkable *theorem*, respecting the *resultant of rotations of a system*, round *any number of successive axes*, all passing *through one fixed point*, but in other respects succeeding each other with any gradual or sudden changes.

(55.) To assist the recollection of this result, it may be stated as follows (comp. 180, (3.) for the definition of an *arcual sum*):—

"*The Arcual Sum of the Halves of the Successive Sides of any Spherical Polygon, is equal to an arc of a Great Circle, which has the Initial (or Final) Point of the Polygon for its Pole, and represents the Semi-area of the Figure*"; it being understood that this *resultant arc* is *reversed* in *direction*, when the *half-sides* are (arcually) *added* in an *opposite order*.

(56.) As regards the *order* thus referred to, it may be observed that in the *arcual addition*, which corresponds to the *quaternion multiplication* in LXXVI., we conceive a point to *move*, *first*, from B to F, through *half* the *arc* DF; which half-side of the triangle DEF answers to the *right-hand factor*, or square-root $(\zeta\delta^{-1})^{\frac{1}{2}}$. We then conceive the same point to move *next* from F to A, through half the arc FE, which answers to the factor placed immediately to the *left* of the former; having thus moved, on the whole, *so far*, through the *resultant arc* BA (as a *transvector*, 180, (3.)), or through any *equal arc* (163), such as ML in fig. 68. And finally, we conceive a motion through half the arc ED, or through any arc *equal* to that half, such as the arc LN in the same figure, to correspond to the extreme *left-hand factor* in the formula; the *final resultant* (or *total transvector arc*), which answers to the *product of the three square roots*, as *arranged* in the formula, being thus represented by the *final arc* MN, which has the point D for its *positive pole*, and the *half-area*, $\frac{1}{2}\Sigma$, for the *angle* (51.) of the *quaternion* (or *versor*) *product* which it *represents*.

(57.) Now the direction of *positive rotation* on the sphere has been supposed to be that round D, from F to E; and therefore *along the perimeter*, in the *order* DFE, as seen* from any point of the surface *within* the triangle: that is, in the order in which the *successive sides* DF, FE, ED have been taken, before *adding* (or *compounding*) their *halves*. And accordingly, in the *conjugate* (or *reciprocal*) formula LXXIX., we took the *opposite order*, DEF, in proceeding as usual from right-hand to left-hand factors, whereof the *former* are supposed to be multiplied *by*† the *latter*; while the result was, as we saw in (52.), a *new*

* In this and other cases of the sort, the spectator is imagined to stand *on the point* of the sphere, *round which* the *rotation on the surface* is conceived to be performed; his body being *outside the sphere*. And similarly when we say, for example, that the *rotation round the line*, or *radius*, OA, *from the line* OB *to the line* OC, is *negative* (or left-handed), as in the recent figures, we mean that such would appear to be the direction of that rotation, to a person standing thus with his *feet on* A, and with his *body* in the direction of OA *prolonged*: or else standing *on the centre* (or origin) O, with his *head at the point* A Compare 174, II.; 177; and the second note to page 152.

† Compare the Notes to pages 147, 159.

versor, in the expression for which, the *area* Σ of the triangle was simply changed to its own *negative*.

(58.) To give an example of the reduction of the area to *zero*, we have only to conceive that the *three points* D, E, F are *co-arcual* (165), or situated *on one great circle*; or that the *three lines* δ, ε, ζ are *complanar*. For this case, by the laws* of *complanar quaternions*, we have the formula,

$$\text{LXXXII.} \ldots (\delta\varepsilon^{-1})^{\frac{1}{2}} (\varepsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}} = 1, \quad \text{if} \quad S\delta\varepsilon\zeta = 0;$$

thus $\cos\frac{1}{2}\Sigma = 1$, and $\Sigma = 0$.

(59.) Again, in (53.) let the point H be co-arcual with D and F, or let $S\delta\zeta\eta = 0$; then, because

$$\text{LXXXII}'. \ldots (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}} = (\zeta\delta^{-1})^{\frac{1}{2}}, \quad \text{if} \quad S\delta\zeta\eta = 0,$$

the product of *four factors* LXXXI. reduces itself to the product of *three* factors LXXVI.; the *geometrical reason* being evidently that in this case the *added area* Σ' *vanishes*; so that the *quadrilateral* DEFH has only the *same* area as the *triangle* DEF.

(60.) But this *added* area (53.) may even have a *negative*† *effect*, as for example when the new point H falls on the old side DE. Accordingly, if we write

$$\text{LXXXIII.} \ldots Q_1 = (\varepsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\varepsilon^{-1})^{\frac{1}{2}},$$

and denote the product LXXXI. of four square-roots by Q_2, we shall have the transformation,

$$\text{LXXXIV.} \ldots Q_2 = (\delta\varepsilon^{-1})^{\frac{1}{2}} Q_1 (\varepsilon\delta^{-1})^{\frac{1}{2}}, \quad \text{if} \quad S\delta\varepsilon\eta = 0;$$

which shows (comp. (15.)) that in this case the *angle* of the *quaternary product* Q_2 is that of the *ternary product* Q_1, or the half-area of the *triangle* EFH (= DEF − DHF), although the *axis* of Q_2 is *transferred* from the position of the axis of Q_1, by a *rotation* round the pole of the arc ED, which brings it from OE to OD.

(61.) From this example, it may be considered to be sufficiently evident, how the formula LXXXI. may be applied and extended, so as to represent (comp. (54.)) the *area of any closed figure on the sphere*, with *any assumed point*

* Compare the Second Chapter of the Second Book.

† In some investigations respecting *areas on a sphere*, it may be convenient to *distinguish* (comp. (28.), (63.) between the *two symbols* DEF and DFE, and to consider them as denoting two *opposite triangles*, of which the *sum* is *zero*. But for the present, we are content to express this *distinction*, by means of the two *conjugate quaternion products* (51.) and (52.).

D *on the surface* as a sort of *spherical origin*; even when this *auxiliary point* is *not* situated *on the perimeter*, but is either *external* or *internal* thereto.

(62.) A *new quaternion* Q_0, with the *same axis* OD as the quaternion Q of (50.), but with a *double angle*, and with a *tensor* equal to *unity*, may be formed by simply *squaring the versor* UQ; and although this *squaring* cannot be effected by *removing the fractional exponents*,* in the formula LXXVI., yet it can easily be accomplished in other ways. For example we have, by LXXIII. LXXIV., and by VII. IX. X., the transformations†:

$$\text{LXXXV.}\ldots\ Q_0 = UQ^2 = r^{-2}(\delta\gamma\alpha\beta)^2 = -\delta^{-2}.\gamma\alpha\beta\delta.\delta\gamma\alpha\beta$$
$$= -(\gamma\alpha\beta)^2 = -(e-\delta)^2 = r^2 - e^2 + 2e\delta;$$

and in fact, because $\delta = r.\text{OD}$, by XII., the trigonometric values LIV. for r and e enable us to write this last result under the form,

$$\text{LXXXVI.}\ldots\ Q_0 = -(\gamma\alpha\beta)^2 = \cos\Sigma + \text{OD}.\sin\Sigma.$$

(63.) To show its geometrical signification, let us conceive that ABC and LMN have the same meanings in the new fig. 70, as in fig. 68; and that $A_1B_1M_1$ are three new points, determined by the three arcual equations (163),

$$\text{LXXXVII.}\ \frown AC = \frown CA_1, \quad \frown BC = \frown CB_1, \quad \frown MN = \frown NM_1;$$

which easily conduct to this fourth equation of the same kind,

$$\text{LXXXVII}'.\ldots \frown LM_1 = \frown B_1A_1.$$

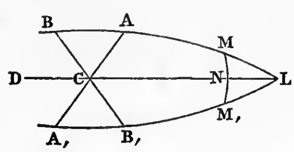

Fig. 70.

This *new arc* LM_1 *represents* thus (comp. 167, and fig. 43) the product $\alpha_1\gamma^{-1}.\gamma\beta_1^{-1} = \gamma\alpha^{-1}.\beta\gamma^{-1}$; while the old arc ML, or its equal BA (31.), represents $\alpha\beta^{-1}$; whence the arc MM_1, which has its *pole* at D, and is numerically equal to the *whole area* Σ of DEF (because MN was seen to be equal (50.) to *half* that area), represents the product $\gamma\alpha^{-1}\beta\gamma^{-1}.\alpha\beta^{-1}$, or $-(\gamma\alpha\beta)^2$, or Q_0. The formula LXXXVI. has therefore been *interpreted*, and may be said to have been *proved anew*, by these simple geometrical considerations.

(64.) We see, at the same time, how to *interpret the symbol*,

$$\text{LXXXVIII.}\ldots\ Q_0 = \frac{\gamma}{\alpha}\frac{\beta}{\gamma}\frac{\alpha}{\beta};$$

namely as denoting a *versor*, of which the *axis* is directed *to*, or *from*, the

* Compare the Note to (54.).

† The equation $\delta\gamma\alpha\beta = \gamma\alpha\beta\delta$ is *not valid generally*; but we have *here* $\delta = -V\gamma\alpha\beta$; and in general, $q\rho = \rho q$, *if* $\rho \parallel Vq$.

corner D of a certain *auxiliary spherical triangle* DEF, whereof the *sides* respectively *opposite* to D, E, F, are *bisected* (5.) by the *given points* A, B, C, according as the *rotation* round α from β to γ is *negative* or *positive*; and of which the *angle* represents, or is numerically equal to, the *area* Σ of that auxiliary triangle, at least if we *still* suppose, as we have hitherto for simplicity done (1.), that the *sides* of the *given* triangle ABC are *each less* than a *quadrant*.

298. The *case* when the *sides* of the *given* triangle are *all greater*, instead of being *all less*, than *quadrants*, may deserve next to be (although more briefly) considered; the case when they are *all equal* to *quadrants*, being reserved for a short subsequent Article: and *other* cases being easily referred to *these*, by *limits*, or by passing from a given line to its *opposite*.

(1.) Supposing now that

$$\text{I.} \ldots l < 0, \quad m < 0, \quad n < 0,$$

or that

$$\text{II.} \ldots a > \frac{\pi}{2}, \quad b > \frac{\pi}{2}, \quad c > \frac{\pi}{2},$$

we may still retain the recent equations IV. to XI.; XIII.; and XV. to XXVI., of 297; but we must *change the sign of the radical*, r, in the equations XII. and XIV., and also the *signs* of the *versors*, $U\delta$, $U\epsilon$, $U\zeta$ in XII., if we desire that the *sides* of the *auxiliary triangle*, DEF, may still be *bisected* (as in figures 67, 68) by the *corners* of the *given* triangle, ABC, of which the sides a, b, c are now each *greater* than a *quadrant*. Thus, r being still the *common tensor* of δ, ϵ, ζ, and therefore being still supposed to be *itself* > 0, we must write now, under these *new conditions* I. or II., the new equations,

$$\text{III.} \ldots \text{OD} = -U\delta = -r^{-1}\delta; \quad \text{OE} = -U\epsilon = -r^{-1}\epsilon; \quad \text{OF} = -U\zeta = -r^{-1}\zeta;$$

$$\text{IV.} \ldots \cos a = -r\cos a', \quad \cos b = -r\cos b', \quad \cos c = -r\cos c'.$$

(2.) The equations IV. and VIII. of 297 still holding good, we may now write,

$$\text{V.} \ldots \pm 2r\cos a'\cos b'\cos c' = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1,$$

according as we adopt positive values (297), or negative values (298), for the cosines l, m, n of the sides of the bisecting triangle; the value of r being still supposed to be positive.

(3.) It is not difficult to prove (comp. 297, LIV., LXIX.), that

$$\text{VI.} \ldots r = \pm\cos\tfrac{1}{2}\Sigma, \quad \text{according as} \quad l > 0, \&\text{c.}, \quad \text{or} \quad l < 0, \&\text{c.};$$

the recent formula V. may therefore be written *unambiguously* as follows:

$$\text{VII.} \ldots 2\cos a'\cos b'\cos c'\cos\tfrac{1}{2}\Sigma = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1;$$

and the formula 297, LV. continues to hold good.

(4.) In like manner, we may write, without an ambiguous sign (comp. 297, LI.), the following *expression for the fourth proportional* $\beta\alpha^{-1}\gamma$ *to three unit-vectors* α, β, γ, the rotation round the first from the second to the third being negative:

$$\text{VIII.} \ldots \beta\alpha^{-1}\gamma = \sin\tfrac{1}{2}\Sigma + \text{OD}\,.\cos\tfrac{1}{2}\Sigma;$$

where the scalar part changes sign, when the rotation is reversed.

(5.) It is, however, to be observed, that although this *formula* VIII. holds good, not only in the cases of the last article and of the present, but also in that which has been reserved for the next, namely when $l = 0$, &c.; yet because, in the *present case* (298) we have the area $\Sigma > \pi$, the *radius* OD is no longer the (positive) *axis* $\mathrm{U}\delta$ of the fourth proportional $\beta\alpha^{-1}\gamma$; nor is $\frac{1}{2}\pi - \frac{1}{2}\Sigma$ any longer, as in 297, L., the (positive) *angle* of that versor. On the contrary we have *now*, for this axis and angle, the expressions:

$$\text{IX.} \ldots \mathrm{Ax}\,.\,\beta\alpha^{-1}\gamma = \text{DO} = -\,\text{OD}; \qquad \text{X.} \ldots \angle\,\beta\alpha^{-1}\gamma = \tfrac{1}{2}(\Sigma - \pi).$$

(6.) To illustrate these results by a construction, we may remark that if, in fig. 67, the bisecting arcs BC, CA, AB be supposed each greater than a quadrant, and if we proceed to form from it a new figure, analogous to 68, the perpendicular CQ will also exceed a quadrant, and the poles P and K will fall *between* the points C and Q; also M and R will fall on the arcs LQ and QL′ *prolonged*: and although the arc KM, or the angle KDM, or L′DR, or EDP, may *still* be considered, as in 297, (34.), to *represent* the *versor* $\beta\alpha^{-1}\gamma$, yet the corresponding *rotation* round the point D is now of a *negative* character.

(7.) And as regards the *quantity* of this rotation, or the magnitude of the *angle* at D, it is again, as in fig. 68, a base-angle of one of three isosceles triangles, with P for their common vertex; but we have now, as in fig. 71, a *new arrangement*, in virtue of which this angle is to be found by halving what remains, when the sum of the supplements of the angles at D and E, in the triangle DEF, is subtracted *from* the angle at F, instead of our subtracting (as in 297, (35.)) the latter angle from the former sum; it is therefore *now*, in agreement with the recent expression X.,

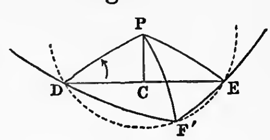

Fig. 71.

$$\text{XI.} \ldots \angle\,\beta\alpha^{-1}\gamma = \tfrac{1}{2}(\text{D} + \text{E} + \text{F}) - \pi.$$

(8.) The negative of the conjugate of the formula VIII. gives,

$$\text{XII.} \ldots \gamma\alpha^{-1}\beta = -\sin\tfrac{1}{2}\Sigma + \text{OD}\,.\cos\tfrac{1}{2}\Sigma;$$

and by taking the negative of the square of this equation, we are conducted to the following:

$$\text{XIII.} \ldots \frac{\gamma}{\alpha}\frac{\beta}{\gamma}\frac{\alpha}{\beta} = -(\gamma\alpha^{-1}\beta)^2 = \cos\Sigma + \text{OD}\,.\sin\Sigma;$$

a result which had only been proved before (comp. 297, (62.), (64.)) for the case $\Sigma < \pi$; and in which it is still supposed that the rotation round α from β to γ is negative.

(9.) With the same direction of rotation, we have also the *conjugate* or *reciprocal* formula,

$$\text{XIV.} \ldots \frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma} = -(\beta\alpha^{-1}\gamma)^2 = \cos\Sigma - \text{OD}\,.\sin\Sigma.$$

(10.) If it happened that only *one* side, as AB, of the *given* triangle ABC, was greater, while each of the two others was less than a quadrant, or that we had $l > 0$, $m > 0$, but $n < 0$; and if we wished to represent the fourth proportional to α, β, γ by means of the foregoing constructions: we should only have to introduce the point C′ *opposite* to C, or to change γ to $\gamma' = -\gamma$; for thus the *new* triangle ABC′ would have *each* side greater than a quadrant, and so would fall under the case of the present Article; after employing the construction for which, we should only have to change the resulting versor to its negative.

(11.) And in like manner, if we had l and m negative, but n positive, we might again substitute for C its opposite point C′, and so fall back on the construction of Art. 297: and similarly in other cases.

(12.) In general, if we *begin* with the equations 297, XII., attributing any *arbitrary* (but positive) value to the *common tensor*, r, of the three co-initial vectors δ, ε, ζ, of which the *versors*, or the *unit-vectors* $\text{U}\delta$, &c., terminate at the corners of a *given* or *assumed triangle* DEF, with sides $= 2a'$, $2b'$, $2c'$, we may then suppose (comp. fig. 67) that *another* triangle ABC, with sides denoted by a, b, c, and with their cosines denoted by l, m, n, is *derived* from this one, by the condition of *bisecting* its *sides*; and therefore by the equations (comp. 297, LVIII.),

$$\text{XV.} \ldots \text{OA} = \alpha = \text{U}(\varepsilon + \zeta), \quad \text{OB} = \beta = \text{U}(\zeta + \delta), \quad \text{OC} = \gamma = \text{U}(\delta + \varepsilon),$$

with the relations 297, IV. V. VI., as before; or by these other equations (comp. 297, XIII. XIV.),

$$\text{XVI.} \ldots \varepsilon + \zeta = 2r\alpha\cos a', \quad \zeta + \delta = 2r\beta\cos b', \quad \delta + \varepsilon = 2r\gamma\cos c'.$$

(13.) When *this* simple construction is adopted, we have at once (comp. 297, LX.), by merely taking *scalars of products of vectors*, and *without any reference to areas* (compare however 297, LXIX., and 298, VII.), the equations,

$$\text{XVII.} \ldots 4 \cos a \cos b' \cos c' = 4 \cos b \cos c' \cos a' = 4 \cos c \cos a' \cos b'$$
$$= - r^{-2} S (\zeta + \delta)(\delta + \varepsilon) = \&c. = 1 + \cos 2a' + \cos 2b' + \cos 2c';$$

or
$$\text{XVIII.} \ldots \frac{\cos a}{\cos a'} = \frac{\cos b}{\cos b'} = \frac{\cos c}{\cos c'} = \frac{\cos a'^2 + \cos b'^2 + \cos c'^2 - 1}{2 \cos a' \cos b' \cos c'};$$

which can indeed be otherwise deduced, by the known formulæ of spherical trigonometry.

(14.) We see, then, that *according as the sum of the squares of the cosines of the half-sides, of a given or assumed spherical triangle*, DEF, *is greater than unity, or equal to unity, or less than unity, the sides of the inscribed and bisecting triangle*, ABC, *are together less than quadrants, or together equal to quadrants, or together greater than quadrants.*

(15.) Conversely, *if* the sides of a *given* spherical triangle ABC be thus *all less*, or *all greater* than *quadrants*, a triangle DEF, but only *one** such triangle, can be *exscribed* to it, so as to have its sides *bisected*, as above: the simplest process being to let fall a perpendicular, such as CQ in fig. 68, from C on AB, &c.; and then to draw new arcs, through C, &c., perpendicular to these perpendiculars, and therefore coinciding in position with the sought sides DE, &c., of DEF.

(16.) The *trigonometrical results* of recent sub-articles, especially as regards the *area*† of a spherical triangle, are probably *all* well known, as certainly *some* of them are; but they are *here* brought forward only in connexion with *quaternion formulæ*; and as one of that class, which is not irrelevant to the present subject, and *includes* the formula 294, LIII., the following may be mentioned, wherein α, β, γ denote *any three vectors*, but the *order of the factors* is important:

$$\text{XIX.} \ldots (\alpha\beta\gamma)^2 = 2\alpha^2\beta^2\gamma^2 + \alpha^2(\beta\gamma)^2 + \beta^2(\alpha\gamma)^2 + \gamma^2(\alpha\beta)^2 - 4\alpha\gamma S\alpha\beta S\beta\gamma.\ddagger$$

* In the next Article, we shall consider a case of *indeterminateness*, or of the existence of indefinitely many exscribed triangles DEF: namely, when the sides of ABC are *all equal* to quadrants.

† This opportunity may be taken of referring to an interesting Note, to pages 96, 97 of *Luby's Trigonometry* (Dublin, 1852); in which an elegant construction, connected with the area of a spherical triangle, is acknowledged as having been mentioned to Dr. Luby, by a since deceased and lamented friend, the Rev. William Digby Sadleir, F.T.C.D. A construction nearly the same, described in the sub-articles to 297, was suggested to the present writer by quaternions, several years ago.

‡ [Using the relation $V\alpha\beta\gamma S\alpha\beta\gamma = \alpha^2 V\beta\gamma S\beta\gamma + \gamma^2 V\alpha\beta S\alpha\beta + V\gamma\alpha(-\beta^2 S\gamma\alpha + 2S\alpha\beta S\beta\gamma)$, this easily follows on squaring $(V + S)\,\alpha\beta\gamma$; or multiply XIX. by β^2 and put $\alpha\beta = r$, $\beta\gamma = p$, and $\beta^2\alpha\gamma = rp$.]

(17.) And if, as in 297, (1.), &c., we suppose that α, β, γ are three *unit-vectors*, OA, OB, OC, and denote, as in 297, (47.), by σ the area of the triangle ABC, the principle expressed by the recent formula XIII. may be stated under this apparently different, but essentially equivalent form:

$$\text{XX.} \ldots \frac{\alpha+\beta}{\beta+\gamma} \cdot \frac{\gamma+\alpha}{\alpha+\beta} \cdot \frac{\beta+\gamma}{\gamma+\alpha} = \cos\sigma + \alpha\sin\sigma;$$

which admits of several verifications.

(18.) We may, for instance, transform it as follows (comp. 297, LXVII.):

$$\text{XXI.} \ldots \frac{-(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{\mathrm{K}(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)} = \frac{-2e+2\alpha(1+l+m+n)}{+2e+2\alpha(1+l+m+n)}$$

$$= \frac{1+l+m+n+e\alpha}{1+l+m+n-e\alpha} = \frac{1+\alpha\tan\frac{\sigma}{2}}{1-\alpha\tan\frac{\sigma}{2}} = \frac{\cos\frac{\sigma}{2}+\alpha\sin\frac{\sigma}{2}}{\cos\frac{\sigma}{2}-\alpha\sin\frac{\sigma}{2}}$$

$$= \left(\cos\frac{\sigma}{2}+\alpha\sin\frac{\sigma}{2}\right)^2 = \cos\sigma+\alpha\sin\sigma, \text{ as above.*}$$

(19.) This seems to be a natural place for observing (comp. (16.)), that if α, β, γ, δ be *any four vectors*, the lately cited equation 294, LIII., and the square of the equation 294, XV., with δ written in it instead of ρ, conduct easily to the following very general and symmetric formula:

$$\begin{aligned} \text{XXII.} \ldots\ & \alpha^2\beta^2\gamma^2\delta^2 + (\mathrm{S}\beta\gamma\mathrm{S}\alpha\delta)^2 + (\mathrm{S}\gamma\alpha\mathrm{S}\beta\delta)^2 + (\mathrm{S}\alpha\beta\mathrm{S}\gamma\delta)^2 \\ & + 2\alpha^2\mathrm{S}\beta\gamma\mathrm{S}\beta\delta\mathrm{S}\gamma\delta + 2\beta^2\mathrm{S}\gamma\alpha\mathrm{S}\gamma\delta\mathrm{S}\alpha\delta + 2\gamma^2\mathrm{S}\alpha\beta\mathrm{S}\alpha\delta\mathrm{S}\beta\delta + 2\delta^2\mathrm{S}\alpha\beta\mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha \\ = {} & 2\mathrm{S}\gamma\alpha\mathrm{S}\alpha\beta\mathrm{S}\beta\delta\mathrm{S}\gamma\delta + 2\mathrm{S}\alpha\beta\mathrm{S}\beta\gamma\mathrm{S}\gamma\delta\mathrm{S}\alpha\delta + 2\mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha\mathrm{S}\alpha\delta\mathrm{S}\beta\delta \\ & + \beta^2\gamma^2(\mathrm{S}\alpha\delta)^2 + \gamma^2\alpha^2(\mathrm{S}\beta\delta)^2 + \alpha^2\beta^2(\mathrm{S}\gamma\delta)^2 \\ & + \alpha^2\delta^2(\mathrm{S}\beta\gamma)^2 + \beta^2\delta^2(\mathrm{S}\gamma\alpha)^2 + \gamma^2\delta^2(\mathrm{S}\alpha\beta)^2.\dagger \end{aligned}$$

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* [Since $\mathrm{U}(\beta+\gamma)$ bisects the angle between β and γ,

$$\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}} = \mathrm{U}\frac{\beta+\gamma}{\gamma} = \mathrm{U}\frac{\beta}{\beta+\gamma}; \text{ and therefore } \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}} = \frac{\mathrm{U}(\alpha+\beta)}{\beta}\,\frac{\beta}{\mathrm{U}(\beta+\gamma)}\,\frac{\mathrm{U}(\gamma+\alpha)}{\alpha}$$

$$= \frac{\alpha}{\mathrm{U}(\alpha+\beta)} \cdot \frac{\mathrm{U}(\beta+\gamma)}{\gamma} \cdot \frac{\gamma}{\mathrm{U}(\gamma+\alpha)} = \left(\frac{\alpha+\beta}{\beta+\gamma} \cdot \frac{\gamma+\alpha}{\alpha+\beta} \cdot \frac{\beta+\gamma}{\gamma+\alpha}\right)^{\frac{1}{2}}.$$

This is a direct transformation from 297, LXXVI. to XX.]

† [This may perhaps be more rapidly derived by operating on $a\alpha + b\beta + c\gamma + d\delta = 0$ by $\mathrm{S}\alpha.$, $\mathrm{S}\beta.$, $\mathrm{S}\gamma.$, and $\mathrm{S}\delta.$, and eliminating a, b, c, and d in the form of a determinant from the four results of operation.]

(20.) If then we take *any spherical quadrilateral* ABCD, and write

$$\text{XXIII.} \ldots l' = \cos \text{AD} = -\text{SU}\alpha\delta, \quad m' = \cos \text{BD} = -\text{SU}\beta\delta, \quad n' = \cos \text{CD} = -\text{SU}\gamma\delta,$$

treating α, β, γ as the unit-vectors of the points A, B, C, and l, m, n as the cosines of the arcs BC, CA, AB, as in 297, (1.), we have the equation,

$$\begin{aligned}\text{XXIV.} \ldots 1 + l^2l'^2 + m^2m'^2 + n^2n'^2 + 2lm'n' + 2mn'l' + 2nl'm' + 2lmn \\ = 2mnm'n' + 2nln'l' + 2lml'm' + l^2 + m^2 + n^2 \\ + l'^2 + m'^2 + n'^2;\end{aligned}$$

which can be confirmed by elementary considerations,* but is here given merely as an *interpretation* of the quaternion formula XXII.

(21.) In squaring the lately cited equation 294, XV., we have used the two following formulæ of transformation (comp. 204, XXII., and 210, XVIII.), in which α, β, γ may be *any three vectors*, and which are often found to be useful:

$$\text{XXV.} \ldots (\text{V}\alpha\beta)^2 = (\text{S}\alpha\beta)^2 - \alpha^2\beta^2; \quad \text{XXVI.} \ldots \text{S}(\text{V}\beta\gamma \,.\, \text{V}\gamma\alpha) = \gamma^2\text{S}\alpha\beta - \text{S}\beta\gamma\text{S}\gamma\alpha.$$

299. The *two cases*, for which the *three sides* a, b, c, of the given triangle ABC, are *all less*, or *all greater*, than *quadrants*, having been considered in the two foregoing Articles, with a reduction, in 298, (10.) and (11.), of certain other cases to these, it only remains to consider that *third* principal *case*, for which the sides of that given triangle are *all equal to quadrants*: or to inquire what is, on our general principles, the *Fourth Proportional to Three Rectangular Vectors*. And we shall find, not only that *this* fourth proportional is *not itself a Vector*, but that it does not even contain any *vector part* (292) different from zero: although, as being found to be equal to a *Scalar*, it is still *included* (131, 276) in the general conception of a *Quaternion*.

(1.) In fact, if we suppose, in 297, (1.), that

$$\text{I.} \ldots l = 0, \ m = 0, \ n = 0, \quad \text{or that} \quad \text{II.} \ldots a = b = c = \frac{\pi}{2},$$

or $\quad \text{III.} \ldots \text{S}\beta\gamma = \text{S}\gamma\alpha = \text{S}\alpha\beta = 0, \quad$ while $\quad \text{IV.} \ldots \text{T}\alpha = \text{T}\beta = \text{T}\gamma = 1,$

the formulæ 297, VII. give,

$$\text{V.} \ldots \delta = 0, \qquad \varepsilon = 0, \qquad \zeta = 0;$$

but these are the *vector parts* of the *three pairs* of *fourth proportionals* to the

* A formula equivalent to this last *equation of seventeen terms*, connecting the *six cosines* of the arcs which join, two by two, the corners of a spherical quadrilateral ABCD, is given at page 407 of Carnot's *Géométrie de Position* (Paris, 1803).

three rectangular unit-lines, α, β, γ, taken in all possible orders; and the same *evanescence of vector parts* must evidently take place, if the three given lines be only *at right angles* to each other, without being *equally long*.

(2.) Continuing, however, for simplicity, to suppose that they are unit lines, and that the rotation round α from β to γ is negative, as before, we see that we have now $r = 0$, and $e = 1$, in 297, (3.); and that thus *the six fourth proportionals reduce themselves to their scalar parts*, namely (here) to *positive or negative unity*. In this manner we find, under the supposed conditions, the values:

$$\text{VI.} \ldots \beta\alpha^{-1}\gamma = \gamma\beta^{-1}\alpha = \alpha\gamma^{-1}\beta = +1; \qquad \text{VI}'. \ldots \gamma\alpha^{-1}\beta = \alpha\beta^{-1}\gamma = \beta\gamma^{-1}\alpha = -1.$$

(3.) For example (comp. 295) we have, by the laws (182) of i, j, k, the values,

$$\text{VII.} \ldots ij^{-1}k = jk^{-1}i = ki^{-1}j = +1; \qquad \text{VII}'. \ldots kj^{-1}i = ik^{-1}j = ji^{-1}k = -1.$$

In fact, the *two fourth proportionals*, $ij^{-1}k$ and $kj^{-1}i$, are respectively equal to the *two ternary products*, $-ijk$ and $-kji$, and therefore to $+1$ and -1, by the laws included in the *Fundamental Formula* A (183).

(4.) To connect this important result with the *constructions* of the two last Articles, we may observe that when we seek, on the general plan of 298, (15.), to *exscribe a spherical triangle*, DEF, *to a given tri-quadrantal* (or *tri-rectangular*) *triangle*, ABC, as for instance to the triangle IJK (or JIK) of 181, in such a manner that the *sides* of the *new* triangle shall be *bisected* by the *corners* of the *old*, the problem is found to admit of *indefinitely many solutions*. *Any point* P may be assumed, in the *interior* of the given triangle ABC; and then, if its *reflexions* D, E, F be taken, with respect to the three *sides*, a, b, c, so that (comp. fig. 72) the *arcs* PD, PE, PF are *perpendicularly bisected* by those three sides, the three *other arcs* EF, FD, DE will be bisected by the *points* A, B, C, as required: because the arcs AE, AF have each the same length as AP, and the angles subtended at A by PE and PF are together equal to two right angles, &c.

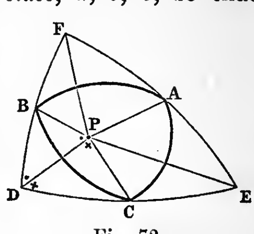

Fig. 72.

(5.) The *positions* of the *auxiliary points*, D, E, F, are therefore, in the present case, *indeterminate*, or *variable*; but the *sum* of the *angles* at those three points is *constant*, and equal to *four right angles*; because, by the *six isosceles triangles* on PD, PE, PF as bases, that sum of the three angles D, E, F is equal to the sum of the angles subtended by the sides of the given triangle ABC, at the assumed interior point P. The *spherical* *excess* of the triangle DEF is therefore equal to *two right angles*, and its area $\Sigma = \pi$; as may be otherwise seen from the same figure 72, and might have been inferred from the formula 297, LV., or LVI.

(6.) The *radius* OD, in the formula 297, XLVII., for the fourth proportional $\beta\alpha^{-1}\gamma$, becomes therefore, in the present case, *indeterminate*; but because the *angle* L′DR, or $\frac{1}{2}(\pi - \Sigma)$, in the same equation, *vanishes*, the formula becomes simply $\beta\alpha^{-1}\gamma = 1$, as in the recent equations VI.; and similarly in other examples, of the class here considered.

(7.) The conclusion, that *the Fourth Proportional to Three Rectangular Lines is a Scalar*, may in several other ways be deduced, from the principles of the present Book. For example, with the recent suppositions, we may write,

$$\text{VIII.} \ldots \beta\alpha^{-1} = -\gamma, \quad \gamma\beta^{-1} = -\alpha, \quad \alpha\gamma^{-1} = -\beta;$$
$$\text{VIII}'. \ldots \gamma\alpha^{-1} = +\beta, \quad \alpha\beta^{-1} = +\gamma, \quad \beta\gamma^{-1} = +\alpha;$$

the three fourth proportionals VI. are therefore equal, respectively, to $-\gamma^2$, $-\alpha^2$, $-\beta^2$, and consequently to $+1$; while the corresponding expressions VI′. are equal to $+\beta^2$, $+\gamma^2$, $+\alpha^2$, and therefore to -1.

(8.) Or (comp. (3.)) we may write *generally* the transformation (comp. 282, XXI.),

$$\text{IX.} \ldots \beta\alpha^{-1}\gamma = \alpha^{-2}.\beta\alpha\gamma, \quad \text{if} \quad \alpha^{-2} = 1 : \alpha^2,$$

in which the factor α^{-2} is *always* a *scalar*, whatever vector α may be; while the *vector part* of the *ternary product* $\beta\alpha\gamma$ *vanishes*, by 294, III., when the recent *conditions of rectangularity* III. are satisfied.

(9.) Conversely, this *ternary product* $\beta\alpha\gamma$, and this *fourth proportional* $\beta\alpha^{-1}\gamma$, can *never reduce themselves to scalars, unless the three vectors* α, β, γ (supposed to be all *actual* (Art. 1)) *are perpendicular each to each.*

SECTION 8.

On an equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book.

300. In the foregoing section, we naturally employed the results of preceding sections of the present Book, to assist ourselves in attaching a definite signification to the Fourth Proportional (297) to Three Diplanar Vectors; and thus, in order to *interpret the symbol* $\beta\alpha^{-1}\gamma$, we availed ourselves of the interpretations *previously* obtained, in this Third Book, of α^{-1} as a *line*, and of

$\alpha\beta$, $\alpha\beta\gamma$ as *quaternions*. But it may be interesting, and not uninstructive, to inquire *how the equivalent symbol*,

$$\text{I.} \ldots (\beta : \alpha) . \gamma, \quad \text{or} \quad \frac{\beta}{\alpha}\gamma, \quad \text{with } \gamma \textit{ not } ||| \ \alpha, \beta,$$

might have been interpreted, on the principles of the Second Book, without at first assuming as known, or even seeking to discover, *any interpretation* of the three lately mentioned symbols,

$$\text{II.} \ldots \alpha^{-1}, \quad \alpha\beta, \quad \alpha\beta\gamma.$$

It will be found that the inquiry conducts to an expression of the form,

$$\text{III.} \ldots (\beta : \alpha) . \gamma = \delta + eu;$$

where δ is the *same vector*, and e is the *same scalar*, as in the recent sub-articles to 297; while u is employed as a temporary symbol, to denote a certain *Fourth Proportional to Three Rectangular Unit Lines*, namely, to the three lines OQ, OL′, and OP in fig. 68;* so that, with reference to the construction represented by that figure, we should be led, by the principles of the Second Book, to write the equation:

$$\text{IV.} \ldots (\text{OB} : \text{OA}) . \text{OC} = \text{OD} . \cos \tfrac{1}{2}\Sigma + (\text{OL}' : \text{OQ}) . \text{OP} . \sin \tfrac{1}{2}\Sigma.$$

And when we proceed to consider *what signification* should be attached, on the principles of the same Second Book, to *that particular fourth proportional*, which is here the coefficient of $\sin \frac{1}{2}\Sigma$, and has been provisionally denoted by u, we find that although it may be regarded as being *in one sense* a *Line*, or at least *homogeneous with a line*, yet it *must not be equated to any Vector*: being rather *analogous, in Geometry, to the Scalar Unit of Algebra*, so that it may be naturally and conveniently *denoted* by the *usual symbol* 1, or + 1, or be *equated to Positive Unity*. But when we thus write $u = 1$, the last term of the formula III. or IV., of the present Article, becomes simply e, or $\sin \frac{1}{2}\Sigma$; and while this *term* (or *part*) of the result comes to be considered as a species of *Geometrical Scalar*, the *complete Expression for the General Fourth Proportional to Three Diplanar Vectors* takes the *Form of a Geometrical Quaternion*: and thus the *formula* 297, XLVII., or 298, VIII., is *reproduced*, at least if we substitute in it, for the present, $(\beta : \alpha) . \gamma$ for $\beta\alpha^{-1}\gamma$, to avoid the necessity of interpreting *here* the recent symbols II.

* ["In the abstract published in the Proceedings (Royal Irish Academy, November 11th, 1844), the words 'South, West, Up' were used at first, instead of the symbols i, j, k; and the sought fourth proportional to jik, which is here denoted by u, was called provisionally, 'Forward.'"—Preface to Lectures, p. (54).]

(1.) The construction of fig. 68 being retained, but no principles peculiar to the Third Book being employed, we may write, with the same significations of c, p, &c., as before,

$$\text{V.} \ldots \text{OB} : \text{OA} = \text{OR} : \text{OQ} = \cos c + (\text{OL}' : \text{OQ}) \sin c;$$

$$\text{VI.} \ldots \text{OC} = \text{OQ} . \cos p + \text{OP} . \sin p.$$

(2.) Admitting then, as is natural, for the purposes of the sought *interpretation*, that *distributive* property which has been *proved* (212) to hold good for the *multiplication of quaternions* (as it does for multiplication in algebra); and writing for abridgment,

$$\text{VII.} \ldots u = (\text{OL}' : \text{OQ}) . \text{OP};$$

we have the *quadrinomial expression*:

$$\text{VIII.} \ldots (\text{OB} : \text{OA}) . \text{OC} = \text{OL}' . \sin c \cos p + \text{OQ} . \cos c \cos p$$
$$+ \text{OP} . \cos c \sin p + u . \sin c \sin p;$$

in which it may be observed that *the sum of the squares of the four coefficients of the three rectangular unit-vectors*, OQ, OL′, OP, *and of their fourth proportional*, u, *is equal to unity*.

(3.) But the coefficient of *this* fourth proportional, which may be regarded as a species of *fourth unit*, is

$$\text{IX.} \ldots \sin c \sin p = \sin \text{MN} = \sin \tfrac{1}{2}\Sigma = e;$$

we must therefore expect to find that the *three other* coefficients in VIII., when divided by $\cos \frac{1}{2}\Sigma$, or by r, give quotients which are the cosines of the arcual distances of some point X upon the unit-sphere, from the three points L′, Q, P; or that a point X can be assigned, for which

$$\text{X.} \ldots \sin c \cos p = r \cos \text{L}'\text{X}; \quad \cos c \cos p = r \cos \text{QX}; \quad \cos c \sin p = r \cos \text{PX}.$$

(4.) Accordingly it is found that these three last equations are satisfied, when we substitute D for X; and therefore that we have the transformation,

$$\text{XI.} \ldots \text{OL}' . \sin c \cos p + \text{OQ} . \cos c \cos p + \text{OP} . \cos c \sin p = \text{OD} . \cos \tfrac{1}{2}\Sigma = \delta,$$

whence follow the equations IV. and III.; and it only remains to study and interpret the *fourth unit*, u, which enters as a factor into the remaining part of the quadrinomial expression VIII., without employing any principles except those of the *Second Book*: and therefore *without using the Interpretations* 278, 284, of $\beta\alpha$, &c.

301. In general, when two sets of three vectors, α, β, γ, and α', β', γ', are connected by the relation,

$$\text{I.} \ldots \frac{\beta}{\alpha} \frac{\gamma}{\gamma'} \frac{\alpha'}{\beta'} = 1, \quad \text{or} \quad \text{II.} \ldots \frac{\beta}{\alpha} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'},$$

it is natural to write this other equation,

$$\text{III.} \ldots \frac{\beta}{\alpha} \gamma = \frac{\beta'}{\alpha'} \gamma';$$

and to say that *these two fourth proportionals* (297), to α, β, γ, and to α', β', γ', are *equal to each other*: whatever the *full signification* of *each* of these two last *symbols* III., supposed for the moment to be *not yet fully known*, may be afterwards found to be. In short, we may propose to make it a *condition of the sought Interpretation*, on the principles of the Second Book, of the *phrase*,

"*Fourth Proportional to three Vectors*,"

and of either of the two equivalent *Symbols* 300, I., that the recent *Equation* III. shall *follow* from I. or II.; just as, at the commencement of that Second Book, and *before* concluding (112) that the general *Geometric Quotient* $\beta : \alpha$ of any *two lines in space* is a *Quaternion*, we made it a *condition* (103) of the *interpretation* of such a *quotient*, that the equation $(\beta : \alpha) . \alpha = \beta$ should be satisfied.

302. There are however *two tests* (comp. 287), to which the recent equation III. must be submitted, before its final adoption; in order that we may be sure of its *consistency*, Ist, with the *previous interpretation* (226) of a Fourth Proportional to *Three Complanar Vectors*, as a *Line* in their common plane; and IInd, with the *general principle* of all *mathematical language* (105), that things *equal* to the *same* thing, are to be considered as equal to *each other*. And it is found, on trial, that *both* these tests *are borne*: so that they form no objection to our *adopting the equation* 301, III., *as true by definition*, whenever the preceding equation II., or I., is satisfied.

(1.) It may happen that the first member of that equation III. is equal to a *line* δ, as in 226; namely, when α, β, γ are *complanar*. In this case, we have by II. the equation,

$$\text{IV.} \ldots \frac{\delta}{\gamma'} = \frac{\delta}{\gamma} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'}, \quad \text{or} \quad \text{IV}'. \ldots \frac{\beta'}{\alpha'} \gamma' = \delta = \frac{\beta}{\alpha} \gamma;$$

so that α', β', γ' are *also* complanar (among themselves), and the line δ is *their* fourth proportional likewise: and the equation III. is satisfied, both members being *symbols for one common line*, δ, which is *in general* situated *in the intersection of the two planes*, $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$; although those planes may happen to *coincide*, without disturbing the truth of the equation.

(2.) Again, for the more general case of *diplanarity* of α, β, γ, we may conceive that the equation* II. co-exists with this other of the same form,

$$\text{V.}\ldots\frac{\beta}{\alpha}\,\frac{\gamma}{\gamma''}=\frac{\beta''}{\alpha''};\quad\text{which gives}\quad\text{VI.}\ldots\frac{\beta}{\alpha}\gamma=\frac{\beta''}{\alpha''}\gamma'',$$

if the definition 301 be adopted. If then that definition be consistent with general principles of equality, we ought to find, by III. and VI., that this third equation between two fourth proportionals holds good:

$$\text{VII.}\ldots\frac{\beta'}{\alpha'}\gamma'=\frac{\beta''}{\alpha''}\gamma'';\quad\text{or that}\quad\text{VIII.}\ldots\frac{\beta'}{\alpha'}\,\frac{\gamma'}{\gamma''}=\frac{\beta''}{\alpha''},$$

when the equations II. and V. are satisfied. And accordingly, those two equations give, by the general principles of the Second Book, respecting quaternions considered as *quotients* of vectors, the transformation,

$$\frac{\beta'}{\alpha'}\,\frac{\gamma'}{\gamma''}=\frac{\beta}{\alpha}\,\frac{\gamma}{\gamma'}\cdot\frac{\gamma'}{\gamma''}=\frac{\beta}{\alpha}\,\frac{\gamma}{\gamma''}=\frac{\beta''}{\alpha''},\ \text{as required.}$$

303. It is then permitted to *interpret the equation* 301, III., on the principles of the Second Book, as being simply a *transformation* (as it is in algebra) of the immediately preceding equation II., or I.; and therefore to write, generally,

$$\text{I.}\ldots q\gamma=q'\gamma',\quad\text{if}\quad\text{II.}\ldots q\,(\gamma:\gamma')=q';$$

where γ, γ' are *any two vectors*, and q, q' are *any two quaternions*, which satisfy this last *condition*. Now, if v and v' be any two *right quaternions*, we have (by 193, comp. 283) the equation,

$$\text{III.}\ldots \mathrm{I}v:\mathrm{I}v'=v:v'=vv'^{-1};$$

or

$$\text{IV.}\ldots v^{-1}(\mathrm{I}v:\mathrm{I}v')=v'^{-1};\quad\text{whence}\quad\text{V.}\ldots v^{-1}.\,\mathrm{I}v=v'^{-1}.\,\mathrm{I}v',$$

by the principle which has just been enunciated. It follows, then, that "*if a right Line* ($\mathrm{I}v$) *be multiplied by the Reciprocal* (v^{-1}) *of the Right Quaternion* (v), *of which it is the Index, the Product* ($v^{-1}\mathrm{I}v$) *is independent of the Length, and of*

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* In this and other cases of reference, the numeral cited is always supposed to be the one which (with the same number) has *last* occurred before, although perhaps it may have been in connexion with a shortly preceding Article. Compare 217, (1.).

the Direction, of the Line thus operated on"; or, in other words, that *this Product has one common Value, for all possible Lines* (a) *in Space*: which *common* or constant *value* may be regarded as a kind of *new Geometrical Unit*, and is equal to what we have lately denoted, in 300, III., and VII., by the temporary symbol u; because, in the last cited formula, the line OP is the index of the right quotient OQ : OL′. Retaining, then, for the moment, this *symbol*, u, we have, *for every line* a *in space, considered as the index of a right quaternion*, v, the *four equations*:

$$\text{VI.} \ldots v^{-1}a = u; \qquad \text{VII.} \ldots a = vu; \qquad \text{VIII.} \ldots v = a : u;$$
$$\text{IX.} \ldots v^{-1} = u : a;$$

in which it is understood that $a = \mathrm{I}v$, and the three last are here regarded as being merely *transformations* of the first, which is deduced and interpreted as above. And hence it is easy to infer, that for *any given system of three rectangular lines* a, β, γ, we have the *general expression*:

$$\text{X.} \ldots (\beta : a) . \gamma = xu, \quad \text{if} \quad a \perp \beta, \ \beta \perp \gamma, \ \gamma \perp a;$$

where the *scalar co-efficient*, x, of the *new unit*, u, is determined by the equation,

$$\text{XI.} \ldots x = \pm (\mathrm{T}\beta : \mathrm{T}a) . \mathrm{T}\gamma, \quad \text{according as} \quad \text{XII.} \ldots \mathrm{U}\gamma = \pm \mathrm{Ax} . (a : \beta).$$

This coefficient x is therefore *always equal*, in *magnitude* (or absolute *quantity*), to the *fourth proportional to the lengths* of the three given lines $a\beta\gamma$; but it is *positively or negatively taken*, according as the *rotation round the third line* γ, *from the second line* β, *to the first line* a, is itself *positive* or *negative*: or in other words, according as the rotation *round the first* line, *from the second to the third*, is on the contrary *negative* or *positive* (compare 294, (3)).

(1.) In illustration of the *constancy* of that fourth proportional which has been, for the present, denoted by u, while the *system* of the three rectangular unit-lines from which it is conceived to be derived is in any manner *turned about*, we may observe that the *three* equations, or proportions,

$$\text{XIII.} \ldots u : \gamma = \beta : a; \quad \gamma : a = a : -\gamma; \quad \beta : -\gamma = \gamma : \beta,$$

conduct immediately to this *fourth* equation of the same kind,

$$\text{XIV.} \ldots u : a = \gamma : \beta, \quad \text{or*} \quad u = (\gamma : \beta) . a;$$

if we admit that this new quantity, or symbol, u, is to be operated *on at all*,

* In equations of this form, the parentheses may be omitted, though for greater clearness they are here retained.

or combined with *other* symbols, according to the general rules of vectors and quaternions.

(2.) It is, then, permitted to change the *three letters* α, β, γ, by a *cyclical permutation*, to the three other letters, β, γ, α (considered again as representing *unit-lines*), without altering the *value* of the *fourth proportional*, u; or in other words, it is allowed to make the *system of the three rectangular lines revolve, through the third part of four right angles, round the interior and co-initial diagonal of the unit-cube*, of which they are three co-initial *edges*.

(3.) And it is still more evident, that no such change of value will take place, if we merely cause the system of the *two first lines* to revolve, *through any angle*, in its own plane, *round the third line* as an axis; since thus we shall merely substitute, for the factor $\beta : \alpha$, another factor *equal* thereto. But by *combining* these two last *modes* of rotation, we can represent *any rotation whatever*, round an origin supposed to be fixed.

(4.) And as regards the *scalar ratio* of any *one* fourth proportional, such as $\beta' : \alpha' . \gamma'$, to any *other*, of the kind here considered, such as $\beta : \alpha . \gamma$, or u, it is sufficient to suggest that, without any real change in the former, we are allowed to suppose it to be so *prepared*, that we shall have

$$\text{XV.} \ldots \alpha' = \alpha; \quad \beta' = \beta; \quad \gamma' = x\gamma;$$

x being some scalar coefficient, and representing the ratio required.

304. In the more general case, when the three given lines are *not* rectangular, *nor* unit-lines, we may on similar principles determine their fourth proportional, without referring to fig. 68 [p. 360], as follows. Without any real loss of generality, we may suppose that the planes of α, β and α, γ are perpendicular to each other; since this comes merely to substituting, if necessary, for the quotient $\beta : \alpha$, another quotient equal thereto. Having thus

$$\text{I.} \ldots \text{Ax} . (\beta : \alpha) \perp \text{Ax} . (\gamma : \alpha), \quad \text{let} \quad \text{II.} \ldots \beta = \beta' + \beta'', \ \gamma = \gamma' + \gamma'',$$

where β' and γ' are parallel to α, but β'' and γ'' are perpendicular to it, and to each other; so that, by 203, I. and II., we shall have the expressions,

$$\text{III.} \ldots \beta' = \text{S}\frac{\beta}{\alpha} . \alpha, \quad \gamma' = \text{S}\frac{\gamma}{\alpha} . \alpha,$$

and

$$\text{IV.} \ldots \beta'' = \text{V}\frac{\beta}{\alpha} . \alpha, \quad \gamma'' = \text{V}\frac{\gamma}{\alpha} . \alpha.$$

We may then deduce, by the distributive principle (300, (2.)), the tranformations,

$$\text{V.} \ldots \frac{\beta}{\alpha} \cdot \gamma = \left(\frac{\beta'}{\alpha} + \frac{\beta''}{\alpha}\right)(\gamma' + \gamma'')$$

$$= \frac{\beta'}{\alpha}\gamma' + \frac{\beta'}{\alpha}\gamma'' + \frac{\beta''}{\alpha}\gamma' + \frac{\beta''}{\alpha}\gamma'' = \delta + xu;$$

where

$$\text{VI.} \ldots \delta = \beta \text{S}\frac{\gamma}{\alpha} + \gamma''\text{S}\frac{\beta}{\alpha} = \gamma\text{S}\frac{\beta}{\alpha} + \beta''\text{S}\frac{\gamma}{\alpha}, \quad \text{and} \quad \text{VII.} \ldots xu = \frac{\beta''}{\alpha}\gamma''.$$

The latter part, xu, is what we have called (300) the (geometrically) *scalar part*, of the sought fourth proportional; while the former part δ may (still) be called its *vector part*: and we see that *this part* is represented by a *line*, which is at once *in the two planes*, of β, γ'', and of γ, β''; or in two planes which may be generally *constructed* as follows, *without now assuming* that the planes $\alpha\beta$ and $\alpha\gamma$ are *rectangular*, as in I. Let γ' be the projection of the line γ on the plane of α, β, and operate on this projection by the quotient $\beta : \alpha$ as a multiplier; the *plane* which is drawn through the line $\beta : \alpha \,.\, \gamma'$ so obtained, at right angles to the plane $\alpha\beta$, is *one locus* for the sought *line* δ: and the plane through γ, which is perpendicular to the plane $\gamma\gamma'$, is *another locus* for that line. And as regards the *length* of this line, or vector part δ, and the *magnitude* (or quantity) of the scalar part xu, it is easy to prove that

$$\text{VIII.} \ldots \text{T}\delta = t \cos s, \qquad \text{and} \qquad \text{IX.} \ldots x = \pm\, t \sin s,$$

where

$$\text{X.} \ldots t = \text{T}\beta : \text{T}\alpha \,.\, \text{T}\gamma, \quad \text{and} \quad \text{XI.} \ldots \sin s = \sin c \sin p,$$

if c denote the angle between the two given lines α, β, and p the inclination of the third given line γ to their plane: the *sign* of the scalar *coefficient*, x, being positive or negative, according as the *rotation* round α from β to γ is negative or positive.

(1.) Comparing the recent construction with fig. 68, we see that when the condition I. is satisfied, the four unit-lines $\text{U}\gamma$, $\text{U}\alpha$, $\text{U}\beta$, $\text{U}\delta$ take the directions of the four radii OC, OQ, OR, OD, which terminate at the four corners of what may be called a *tri-rectangular quadrilateral* CQRD on the sphere.

(2.) It may be remarked that the *area* of this *quadrilateral* is exactly equal to *half* the area Σ of the *triangle* DEF; which may be inferred, either from the circumstance that its *spherical excess* (over *four* right angles) is constructed by the angle MDN; or from the triangles DBR and EAS being together equal to the triangle ABF, so that the area of DESR is Σ, and therefore that of CQRD is $\frac{1}{2}\Sigma$, as before.

(3.) The two *sides* CQ, QR of this quadrilateral, which are *remote* from the *obtuse* angle at D, being still called p and c, and the side CD which is *opposite* to c being still denoted by c', let the side DR which is opposite to p be now called p'; also let the *diagonals* CR, QD be denoted by d and d'; and let s denote the *spherical excess* (CDR $- \frac{1}{2}\pi$), or the *area* of the quadrilateral. We shall then have the relations,

$$\text{XII.}\ldots\begin{cases}\cos d = \cos p\,\cos c; \quad \cos d' = \cos p\,\cos c';\\ \tan c' = \cos p\,\tan c; \quad \tan p' = \cos c\,\tan p;\\ \cos s = \cos p\,\sec p' = \cos c\,\sec c' = \cos d\,\sec d';\end{cases}$$

of which some have virtually occurred before, and all are easily proved by right-angled triangles, arcs being when necessary prolonged.

(4.) If we take now two points, A and B, on the side QR, which satisfy the arcual equation (comp. 297, XL., and fig. 68),

$$\text{XIII.}\ldots \frown \text{AB} = \frown \text{QR};$$

and if we then join AC, and let fall on this new arc the perpendiculars BB′, DD′; it is easy to prove that the *projection* B′D′ of the side BD on the arc AC is *equal* to that arc, and that the *angle* DBB′ is right: so that we have the two new equations,

$$\text{XIV.}\ldots \frown \text{B'D'} = \frown \text{AC}; \quad \text{XV.}\ldots \text{DBB'} = \tfrac{1}{2}\pi;$$

and the *new quadrilateral* BB′D′D is also *tri-rectangular*.

(5.) Hence *the point* D *may be derived from the three points* ABC, *by any two of the four following conditions*: Ist, the *equality* XIII. of the *arcs* AB, QR; IInd, the corresponding equality XIV. of the arcs AC, B′D′; IIIrd, the *tri-rectangular character* of the *quadrilateral* CQRD; IVth, the corresponding character of BB′D′D.

(6.) In other words, this *derived point* D is the *common intersection of the four perpendiculars, to the four arcs* AB, AC, CQ, BB′, erected *at the four points* R, D′, C, B; CQ, BB′ being still the perpendiculars from C and B, on AB and AC; and R and D′ being deduced from Q and B′, by equal arcs, as above.

305. These consequences of the construction employed in 297, &c., are here mentioned merely in connexion with that theory of *fourth proportionals to vectors*, which they have thus served to illustrate; but they are perhaps

numerous and interesting enough, to justify us in suggesting the *name*, "*Spherical Parallelogram*,"* for the quadrilateral CABD, or BACD, in fig. 68 (or 67), p. 360; and in proposing to say that D is the *Fourth Point*, which *completes* such a *parallelogram*, when the *three points* C, A, B, or B, A, C, are *given* upon the sphere, as *first*, *second*, and *third*. It must however be carefully observed, that the *analogy to the plane* is here thus far *imperfect*, that in the *general case*, when the *three given points* are *not co-arcual*, but on the contrary are corners of a *spherical triangle* ABC, then if we take C, D, B, or B, D, C, for the *three first points* of a *new spherical parallelogram*, of the kind here considered, the *new fourth point*, say A_1, will *not coincide* with the *old second point* A; although it will *very nearly* do so, if the *sides* of the triangle ABC be *small*: the *deviation* AA_1 being in fact found to be small of the *third order*, if those sides of the given triangle be supposed to be small of the *first* order; and being always directed *towards the foot of the perpendicular*, let fall from A on BC.

(1.) To investigate the *law* of this *deviation*, let β, γ be still any two given unit-vectors, OB, OC, making with each other an angle equal to a, of which the cosine is l; and let ρ or OP be any third vector. Then, if we write,

$$\text{I.} \ldots \rho_1 = \phi(\rho) = \tfrac{1}{2}\mathrm{N}\rho \,.\left(\frac{\beta}{\rho}\gamma + \frac{\gamma}{\rho}\beta\right), \quad \mathrm{OQ} = \mathrm{U}\rho, \quad \mathrm{OQ}_1 = \mathrm{U}\rho_1,$$

the new or *derived vector*, $\phi\rho$ or ρ_1, or OP_1, will be the *common vector part* of the *two fourth proportionals*, to ρ, β, γ, and to ρ, γ, β, *multiplied by the square of the length of* ρ; and BQCQ_1 will be what we have lately called a *spherical parallelogram*. We shall also have the transformation (compare 297, (2.)),

$$\text{II.} \ldots \rho_1 = \phi\rho = \beta \mathrm{S}\frac{\rho}{\gamma} + \gamma\mathrm{S}\frac{\rho}{\beta} - \rho\mathrm{S}\frac{\gamma}{\beta};$$

and the *distributive symbol of operation* ϕ will be such that

$$\text{III.} \ldots \phi\rho \parallel\!\!| \beta, \gamma, \quad \text{and} \quad \phi^2\rho = \rho, \quad \text{if} \quad \rho \parallel\!\!| \beta, \gamma;\dagger$$

but

$$\text{IV.} \ldots \phi\rho = -l\rho, \quad \text{if} \quad \rho \parallel \mathrm{Ax}\,.(\gamma : \beta).$$

(2.) This being understood, let

$$\text{V.} \ldots \rho = \rho' + \rho''; \quad \phi\rho' = \rho'_1; \quad \rho' \parallel\!\!| \beta, \gamma; \quad \rho'' \parallel \mathrm{Ax}\,.(\gamma : \beta);$$

---

* By the same analogy, the quadrilateral CQBD, in fig. 68, may be called a *Spherical Rectangle*.

† [In fact $\phi\beta = \gamma$ and $\phi\gamma = \beta$. So $\phi(y\beta + z\gamma) = z\beta + y\gamma$.]

so that ρ', or OP$'$, is the projection of ρ on the plane of $\beta\gamma$; and ρ'', or OP$''$, is the part (or component) of ρ, which is perpendicular to that plane. Then we shall have an indefinite *series of derived vectors*, $\rho_1, \rho_2, \rho_3, \ldots$ or rather *two* such series, succeeding each other *alternately*, as follows:

$$\text{VI.} \ldots \begin{cases} \rho_1 = \phi\rho = \rho'_1 - l\rho''; & \rho_2 = \phi^2\rho = \rho' + l^2\rho''; \\ \rho_3 = \phi^3\rho = \rho'_1 - l^3\rho''; & \rho_4 = \phi^4\rho = \rho' + l^4\rho''; \ \&\text{c.}; \end{cases}$$

the *two series of derived points*, $P_1, P_2, P_3, P_4, \ldots$ being thus *ranged*, *alternately*, *on the two perpendiculars*, PP$'$ and $P_1P'_1$, which are let fall *from the points* P and P_1, *on the given plane* BOC; and the intervals, $PP_2, P_1P_3, P_2P_4, \ldots$ forming a *geometrical progression*, in which each is equal to the one before it, multiplied by the *constant factor* $-l$, or by the *negative of the cosine* of the *given angle* BOC.

(3.) If then this angle be still supposed to be distinct from 0 and π, and also in general from the intermediate value $\frac{1}{2}\pi$, we shall have the *two limiting values*,

$$\text{VII.} \ldots \rho_{2n} = \rho', \quad \rho_{2n+1} = \rho'_1, \quad \text{if} \quad n = \infty;$$

or in words, *the derived points* $P_2, P_4, \ldots$ of *even orders*, *tend to the point* P$'$, and the *other derived points*, $P_1, P_3, \ldots$ of *odd orders*, tend to the *other point* P'_1, *as limiting positions*; these *two limit points* being the *feet of the two* (rectilinear) *perpendiculars*, let fall (as above) from P and P$'$ on the plane BOC.

(4.) But even the *first deviation* PP_2 is *small of the third order*, if the *length* $T\rho$ of the *line* OP be considered as *neither* large nor small, and if the *sides* of the spherical *triangle* BQC be *small* of the *first* order. For we have by VI. the following expressions for that deviation,

$$\text{VIII.} \ldots PP_2 = \rho_2 - \rho = (l^2 - 1)\rho'' = -\sin^2 a \,.\, \sin p_a \,.\, T\rho \,.\, U\rho'';$$

where p_a denotes the *inclination* of the *line* ρ to the *plane* $\beta\gamma$; or the *arcual perpendicular* from the *point* Q on the side BC, or a, of the triangle. The statements lately made (305) are therefore proved to have been correct.

(5.) And if we now resume and extend the *spherical construction*, and conceive that D_1 is deduced from BA_1C, as A_1 was from BDC, or D from BAC; while A_2 may be supposed to be deduced by the same rule from BD_1C, and D_2 from BA'_2C, &c., through an indefinite *series of spherical parallelograms*, in which the *fourth point* of any *one* is treated as the *second point* of the *next*, while the *first* and *third* points remain *constant*: we see that the points $A_1, A_2, \ldots$ are all situated *on the arcual perpendicular* let fall from A on BC; and that in like manner the points $D_1, D_2, \ldots$ are all situated on that *other* arcual perpendicular, which is let fall from D on BC. We see also that *the ultimate positions*, A_∞ and D_∞, *coincide precisely with the feet of those two perpendiculars*: a remarkable theorem, which it would perhaps be difficult to prove, by any other method than that of the Quaternions, at least with calculations so simple as those which have been employed above.

(6.) It may be remarked that the construction of fig. 68 might have been otherwise suggested (comp. 223, IV.), by the principles of the Second Book, if we had sought to assign the *fourth proportional* (297) *to three right quaternions*; for example, to *three right versors*, v, v', v'', whereof the *unit lines* α, β, γ should be supposed to be the *axes*. For the result would be in general a *quaternion* $v'v^{-1}v''$, with e for its *scalar part*, and with δ for the *index of its right part*: e and δ denoting the *same scalar*, and the *same vector*, as in the sub-articles to 297.

306. Quaternions may also be employed to furnish a *new construction*, which shall complete (comp. 305, (5.)) the *graphical* determination of the *two series* of *derived points*,

$$\text{I.} \ldots \text{D}, \ \text{A}_1, \ \text{D}_1, \ \text{A}_2, \ \text{D}_2, \ \&\text{c.},$$

when the *three* points A, B, C are *given* upon the unit-sphere; and thus shall render *visible* (so to speak), with the help of a new figure, the tendencies of those derived points to *approach*, alternately and indefinitely, to the *feet*, say D′ and A′, of the two *arcual perpendiculars* let fall from the two *opposite corners*, D and A, of the *first spherical parallelogram*, BACD, on its *given diagonal* BC; which *diagonal* (as we have seen) is common to *all* the successive parallelograms.

(1.) The given triangle ABC being supposed for simplicity to have its sides *abc* less than quadrants, as in 297, so that their cosines *lmn* are positive, let A′, B′, C′ be the feet of the perpendiculars let fall on these three sides from the points A, B, C; also let M and N be two auxiliary points, determined by the equations,

$$\text{II.} \ldots \frown \text{BM} = \frown \text{MC}, \quad \frown \text{AM} = \frown \text{MN};$$

so that the arcs AN and BC bisect each other in M. Let fall from N a perpendicular ND′ on BC, so that

$$\text{III.} \ldots \frown \text{BD}' = \frown \text{A}'\text{C};$$

and let B″, C″ be two other auxiliary points, on the sides b and c, or on those sides prolonged, which satisfy these two other equations,

$$\text{IV.} \ldots \frown \text{B}'\text{B}'' = \frown \text{AC}, \quad \frown \text{C}'\text{C}'' = \frown \text{AB}.$$

(2.) Then the *perpendiculars to these last sides,* CA and AB, erected *at these last points,* B″ and C″, will *intersect each other in the point* D, which *completes* (305) the *spherical parallelogram* BACD; and the *foot of the perpendicular from this point* D, on the third side BC of the given triangle, will *coincide* (comp. 305, (2.)) *with the foot* D′ *of the perpendicular on the same side from* N; so that this last perpendicular ND′ is *one locus of the point* D.

(3.) To obtain *another locus* for that point, adapted to our present purpose, let E denote now* that new point in which the two diagonals, AD and BC, intersect each other; then because (comp. 297, (2.)) we have the expression,

$$\text{V.} \ldots \text{OD} = \text{U}(m\beta + n\gamma - l\alpha),$$

we may write (comp. 297, (25.), and (30.)),

$$\text{VI.} \ldots \text{OE} = \text{U}(m\beta + n\gamma), \quad \text{whence} \quad \text{VII.} \ldots \sin \text{BE} : \sin \text{EC} = n : m = \cos \text{BA}' : \cos \text{A}'\text{C};$$

the *diagonal* AD thus dividing the arc BC into *segments*, of which the *sines* are proportional to the *cosines* of the *adjacent sides* of the given triangle, or to the cosines of their *projections* BA′ and A′C on BC; so that the *greater segment* is adjacent to the *lesser side*, and the *middle point* M of BC (1.) lies *between the points* A′ and E.

(4.) The intersection E is therefore a known point, and the great circle through A and E is a *second known locus* for D; which point may therefore be found, as the *intersection of the arc* AE prolonged, *with the perpendicular* ND′ from N (1.). And because E lies (3.) *beyond the middle point* M of BC, with respect to the foot A′ of the perpendicular on BC from A, but (as it is easy to prove) *not so far beyond* M *as the point* D′, or in other words falls *between* M *and* D′ (when the arc BC is, as above supposed, *less than a quadrant*), the prolonged arc AE cuts ND′ *between* N *and* D′; or in other words, *the perpendicular distance of the sought fourth point* D, *from the given diagonal* BC *of the parallelogram, is less than the distance of the given second point* A, from the same given diagonal. (Compare the annexed fig. 73.)

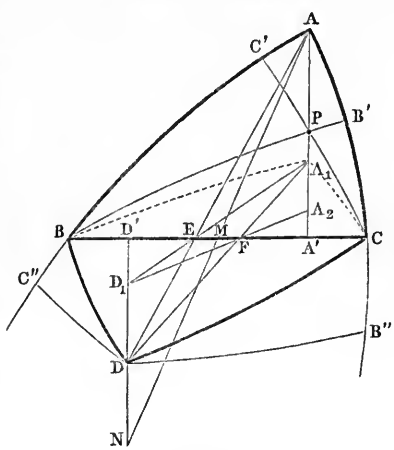

Fig. 73.

* It will be observed that M, N, E have not here the same significations as in fig. 68; and that the present letters C′ and C″ correspond to Q and R in that figure.

(5.) Proceeding next (305) to *derive* a *new point* A_1 from B, D, C, as D has been derived from B, A, C, we see that we have only to determine a *new** *auxiliary point* F, by the equation,

$$\text{VIII.} \ldots \frown \text{EM} = \frown \text{MF};$$

and then to draw DF, and prolong it till it meets AA′ in the required point A_1, which will thus *complete the second parallelogram*, $BDCA_1$, with BC (as before) for a *given diagonal*.

(6.) In like manner, to complete (comp. 305, (5.)), the *third* parallelogram, BA_1CD_1, with the *same* given diagonal BC, we have only to draw the arc A_1E, and prolong it till it cuts ND′ in D_1; after which we should find the point A_2 of a *fourth* successive parallelogram BD_1CA_2, by drawing D_1F, and so on for ever.

(7.) The constant and indefinite *tendency*, of the *derived points* D, D_1 . . . to the *limit-point* D′, and of the *other* (or *alternate*) derived points A_1, A_2, . . . to the *other limit-point* A′, becomes therefore evident from this new construction; the *final* (or *limiting*) *results* of which, we may express by these two equations (comp. again 305, (5.)),

$$\text{IX.} \ldots D_\infty = D'; \quad A_\infty = A'.$$

(8.) But the *smallness* (305) of the *first deviation* AA_1, when the *sides* of the given triangle ABC are *small*, becomes at the same time evident, by means of the same construction, with the help of the formula VII.; which shows that the *interval*† EM, or the *equal* interval MF (5), is *small of the third order*, when the *sides* of the given triangle are supposed to be small of the *first* order: agreeing thus with the equation 305, VIII.

(9.) The theory of such *spherical parallelograms* admits of some interesting applications, especially in connexion with *spherical conics*; on which however we cannot enter here, beyond the mere *enunciation of a Theorem*,‡ of which (comp. 271) the proof by quaternions is easy:—

* This new point, and the intersection of the perpendiculars of the given triangle, are evidently not the same in the new figure 73, as the points denoted by the same letters, F and P, in the former figure 68; although the four points A, B, C, D are conceived to bear to each other the same relations in the two figures, and indeed in fig. 67 also; BACD being, in *that* figure also, what we have proposed to call a *spherical parallelogram*. Compare the Note to (3.).

† The formula VII. gives easily the relation,

$$\text{VII}' \ldots \tan \text{EM} = \tan \text{MA}' \left(\tan \frac{a}{2}\right)^2;$$

hence the interval EM is small of the third order, in the case (8.) here supposed; and generally, if $a < \frac{\pi}{2}$, as in (1.), while b and c are unequal, the formula shows that this interval EM is less than MA′, or than D′M, so that E falls between M and D′, as in (4.).

‡ This Theorem was communicated to the Royal Irish Academy in June, 1845, as a consequence of the principles of Quaternions. See the *Proceedings* of that date (Vol. III., page 109).

"*If* KLMN *be any spherical quadrilateral, and* I *any point on the sphere; if also we complete the spherical parallelograms,*

$$\text{X.}\ldots \text{KILA}, \quad \text{LIMB}, \quad \text{MINC}, \quad \text{NIKD},$$

and determine the poles E *and* F *of the diagonals* KM *and* LN *of the quadrilateral: then these two poles are the foci* of a spherical conic, inscribed in the derived quadrilateral* ABCD, *or touching its four sides.*" †

(10.) Hence, in a notation‡ elsewhere proposed, we shall have, under these conditions of construction, the formula:

$$\text{XI.}\ldots \text{EF}\,(\,.\,.\,)\,\text{ABCD}; \quad \text{or} \quad \text{XI}'\ldots \text{EF}\,(\,.\,.\,)\,\text{BCDA}; \ \&\text{c.}$$

(11.) Before closing this article and section, it seems not irrelevant to remark, that the projection γ' of the unit-vector γ, on the plane of α and β, is given by the formula,

$$\text{XII.}\ldots \gamma' = \frac{\alpha \sin a \cos B + \beta \sin b \cos A}{\sin c};$$

and that therefore the point P, in which (see again fig. 73) the three arcual perpendiculars of the triangle ABC intersect, is on the vector,

$$\text{XIII.}\ldots \rho = \alpha \tan A + \beta \tan B + \gamma \tan C.$$

(12.) It may be added, as regards the construction in 305, (2.), that *the right lines,*

$$\text{XIV.}\ldots \text{PP}_1, \quad \text{P}_1\text{P}_2, \quad \text{P}_2\text{P}_3, \quad \text{P}_3\text{P}_4, \ldots$$

however far their series may be continued, *intersect the given plane* BOC, *alternately, in two points* S and T, of which the vectors are,

$$\text{XV.}\ldots \text{OS} = \frac{\rho'_1 + l\rho'}{1 + l}, \quad \text{OT} = \frac{\rho' + l\rho'_1}{1 + l};$$

and which thus become *two fixed points in the plane*, when the position of the point P *in space* is *given*, or assumed.

---

* In the language of modern geometry, the *conic* in question may be said to *touch eight given arcs*; *four real*, namely the *sides* AB, BC, CD, DA; and *four imaginary*, namely *two* from *each* of the *focal points*, E and F.

† [Take $q = \lambda\iota^{-1}\kappa$, $r = \mu\iota^{-1}\lambda$, $s = \nu\iota^{-1}\mu$, and $t = \kappa\iota^{-1}\nu$; then $srq = -\nu\iota^{-1}\kappa = Kt$, $rq = \mu\kappa$, $sr = \nu\lambda$, $ts = \kappa\mu$, and $qt = \lambda\nu$. On reference to fig. 60, p. 304, there is no difficulty in seeing that a conic having the given foci may be drawn to touch the four sides, produced when necessary.]

‡ Compare the second Note to page 310.

SECTION 9.

On a Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book.

307. The *Conception* of the *Fourth Proportional to three Rectangular Unit-Lines*, as being itself a species of *Fourth Unit in Geometry*, is eminently *characteristic of the present Calculus*; and offers a *Third Method of interpreting a Product of two Vectors* as a *Quaternion*: which is however found to be *consistent*, in all its *results*, with the two former methods (278, 284) of the present Book; and admits of being easily extended to products of *three* or more *lines in space*, and generally to *Functions of Vectors* (289). In fact we have only to conceive* that *each proposed vector*, a, is *divided* by the *new* or *fourth unit*, u, above alluded to; and that the *quotient* so obtained, which is always (by 303, VIII.) the *right quaternion* $\mathrm{I}^{-1}a$, whereof the vector a is the *index*, is *substituted* for that vector: the *resulting quaternion* being finally, if we think it convenient, *multiplied into the same fourth unit*. For in this way we shall merely *reproduce the process* of 284, or 289, although now as a consequence of a *different train of thought*, or of a *distinct* but *Consistent Interpretation*: which thus conducts, by a *new Method*, to the *same Rules of Calculation* as before.

(1.) The *equation of the unit-sphere*, $\rho^2 + 1 = 0$ (282, XIV.), may thus be conceived to be an *abridgment* of the following fuller equation:

$$\mathrm{I.} \ldots \left(\frac{\rho}{u}\right)^2 = -1;$$

---

* It was in a somewhat analogous way that *Des Cartes* showed, in his *Geometria* (Schooten's Edition, Amsterdam, 1659), that all *products* and *powers* of *lines*, considered relatively to their *lengths* alone, and without any reference to their *directions*, could be *interpreted as lines*, by the suitable introduction of a line taken for *unity*, however high the *dimension* of the product or power might be. Thus (at page 3 of the cited work) the following remark occurs:—

"Ubi notandum est, quòd per a^2 vel b^3, similésve, communiter, non nisi lineas omnino simplices concipiam, licèt illas, ut nominibus in Algebra usitatis utar, Quadrata aut Cubos, &c. appellem."

But it was much more difficult to accomplish the corresponding *multiplication of directed lines in space*; on account of the *non-existence of any such line*, which is *symmetrically related to all other lines*, or *common to all possible planes* (comp. the Note to page 258). The *Unit of Vector-Multiplication* cannot properly be *itself* a *Vector*, if the *conception* of the *Symmetry of Space* is to be retained, and duly combined with the other elements of the question. This difficulty however disappears, at least in theory, when we come to consider that *new Unit*, of a *scalar* kind (300), which has been above denoted by the temporary symbol u, and has been obtained, in the foregoing section, as a certain *Fourth Proportional to Three Rectangular Unit-Lines*, such as the *three co-initial edges*, AB, AC, AD of the *quotient* $\rho : u$ being considered as equal (by 303) to the *right quaternion*, $I^{-1}\rho$, which must *here* be a *right versor* (154), because its *square* is negative unity.

(2.) The *equation of the ellipsoid*,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2 \text{ (282, XIX.)},$$

may be supposed, in like manner, to be *abridged* from this other equation:

$$\text{II.} \ldots T\left(\frac{\iota}{u}\frac{\rho}{u} + \frac{\rho}{u}\frac{\kappa}{u}\right) = \left(\frac{\kappa}{u}\right)^2 - \left(\frac{\iota}{u}\right)^2;$$

and similarly in other cases.

(3.) We might also write these equations, of the sphere and ellipsoid, under these other, but connected forms:

$$\text{III.} \ldots \frac{\rho}{u}\rho = -u; \quad \text{IV.} \ldots T\left(\frac{\iota}{u}\rho + \frac{\rho}{u}\kappa\right) = \frac{\kappa}{u}\kappa - \frac{\iota}{u}\iota;$$

with interpretations which easily offer themselves, on the principles of the foregoing section.

(4.) It is, however, to be distinctly understood, that *we do not propose to adopt this Form of Notation*, in the *practice* of the present *Calculus*: and that we merely *suggest* it, in passing, as one which may serve to throw some additional light on the *Conception*, introduced in this Third Book, of a *Product of two Vectors* as a *Quaternion*.

(5.) In general, the *Notation of Products*, which has been employed throughout the greater part of the present Book and Chapter, appears to be

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what we have called an *Unit-Cube*: for *this* fourth proportional, by the proposed *conception* of it, undergoes *no change*, when the cube ABCD is in any manner *moved*, or *turned*; and therefore may be considered to be *symmetrically related to all directions of lines in space*, or to all possible *vections* (or *translations*) of a *point*, or *body*. In fact, we *conceive* its *determination*, and the *distinction* of it (as $+u$) from the *opposite unit* of the *same kind* ($-u$), to depend *only* on the *usual assumption* of an *unit of length*, combined with the *selection of a hand* (as, for example, the *right* hand), *rotation towards which hand* shall be considered to be *positive*, and *contrasted* (as such) with rotation towards the *other hand*, round the *same arbitrary axis*. Now in whatever manner the supposed *cube* may be thrown about in space, the *conceived rotation round the edge* AB, *from* AC *to* AD, will have the *same character*, as *right-handed* or *left-handed*, at the *end* as at the *beginning* of the motion. If then the *fourth proportional* to these *three edges*, taken *in this order*, be *denoted* by $+u$, or simply by $+1$, *at one stage* of that arbitrary *motion*, it may (on the plan here considered) be denoted by the *same symbol*, at *every other stage*: while the *opposite character* of the (conceived) *rotation*, round the *same edge* AB, *from* AD *to* AC, leads us to regard the fourth proportional to AB, AD, AC as being on the contrary equal to $-u$, or to -1. It is true that this *conception* of a *new unit for space*, *symmetrically related* (as above) *to all linear directions* therein, may appear somewhat abstract and metaphysical; but readers who think it such can of course confine their attention to the *rules of calculation*, which have been above derived from it, and from other connected considerations: and which have (it is hoped) been stated and exemplified, in this and in a former volume, with sufficient clearness and fulness.

much more convenient, for actual *use* in *calculation*, than any *Notation of Quotients*: either such as has been just now suggested for the sake of illustration, or such as was employed in the Second Book, in connexion with that *First Conception of a Quaternion* (112), to which that Book mainly related, as the *Quotient of two Vectors* (or of two directed lines in space). The *notations* of the two Books are, however, intimately *connected*, and the former was judged to be an useful preparation for the latter, even as regarded the *quotient-forms* of many of the expressions used: while the *Characteristics of Operation*, such as

$$\mathrm{S},\ \mathrm{V},\ \mathrm{T},\ \mathrm{U},\ \mathrm{K},\ \mathrm{N},$$

are employed according to exactly the *same laws* in both. In short, a reader of the Second Book has *nothing to unlearn* in the Third; although he may be supposed to have become prepared for the use of somewhat shorter and more convenient *processes*, than those before employed.

SECTION 10.

On the Interpretation of a Power of a Vector as a Quaternion.

308. The only symbols, of the kinds mentioned in 277, which we have not yet interpreted, are the *cube* α^3, and the *general power* α^t, of an *arbitrary vector base*, α, with an *arbitrary scalar exponent*, t; for we have already assigned interpretations (282, (1.), (14.), and 299, (8.)) for the *particular symbols* α^2, α^{-1}, α^{-2}, which are *included* in this last *form*. And we shall preserve those particular interpretations if we now *define*, in full consistency with the principles of the present and preceding Books, that this *Power* α^t is generally a *Quaternion*, which may be decomposed into *two factors*, of the *tensor* and *versor* kinds, as follows:

$$\mathrm{I.} \ldots \alpha^t = \mathrm{T}\alpha^t . \mathrm{U}\alpha^t;$$

$\mathrm{T}\alpha^t$ denoting the *arithmetical value* of the t^{th} *power* of the *positive number* $\mathrm{T}\alpha$, which represents (as usual) the *length* of the *base-line* α; and $\mathrm{U}\alpha^t$ denoting a *versor*, which *causes any line* ρ, *perpendicular to that line* α, to *revolve round it as an axis, through* t *right angles*, or quadrants, and in a *positive* or *negative direction*, according as the *scalar exponent*, t, is *itself* a *positive* or *negative number* (comp. 234, (5.)).

(1.) As regards the omission of parentheses in the formula I., we may observe that the recent *definition*, or *interpretation*, of the symbol α^t, enables us to write (comp. 237, II. III.),

$$\text{II.} \ldots \mathrm{T}(\alpha^t) = (\mathrm{T}\alpha)^t = \mathrm{T}\alpha^t; \qquad \text{III.} \ldots \mathrm{U}(\alpha^t) = (\mathrm{U}\alpha)^t = \mathrm{U}\alpha^t.$$

(2.) The *axis* and *angle* of the *power* α^t, considered as a quaternion, are generally determined by the two following formulæ:

$$\text{IV.} \ldots \mathrm{Ax}.\,\alpha^t = \pm\,\mathrm{U}\alpha; \qquad \text{V.} \ldots \angle\,.\,\alpha^t = 2n\pi \pm \tfrac{1}{2}t\pi;$$

the *signs* accompanying each other, and the (positive or negative or null) integer, n, being so chosen as to bring the *angle* within the usual limits, 0 and π.

(3.) In general (comp. 235), we may speak of the (positive or negative) product, $\frac{1}{2}t\pi$, as being the *amplitude* of the same *power*, with reference to the line α as an *axis of rotation*; and may write accordingly,

$$\text{VI.} \ldots \mathrm{am}.\,\alpha^t = \tfrac{1}{2}t\pi.$$

(4.) We may write also (comp. 234, VII. VIII.),

$$\text{VII.} \ldots \mathrm{U}\alpha^t = \cos\frac{t\pi}{2} + \mathrm{U}\alpha\,.\sin\frac{t\pi}{2}; \ \text{ or briefly, } \quad \text{VIII.} \ldots \mathrm{U}\alpha^t = \mathrm{cas}\,\frac{t\pi}{2}.$$

(5.) In particular,

$$\text{IX.} \ldots \mathrm{U}\alpha^{2n} = \mathrm{cas}\,n\pi = \pm 1; \qquad \text{IX}'. \ldots \mathrm{U}\alpha^{2n+1} = \pm\,\mathrm{U}\alpha;$$

upper or lower signs being taken, according as the number n (supposed to be whole) is even or odd. For example, we have thus the *cubes*,

$$\text{X.} \ldots \mathrm{U}\alpha^3 = -\,\mathrm{U}\alpha; \qquad \text{X}'. \ldots \alpha^3 = -\,\alpha\mathrm{N}\alpha.$$

(6.) The *conjugate* and *norm* of the power α^t may be thus expressed (it being remembered that to turn a line $\perp\ \alpha$ through $-\frac{1}{2}t\pi$ round $+\alpha$, is equivalent to turning that line through $+\frac{1}{2}t\pi$ round $-\alpha$):

$$\text{XI.} \ldots \mathrm{K}\alpha^t = \mathrm{T}\alpha^t.\mathrm{U}\alpha^{-t} = (-\alpha)^t; \qquad \text{XII.} \ldots \mathrm{N}\alpha^t = \mathrm{T}\alpha^{2t};$$

parentheses being unnecessary, because (by 295, VIII.) $\mathrm{K}\alpha = -\alpha$.

(7.) The *scalar*, *vector*, and *reciprocal* of the same power are given by the formulæ:

$$\text{XIII.} \ldots \mathrm{S}.\,\alpha^t = \mathrm{T}\alpha^t.\cos\frac{t\pi}{2}; \qquad \text{XIV.} \ldots \mathrm{V}.\,\alpha^t = \mathrm{T}\alpha^t.\mathrm{U}\alpha\,.\sin\frac{t\pi}{2};$$

$$\text{XV.} \ldots 1 : \alpha^t = \mathrm{T}\alpha^{-t}.\,\mathrm{U}\alpha^{-t} = \alpha^{-t} = \mathrm{K}\alpha^t : \mathrm{N}\alpha^t \text{ (comp. 190, (3.))}.$$

(8.) If we decompose any vector ρ into parts ρ' and ρ'', which are respectively parallel and perpendicular to α, we have the general transformation:*

$$\text{XVI.} \ldots \alpha^t\rho\alpha^{-t} = \alpha^t(\rho' + \rho'')\alpha^{-t} = \rho' + \mathrm{U}\alpha^{2t}.\rho'',$$

= *the new vector obtained by causing* ρ *to revolve conically through an angular quantity expressed by* $t\pi$, *round the line* α *as an axis* (comp. 297, (15.)).

(9.) More generally (comp. 191, (5.)), if q be any *quaternion*, and if

$$\text{XVII.} \ldots \alpha^t q\alpha^{-t} = q',$$

the *new quaternion* q' is formed from q by such a *conical rotation of its own axis* $\mathrm{Ax}.q$, through $t\pi$, round α, *without any change of its angle* $\angle q$, or of its tensor $\mathrm{T}q$.

(10.) Treating ijk as *three rectangular unit-lines* (295), the *symbol*, or *expression*,

$$\text{XVIII.} \ldots \rho = rk^tj^skj^{-s}k^{-t}, \quad \text{or} \quad \text{XIX.} \ldots \rho = rk^tj^{2s}k^{1-t},$$

in which

$$\text{XX.} \ldots r \geqq 0, \quad s \geqq 0, \quad s \leqq 1, \quad t \geqq 0, \quad t \leqq 2,$$

may represent any vector; the *length* or *tensor* of this line ρ being r; its *inclination*† to k being $s\pi$; and the *angle* through which the variable *plane* $k\rho$ may be conceived to have *revolved*, from the initial position ki, with an initial direction towards the position kj, being $t\pi$.

(11.) In accomplishing the transformation XVI., and in passing from the expression XVIII. to the less symmetric but equivalent expression XIX., we employ the principle that

$$\text{XXI.} \ldots kj^{-s} = \mathrm{S}^{-1}0 = -\mathrm{K}(kj^{-s}) = j^sk;$$

which easily admits of extension, and may be confirmed by such transformations as VII. or VIII.

(12.) It is scarcely necessary to remark, that the definition or interpretation I., of the *power* α^t of *any vector* α, gives (as in algebra) the *exponential property*,

$$\text{XXII.} \ldots \alpha^s\alpha^t = \alpha^{s+t},$$

whatever scalars may be denoted by s and t; and similarly when there are more than two factors of this form.

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* Compare the shortly following sub-article (11.).

† If we conceive (compare the first Note to page 345) that the two lines i and j are directed respectively towards the *south* and *west* points of the horizon, while the third line k is directed towards the *zenith*, then $s\pi$ is the *zenith-distance* of ρ; and $t\pi$ is the *azimuth* of the same line, measured *from south to west*, and thence (if necessary) through north and east, to south again.

(13.) As verifications of the expression XVIII., considered as representing a *vector*, we may observe that it gives,

$$\text{XXIII.} \ldots \rho = -\mathrm{K}\rho; \quad \text{and} \quad \text{XXIV.} \ldots \rho^2 = -r^2.$$

(14.) More generally, it will be found that if u* be *any scalar*, we have the eminently simple transformation:

$$\text{XXV.} \ldots \rho^u = (rk^t j^s k j^{-s} k^{-t})^u = r^u k^t j^s k^u j^{-s} k^{-t}.$$

In fact, the two last expressions denote generally two *equal quaternions*, because they have, Ist, *equal tensors*, each $= r^u$; IInd, *equal angles*, each $= \angle\,(k^u)$; and IIIrd, *equal* (or coincident) *axes*, each formed from $\pm k$ by one *common system* of two *successive rotations*, one through $s\pi$ round j, and the other through $t\pi$ round k.

309. *Any quaternion*, q, which is not simply a *scalar*, may be brought to the *form* α^t, by a suitable choice of the *base*, α, and of the *exponent*, t; which latter may moreover be supposed to fall between the limits 0 and 2; since for this purpose we have only to write,

$$\text{I.} \ldots t = \frac{2\angle q}{\pi}; \quad \text{II.} \ldots \mathrm{T}\alpha = \mathrm{T}q^{\frac{1}{t}}; \quad \text{III.} \ldots \mathrm{U}\alpha = \mathrm{Ax}\,.\,q;$$

and thus *the general dependence of a Quaternion, on a Scalar and a Vector Element, presents itself in a new way* (comp. 17, 207, 292). When the proposed quaternion is a *versor*, $\mathrm{T}q = 1$, we have thus $\mathrm{T}\alpha = 1$; or in other words, the *base* α, of the equivalent *power* α^t, is an *unit-line*. Conversely, *every versor* may be considered as a *power of an unit-line*, with a *scalar exponent*, t, which may be supposed to be in *general positive*, and *less than two*; so that we may write *generally*,

$$\text{IV.} \ldots \mathrm{U}q = \alpha^t, \quad \text{with} \quad \text{V.} \ldots \alpha = \mathrm{Ax}\,.\,q = \mathrm{T}^{-1}1,$$

and

$$\text{VI.} \ldots t > 0, \quad t < 2;$$

although if this versor *degenerate* into 1 or -1, the *exponent* t becomes 0 or 2, and the *base* α has an indeterminate or *arbitrary direction*. And from such *transformations of versors* new methods may be deduced, for treating questions of *spherical trigonometry* and generally of *spherical geometry*.

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* The employment of this letter u, to denote what we called, in the two preceding sections, a *fourth unit*, &c., was stated to be a merely temporary one. In general, we shall henceforth simply *equate* that *scalar unit* to the *number one*; and *denote* it (when necessary to be denoted at all) by the *usual symbol*, 1, for that number.

(1.) Conceive that P, Q, R, in fig. 46 [p. 153] are replaced by A, B, C, with unit-vectors, α, β, γ as usual; and let x, y, z be three scalars between 0 and 2, determined by the three equations,

$$\text{VII.}\ldots x\pi = 2\text{A}, \quad y\pi = 2\text{B}, \quad z\pi = 2\text{C};$$

where A, B, C denote the angles of the spherical triangle. The three versors, indicated by the three arrows in the upper part of the figure, come then to be thus denoted:

$$\text{VIII.}\ldots q = \alpha^x; \quad q' = \beta^y; \quad q'q = \gamma^{2-z};$$

so that we have the equation,

$$\text{IX.}\ldots \beta^y\alpha^x = \gamma^{2-z}; \quad \text{or} \quad \text{X.}\ldots \gamma^z\beta^y\alpha^x = -1;$$

from which last, by easy divisions and multiplications, these two others immediately follow:

$$\text{X}'.\ldots \alpha^x\gamma^z\beta^y = -1; \quad \text{X}''.\ldots \beta^y\alpha^x\gamma^z = -1;$$

the rotation round α from β to γ being again supposed to be negative.

(2.) In X. we may write (by 308, VIII.),

$$\text{XI.}\ldots \alpha^x = \text{c}\alpha\text{sA}; \quad \beta^y = \text{c}\beta\text{sB}; \quad \gamma^z = \text{c}\gamma\text{sC};$$

and then the formula becomes, for *any spherical triangle*, in which the *order of rotation* is as above:

$$\text{XII.}\ldots \text{c}\gamma\text{sC}\,.\,\text{c}\beta\text{sB}\,.\,\text{c}\alpha\text{sA} = -1;$$

or (comp. IX.),

$$\text{XIII.}\ldots -\cos\text{C} + \gamma\sin\text{C} = (\cos\text{B} + \beta\sin\text{B})(\cos\text{A} + \alpha\sin\text{A}).$$

(3.) Taking the scalars on both sides of this last equation, and remembering that $\text{S}\beta\alpha = -\cos c$, we thus immediately derive *one form* of the *fundamental equation of spherical trigonometry*; namely, the equation,

$$\text{XIV.}\ldots \cos\text{C} + \cos\text{A}\cos\text{B} = \cos c \sin\text{A}\sin\text{B}.$$

(4.) Taking the vectors, we have this other formula:

$$\text{XV.}\ldots \gamma\sin\text{C} = \alpha\sin\text{A}\cos\text{B} + \beta\sin\text{B}\cos\text{A} + \text{V}\beta\alpha\sin\text{A}\sin\text{B};$$

which is easily seen to agree with 306, XII., and may also be usefully compared with the equation 210, XXXVII.

(5.) The result XV. may be enunciated in the form of a *Theorem*, as follows:—

"*If there be any spherical triangle* ABC, *and three lines be drawn from the centre* O *of the sphere, one towards the point* A, *with a length* = sin A cos B; *another* *towards the point* B, *with a length* = sin B cos A ; *and the third perpendicular to the plane* AOB, *and towards the same side of it as the point* C, *with a length* = sin c sin A sin B ; *and if, with these three lines as edges, we construct a parallelepiped*: *the intermediate diagonal from* O *will be directed towards* C, *and will have a length* = sin C."

(6.) Dividing both members of the same equation XV. by ρ, and taking scalars, we find that if P be *any fourth point* on the sphere, and Q the *foot of the perpendicular* let fall from this point *on the arc* AB, this perpendicular PQ being considered as *positive* when C and P are situated *at one common side* of that arc (or in one *common hemisphere*, of the two into which the great circle through A and B divides the spheric surface), we have then,

$$\text{XVI.} \ldots \sin \text{C} \cos \text{PC} = \sin \text{A} \cos \text{B} \cos \text{PA} + \sin \text{B} \cos \text{A} \cos \text{PB} + \sin \text{A} \sin \text{B} \sin c \sin \text{PQ};$$

a formula which might have been derived from the equation 210, XXXVIII., by first cyclically changing abcABC to bcaBCA, and then passing from the former triangle to its *polar* or *supplementary* : and from which many less general equations may be deduced, by assigning particular positions to P.

(7.) For example, if we conceive the point P to be the *centre of the circumscribed small circle* ABC, and denote by R the *arcual radius* of that circle, and by S the *semisum of the three angles*, so that $2\text{S} = \text{A} + \text{B} + \text{C} = \pi + \sigma$, if σ again denote, as in 297, (47.), the *area* of the triangle* ABC, whence

$$\text{XVII.} \ldots \text{PA} = \text{PB} = \text{PC} = R, \quad \text{and} \quad \sin \text{PQ} = \sin R \sin (\text{S} - \text{C}),$$

the formula XVI. gives easily,

$$\text{XVIII.} \ldots 2 \cot R \sin \frac{\sigma}{2} = \sin \text{A} \sin \text{B} \sin c;$$

a relation between radius and area, which agrees with known results, and from which we may, by 297, LXX., &c., deduce the known equation :

$$\text{XIX.} \ldots e \tan R = 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2};$$

in which we have still, as in 297, (47.), &c.,

$$\text{XX.} \ldots e = (\text{S}\alpha\beta\gamma =) \sin a \sin b \sin \text{C} = \&\text{c}.$$

(8.) In like manner we might have supposed, in the corresponding general equation 210, XXXVIII., that P was placed at the centre of the *inscribed*

* Compare the Note to the cited sub-article.

small circle, and that the arcual radius of *that* circle was r, the semisum of the *sides* being s; and thus should have with ease deduced this other known relation, which is a sort of polar reciprocal of XVIII.,

$$\text{XXI.} \ldots 2 \tan r \,.\, \sin s = e.$$

But these results are mentioned here, only to exemplify the fertility of the formulæ, to which the present calculus conducts, and from which the theorem in (5.) was early seen to be a consequence.

(9.) We might *develop the ternary product* in the equation XII., as we developed the *binary product* XIII.; compare scalar and vector parts; and operate on the latter, by the symbol $\mathrm{S} \,.\, \rho^{-1}$. *New general theorems*, or at least new general *forms*, would thus arise, of which it may be sufficient in this place to have merely suggested the investigation.

(10.) As regards the *order of rotation* (1.) (2.), it is clear, from a mere *inspection* of the formula XV., that the rotation round γ from β to α, or that round C from B to A, *must be positive, when that equation* XV. *holds good*; at least if the *angles* A, B, C, of the *triangle* ABC, be (as usual) treated as *positive*: because the rotation round the *line* $\mathrm{V}\beta\alpha$ from β to α is *always* positive (by 281, (3.)).

(11.) If, then, for any *given spherical triangle*, ABC, with *angles* still supposed to be *positive*, the *rotation* round C from B to A should happen to be (on the contrary) *negative*, we should be obliged to *modify the formula* XV.; which could be done, for example, so as to restore its correctness, by *interchanging* α *with* β, and at the same time A with B.

(12.) There is, however, a *sense* in which the *formula* might be considered as still *remaining true*, without any change in the mode of *writing* it; namely, if we were to *interpret the symbols*, A, B, C as denoting *negative angles*, for the *case* last supposed (11.). Accordingly, if we take the *reciprocal* of the equation X., we get this other equation,

$$\text{XXII.} \ldots \alpha^{-x}\beta^{-y}\gamma^{-z} = -1;$$

where x, y, z are *positive*, as before, and therefore the *new exponents*, $-x, -y, -z$, are *negative*, if the rotation round α from β to γ be *itself* negative, as in (1.).

(13.) On the whole, then, if α, β, γ be *any given system of three co-initial and diplanar unit-lines*, OA, OB, OC, we can *always assign a system of three scalars*, x, y, z, which shall satisfy the *exponential equation* X., and shall have *relations of the form* VII. *to the spherical angles* A, B, C; but these three scalars, if determined so as to fall *between the limits* ± 2, will be *all positive*, or *all negative*, according as the *rotation* round α from β to γ is *negative*, as in (1.), or *positive*, as in (11.).

(14.) As regards the *limits* just mentioned, or the *inequalities*,

$$\text{XXIII.} \ldots x < 2, \quad y < 2, \quad z < 2; \qquad x > -2, \quad y > -2, \quad z > -2,$$

they are introduced with a view to render the problem of finding the *exponents xyz* in the formula X. *determinate*; for since we have, by 308,

$$\text{XXIV.} \ldots \alpha^4 = \beta^4 = \gamma^4 = +1, \quad \text{if} \quad \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = 1,$$

we might otherwise *add any multiple* (positive or negative) *of the number four*, to the value of the exponent of *any unit-line*, and the value of the resulting *power* would not be altered.

(15.) If we admitted exponents $= \pm 2$, we might render the problem of satisfying the equation X. *indeterminate* in another way; for it would then be sufficient to suppose that *any one* of the three exponents was thus equal to $+2$, or -2, and that the *two others* were each $= 0$; or else that *all three* were of the form ± 2.

(16.) When it was lately said (13.), that the *exponents*, x, y, z, in the formula X., if *limited* as above, would have one *common sign*, the *case* was tacitly excluded, for which those exponents, or *some* of them, when multiplied each by a quadrant, give angles *not* equal to those of the spherical *triangle* ABC, whether positively *or* negatively taken; *but* equal to the *supplements* of those angles, or to the *negatives* of those supplements.

(17.) In fact, it is evident (because $\alpha^2 = \beta^2 = \gamma^2 = -1$), that the equation X., or the reciprocal equation XXII., if it be satisfied by any *one system* of values of xyz, will *still* be satisfied, when we divide or multiply *any two* of the three exponential *factors*, by the *squares* of the two *unit-vectors*, of which those factors are supposed to be *powers*: or in other words, if we *subtract* or *add* the *number two*, in *each of two exponents*.

(18.) We may, for example, derive from XXII. this other equation:

$$\text{XXV.} \ldots \alpha^{2-x}\beta^{2-y}\gamma^{-z} = -1; \quad \text{or} \quad \text{XXVI.} \ldots \alpha^{2-x}\beta^{2-y} = \gamma^{z-2};$$

which, when the rotation is as supposed in (1.), so that xyz are *positive*, may be interpreted as follows.

(19.) Conceive a *lune* CC′, with points A and B on its two bounding semicircles, and with a negative rotation round A from B to C; or, what comes to the same thing, with a positive rotation round A from B to C′. Then, on the plan illustrated by figures 45 and 46, the *supplements* $\pi - \text{A}$, $\pi - \text{B}$, of the angles A and B in the triangle ABC, or the angles at the *same points* A and B in the *co-lunar* triangle ABC′, will *represent two versors*, a *multiplier*, and a *multiplicand*, which are precisely those denoted, in XXVI., by the *two factors*, α^{2-x} and β^{2-y}; and the *product* of these two factors, taken in *this order*, is that *third versor*, which has its *axis* directed to C′, and is *represented*, on the same general plan (177), by the *external angle of the lune, at that point*, C′; which, in *quantity*, is equal to the external angle of the same lune at C, or to the angle $\pi - \text{C}$. This *product* is therefore equal to that *power of the unit-line* OC′, or $-\gamma$, which has its *exponent* $= \frac{2}{\pi}(\pi - \text{C}) = 2 - z$; we have therefore, by this *construction*, the equation,

$$\text{XXVII.} \ldots \alpha^{2-x}\beta^{2-y} = (-\gamma)^{2-z};$$

which (by 308, (6.)) agrees with the recent formula XXVI.

310. The equation,

$$\text{I.} \ldots \gamma^{\frac{2\text{C}}{\pi}} \beta^{\frac{2\text{B}}{\pi}} \alpha^{\frac{2\text{A}}{\pi}} = -1,$$

which results from 309, (1.), and in which α, β, γ are the unit-vectors OA, OB, OC of any three points on the unit-sphere; while the three scalars A, B, C, in the exponents of the three factors, represent generally the angular quantities of rotation, round those three unit-lines, or radii, α, β, γ, from the plane AOC to the plane AOB, from BOA to BOC, and from COB to COA, and are positive or negative according as these rotations of planes are themselves positive or negative: must be regarded as an important formula, in the applications of the present Calculus. It *includes*, for example, the whole doctrine of *Spherical Triangles*; not merely because it conducts, as we have seen (309, (3.)), to *one form of the fundamental scalar equation of spherical trigonometry*, namely to the equation,

$$\text{II.} \ldots \cos \text{C} + \cos \text{A} \cos \text{B} = \cos c \sin \text{A} \sin \text{B};$$

but also because it gives a *vector equation* (309, (4.)), which serves to *connect* the *angles*, or the *rotations*, A, B, C, with the *directions** *of the radii*, α, β, γ, or OA, OB, OC, for any system of *three diverging right lines* from one origin. It

* This may be considered to be another instance of that habitual reference to *direction*, as *distinguished* from mere *quantity* (or magnitude), although *combined* therewith, which pervades the present Calculus, and is eminently *characteristic* of it; whereas *Des Cartes*, on the contrary, had aimed to reduce all problems of geometry to the determination of the *lengths of right lines*; although (as all who use his *co-ordinates* are of course well aware) a certain reference to *direction* is even in *his* theory inevitable, in connexion with the interpretation of *negative roots* (by him called *inverse* or *false* roots)

may, therefore, be not improper to make here a few additional remarks, respecting the nature, evidence, and extension of the recent formula I.

(1.) Multiplying both members of the equation I., by the inverse exponential $\gamma^{-\frac{2C}{\pi}}$, we have the transformation (comp. 309, (1.)):

$$\text{III.} \ldots \beta^{\frac{2B}{\pi}} \alpha^{\frac{2A}{\pi}} = -\gamma^{-\frac{2C}{\pi}} = \gamma^{\frac{2(\pi - C)}{\pi}}.$$

(2.) Again, multiplying both members of I. *into** $\alpha^{-\frac{A}{\pi}}$, we obtain this other formula:

$$\text{IV.} \ldots \gamma^{\frac{2C}{\pi}} \beta^{\frac{2B}{\pi}} = -\alpha^{-\frac{2A}{\pi}} = \alpha^{\frac{2(\pi - A)}{\pi}}.$$

(3.) Multiplying this last equation IV. *by* $\alpha^{\frac{2A}{\pi}}$, and the equation III. *into* $\gamma^{\frac{2C}{\pi}}$, we derive these other forms:

$$\text{V.} \ldots \alpha^{\frac{2A}{\pi}} \gamma^{\frac{2C}{\pi}} \beta^{\frac{2B}{\pi}} = -1; \quad \text{VI.} \ldots \beta^{\frac{2B}{\pi}} \alpha^{\frac{2A}{\pi}} \gamma^{\frac{2C}{\pi}} = -1;$$

so that *cyclical permutation of the letters*, α, β, γ, and A, B, C, *is allowed in the equation* I.; as indeed was to be expected, from the nature of the theorem which that equation expresses.

(4.) From either V. or VI. we can deduce the formula:

$$\text{VII.} \ldots \alpha^{\frac{2A}{\pi}} \gamma^{\frac{2C}{\pi}} = -\beta^{-\frac{2B}{\pi}} = \beta^{\frac{2(\pi - B)}{\pi}};$$

by comparing which with III. and IV., we see that *cyclical permutation* of letters is permitted, in *these* equations also.

(5.) Taking the *reciprocal* (or *conjugate*) of the equation I., we obtain (compare 309, XXII.) this other equation:

$$\text{VIII.} \ldots \alpha^{-\frac{2A}{\pi}} \beta^{-\frac{2B}{\pi}} \gamma^{-\frac{2C}{\pi}} = -1;$$

or

$$\text{IX.} \ldots \alpha^{\frac{2(\pi - A)}{\pi}} \beta^{\frac{2(\pi - B)}{\pi}} \gamma^{\frac{2(\pi - C)}{\pi}} = +1;$$

---

of equations. Thus in the first sentence of Schooten's recently cited translation (1659) of the *Geometry* of Des Cartes, we find it said: "Omnia Geometriæ Problemata facilè ad hujusmodi terminos reduci possunt, ut deinde ad illorum constructionem, opus tantum sit rectarum quarundam longitudinem cognoscere."

The very different *view of geometry*, to which the present writer has been led, makes it the more proper to express here the profound admiration with which he regards the cited Treatise of Des Cartes: containing as it does the germs of so large a portion of all that has since been done in mathematical science, even as concerns *imaginary roots* of equations, considered as marks of *geometrical impossibility*.

* For the distinction between multiplying a quaternion *into* and *by* a factor, see the Notes to pages 147, 159.

in which cyclical permutation of letters is again allowed, and from which (or from III.) we can at once derive the formula,

$$\text{X.} \ldots \alpha^{-\frac{2\text{A}}{\pi}} \beta^{-\frac{2\text{B}}{\pi}} = -\gamma^{\frac{2\text{C}}{\pi}}.$$

(6.) The equation X. may also be thus written (comp. 309, XXVII.):

$$\text{XI.} \ldots \alpha^{\frac{2(\pi-\text{A})}{\pi}} \beta^{\frac{2(\pi-\text{B})}{\pi}} = \gamma^{-\frac{2(\pi-\text{C})}{\pi}} = (-\gamma)^{\frac{2(\pi-\text{C})}{\pi}}.$$

(7.) And all the foregoing equations may be *interpreted* (comp. 309, (19.)), and at the same time *proved*, by a reference to that general *construction* (177) for the *multiplication of versors*, which the figures 45 and 46 were designed to illustrate; if we bear in mind that a *power* α^t, of an *unit-line* α, with a *scalar exponent*, t, is (by 308, 309) a *versor*, which has the *effect* of *turning a line* $\perp$ α, *through* t *right angles*, *round* α as an *axis of rotation*.

(8.) The principle expressed by the equation I., from which all the subsequent equations have been deduced, may be stated in the following manner, if we adopt the *definition* proposed in an earlier part of this work (180, (4.)), for the *spherical sum* of two angles on a spheric surface:

"*For any spherical triangle, the Spherical Sum of the three angles, if taken in a suitable Order, is equal to Two Right Angles.*"

(9.) In fact, when the rotation round A from B to C is negative, if we *spherically add* the angle B to the angle A, the *spherical sum* so obtained is (by the definition referred to) equal to the *external angle at* C; if then we *add* to *this sum*, or *supplement* of C, the angle C *itself*, we get a *final* or *total sum*, which is exactly equal to π; *addition of spherical angles* at *one vertex*, and therefore in *one plane*, being accomplished in the *usual* manner; but the *spherical summation* of angles with *different vertices* being performed according to those *new rules*, which were deduced in the Ninth Section of Book II., Chapter I.; and were connected (180, (5.)) with the conception of *angular transvection*, or of the *composition of angular motions, in different and successive planes.*

(10.) Without pretending to attach importance to the following *notation*, we may just propose it in passing, as one which may serve to recall and represent the *conception* here referred to. Using a *plus in parentheses*, as a *symbol* or *characteristic* of such *spherical addition of angles*, the formula I. may be *abridged* as follows:

$$\text{XII.} \ldots \text{C} \, (+) \, \text{B} \, (+) \, \text{A} = \pi;$$

the *symbol of an added angle* being written to the *left* of the symbol of the *angle to which it is added* (comp. 264, (4.)); because *such addition corresponds* (as above) to a *multiplication of versors*, and we have agreed to write the *symbol of the multiplier* to the *left** of the symbol of the *multiplicand*, in every *multiplication of quaternions*.

311. There is, however, *another view* of the important equation 310, I., according to which it is connected rather with *addition of arcs* (180, (3.)), than with *addition of angles* (180, (4.)); and may be *interpreted*, and *proved anew*, with the help of the *supplementary* or *polar triangle*, A′B′C′, as follows.

(1.) The rotation round A from B to C being still supposed to be negative, let A′, B′, C′ be (as in 175) the positive poles of the sides BC, CA, AB; and let α', β', γ' be their unit-vectors. Then, because the rotation round α from γ' to β' is positive (by 180, (2.)), and is in quantity the supplement of the spherical angle A, the *product* $\gamma'\beta'$ will be (by 281, (2.), (3.)) a *versor*, of which α is the *axis*, and A the *angle*; with similar results for the two other products, $\alpha'\gamma'$, $\beta'\alpha'$.

(2.) If then we write (comp. 291),

$$\text{I.} \ldots \alpha' = \mathrm{UV}\beta\gamma, \quad \beta' = \mathrm{UV}\gamma\alpha, \quad \gamma' = \mathrm{UV}\alpha\beta,$$

supposing that

$$\text{II.} \ldots \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = 1, \quad \text{and} \quad \text{III.} \ldots \mathrm{S}\alpha\beta\gamma > 0$$

we shall have (comp. again 180, (2.)),

$$\text{IV.} \ldots \alpha = \mathrm{UV}\gamma'\beta', \quad \beta = \mathrm{UV}\alpha'\gamma', \quad \gamma = \mathrm{UV}\beta'\alpha', \dagger$$

and

$$\text{V.} \ldots \mathrm{A} = \angle\gamma'\beta', \quad \mathrm{B} = \angle\alpha'\gamma', \quad \mathrm{C} = \angle\beta'\alpha';$$

whence (by 308 or 309) we have the following *exponential expressions* for these three last *products of unit-lines*.

$$\text{VI.} \ldots \gamma'\beta' = \alpha^{\frac{2\mathrm{A}}{\pi}}; \quad \alpha'\gamma' = \beta^{\frac{2\mathrm{B}}{\pi}}; \quad \beta'\alpha' = \gamma^{\frac{2\mathrm{C}}{\pi}}.$$

(3.) Multiplying these three expressions, in an inverted order, we have, therefore, the new product:

$$\text{VII.} \ldots \gamma^{\frac{2\mathrm{C}}{\pi}}\beta^{\frac{2\mathrm{B}}{\pi}}\alpha^{\frac{2\mathrm{A}}{\pi}} = \beta'\alpha' . \alpha'\gamma' . \gamma'\beta' = \gamma'^2\beta'^2\alpha'^2 = -1;$$

and the equation 310, I. is in this way *proved anew*.

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* Compare the Note to page 147.

† [Here $\quad \mathrm{UV}\beta'\gamma' = \mathrm{UVV}\gamma\alpha\mathrm{V}\alpha\beta = \mathrm{U}(-\alpha\mathrm{S}\alpha\beta\gamma) = -\alpha\mathrm{US}\alpha\beta\gamma.$]

(4.) And because, instead of VI., we might have written,

$$\text{VIII.} \ldots \alpha^{\frac{2A}{\pi}} = -\frac{\gamma'}{\beta'}; \quad \beta^{\frac{2B}{\pi}} = -\frac{\alpha'}{\gamma'}; \quad \gamma^{\frac{2C}{\pi}} = -\frac{\beta'}{\alpha'},$$

we see that the *equation* to be proved may be reduced to the form of the *identity*

$$\text{IX.} \ldots \frac{\beta'}{\alpha'} \frac{\alpha'}{\gamma'} \frac{\gamma'}{\beta'} = +1;$$

and may be *interpreted* as expressing, what is evident, that if a point be supposed to move first along the side B′C′, of the polar triangle A′B′C′, from B′ to C′; then along the successive side C′A′, from C′ to A′; and finally along the remaining side A′B′, from A′ to B′, it will thus have *returned* to the position from which it *set out*, or will *on the whole* have *not changed place* at all.

(5.) In *this view*, then, we perform what we have elsewhere called an *addition of arcs* (instead of *angles* as in 310); and in a *notation* already used (264, (4.)), we may express the result by the formula,

$$\text{X.} \ldots \frown \mathrm{A'B'} + \frown \mathrm{C'A'} + \frown \mathrm{B'C'} = 0;$$

each of the two *left-handed symbols* denoting an *arc*, which is conceived to be *added* (as a *successive vector-arc*, 180, (3.)), to the arc whose symbol immediately *follows* it, or is written *next* it, but towards the *right-hand*.

(6.) The expressions VI. or VIII., for the *exponential factors* in 310, I., show in a new way the necessity of attending to the *order* of those factors, in that formula: for if we should *invert that order*, *without altering* (as in 310, VIII.) *the exponents*, we may now see that we should obtain this *new product*:

$$\text{XI.} \ldots \alpha^{\frac{2A}{\pi}} \beta^{\frac{2B}{\pi}} \gamma^{\frac{2C}{\pi}} = -\frac{\gamma'}{\beta'} \frac{\alpha'}{\gamma'} \frac{\beta'}{\alpha'} = +(\gamma'\beta'\alpha')^2;$$

which, on account of the *diplanarity* of the lines α', β', γ', is *not equal to negative unity*, but to a certain *other versor*; the properties of which may be inferred from what was shown in 297, (64.), and in 298, (8.), but upon which we cannot here delay.

312. In general (comp. 221), *an equation*, such as

$$\text{I.} \ldots q' = q,$$

between two quaternions, includes a system of four scalar equations*, such as the following:

$$\text{II.} \ldots \mathrm{S}q' = \mathrm{S}q; \quad \mathrm{S}\alpha q' = \mathrm{S}\alpha q; \quad \mathrm{S}\beta q' = \mathrm{S}\beta q; \quad \mathrm{S}\gamma q' = \mathrm{S}\gamma q;$$

* The *propriety*, which such results as this establish, for the use of the *name*, QUATERNIONS, as applied to this whole Calculus, on account of its essential connexion with the *number* FOUR, does not require to be again insisted on.

where α, β, γ may be *any three actual and diplanar vectors*: and conversely, if α, β, γ be any three *such* vectors, then the *four* scalar equations II. reproduce, and are sufficiently replaced by, the *one* quaternion equation I. But an *equation between two vectors* is equivalent only to a system of *three scalar equations*, such as the *three last* equations II.; for example, in 294, (12.), the *one vector equation* XXII. is equivalent to the *three scalar equations* XXI., under the immediately preceding *condition of diplanarity* XX. In like manner, an *equation between two versors of quaternions*,* such as the equation

$$\text{III.} \ldots \mathrm{U}q' = \mathrm{U}q,$$

includes generally a system of *three*, but of *not more than three*, *scalar* equations; because the *versor* Uq *depends* generally (comp. 157) on a *system of three scalars*, namely the *two* which determine its *axis* Ax . q, and the *one* which determines its *angle* $\angle\, q$; or because the *versor equation* III. requires to be *combined* with the *tensor equation*,

$$\text{IV.} \ldots \mathrm{T}q' = \mathrm{T}q, \qquad \text{compare 187 (13.)},$$

in order to reproduce the *quaternion equation* I. Now the recent equation, 310, I., is evidently of this *versor-form* III., if α, β, γ be still supposed to be *unit-lines*. If then we *met that equation*, or if one of its *form* had occurred to us, without any knowledge of its *geometrical signification*, we might propose to *resolve it*, with respect to the *three scalars* A, B, C, treated as *three unknown quantities*. The few following remarks, on the problem thus proposed, may be not out of place, nor uninstructive, here.

(1.) Writing for abridgment,

$$\text{V.} \ldots \cot \mathrm{A} = t, \quad \cot \mathrm{B} = u, \quad \cot \mathrm{C} = v,$$

and

$$\text{VI.} \ldots s = -\operatorname{cosec} \mathrm{A} \operatorname{cosec} \mathrm{B} \operatorname{cosec} \mathrm{C},$$

the equation to be resolved becomes (by 308, VII., or 309, XII.),

$$\text{VII.} \ldots (v + \gamma)(u + \beta)(t + \alpha) = s;$$

in which the *tensors* on both sides are already equal, because

$$\text{VIII.} \ldots s^2 = (v^2 + 1)(u^2 + 1)(t^2 + 1).$$

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* An equation, $\mathrm{U}\rho' = \mathrm{U}\rho$, or $\mathrm{UV}q' = \mathrm{UV}q$, between two *versors of vectors* (156), or between the *axes* of two quaternions (291), is equivalent only to a system of *two scalar equations*; because the *direction of an axis*, or of a *vector*, depends on a system of *two angular elements* (111).

(2.) Multiplying the equation VII. *by* $t + \alpha$, and *into* $t - \alpha$, and dividing the result by $t^2 + 1$, we have this new equation of the same *form*, but differing by *cyclical permutation* (comp. 310, (3.)):

$$\text{IX.} \ldots (t + \alpha)(v + \gamma)(u + \beta) = s;$$

and in like manner,

$$\text{X.} \ldots (u + \beta)(t + \alpha)(v + \gamma) = s.$$

(3.) Taking the half difference of the two last equations, and observing that (by 279, IV., and 294, II.)

$$\text{XI.} \ldots \begin{cases} \frac{1}{2}(\beta\alpha\gamma - \alpha\gamma\beta) = \text{V}.\beta\text{V}\alpha\gamma = \gamma\text{S}\alpha\beta - \alpha\text{S}\beta\gamma, \\ \frac{1}{2}(\beta\alpha - \alpha\beta) = \text{V}\beta\alpha, \quad \frac{1}{2}(\beta\gamma - \gamma\beta) = \text{V}\beta\gamma, \end{cases}$$

we arrive at this new equation, of *vector form*:

$$\text{XII.} \ldots 0 = v\text{V}\beta\alpha + t\text{V}\beta\gamma + \gamma\text{S}\alpha\beta - \alpha\text{S}\beta\gamma;$$

which is equivalent only to a system of *two scalar equations*, because it gives $0 = 0$, when operated on by $\text{S}.\beta$ (comp. 294, (9.)).

(4.) It enables us, however, to *determine the two scalars*, t and v; for if we operate on it by $\text{S}.\alpha$, we get (comp. 298, XXVI.),

$$\text{XIII.} \ldots t\text{S}\alpha\beta\gamma = \alpha^2\text{S}\beta\gamma - \text{S}\beta\alpha\,\text{S}\alpha\gamma = \text{S}(\text{V}\beta\alpha.\text{V}\alpha\gamma);$$

and if we operate on the same equation XII. by $\text{S}.\gamma$, we get in like manner,

$$\text{XIV.} \ldots v\text{S}\alpha\beta\gamma = \gamma^2\text{S}\alpha\beta - \text{S}\alpha\gamma\,\text{S}\gamma\beta = \text{S}(\text{V}\alpha\gamma.\text{V}\gamma\beta).$$

(5.) Processes quite similar give the analogous result,

$$\text{XV.} \ldots u\text{S}\alpha\beta\gamma = \beta^2\text{S}\gamma\alpha - \text{S}\gamma\beta\,\text{S}\beta\alpha = \text{S}(\text{V}\gamma\beta.\text{V}\beta\alpha):$$

and thus the *problem* is *resolved*, in the sense that *expressions have been found* for the *three sought scalars*, t, u, v, or for the *cotangents* V. of the *three sought angles* A, B, C: whence the *fourth scalar*, s, in the quaternion equation VII., can easily be deduced, as follows.

(6.) Since (by 294, (6.), changing δ to α, and afterwards cyclically permuting) we have, for *any three vectors* α, β, γ, the general transformations,

$$\text{XVI.} \ldots \begin{cases} \alpha\text{S}\alpha\beta\gamma = \text{V}(\text{V}\beta\alpha.\text{V}\alpha\gamma), \\ \beta\text{S}\alpha\beta\gamma = \text{V}(\text{V}\gamma\beta.\text{V}\beta\alpha), \\ \gamma\text{S}\alpha\beta\gamma = \text{V}(\text{V}\alpha\gamma.\text{V}\gamma\beta), \end{cases}$$

the expressions XIII. XV. XIV. give,

$$\text{XVII.} \ldots \begin{cases} (t + a)\,\mathrm{S}\alpha\beta\gamma = \mathrm{V}\beta\alpha\,.\,\mathrm{V}\alpha\gamma\,; \\ (u + \beta)\,\mathrm{S}\alpha\beta\gamma = \mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha\,; \\ (v + \gamma)\,\mathrm{S}\alpha\beta\gamma = \mathrm{V}\alpha\gamma\,.\,\mathrm{V}\gamma\beta\,; \end{cases}$$

whence, by VII.,

$$\text{XVIII.} \ldots s\,(\mathrm{S}\alpha\beta\gamma)^3 = (\mathrm{V}\gamma\beta)^2(\mathrm{V}\beta\alpha)^2(\mathrm{V}\alpha\gamma)^2\,;$$

and thus the remaining scalar, s, is also entirely determined.

(7.) And the equation VIII. may be verified, by observing that the expressions XVII. give,

$$\text{XIX.} \ldots \begin{cases} (t^2 + 1)\,(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\beta\alpha)^2\,(\mathrm{V}\alpha\gamma)^2\,; \\ (u^2 + 1)\,(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\gamma\beta)^2(\mathrm{V}\beta\alpha)^2\,; \\ (v^2 + 1)\,(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\alpha\gamma)^2\,(\mathrm{V}\gamma\beta)^2. \end{cases}$$

(8.) The equations XIII. XIV. XV. XVI. give, by elimination of $\mathrm{S}\alpha\beta\gamma$, these new expressions:

$$\text{XX.} \ldots \alpha t^{-1} = (\mathrm{V} : \mathrm{S})(\mathrm{V}\beta\alpha\,.\,\mathrm{V}\alpha\gamma)\,; \quad \beta u^{-1} = (\mathrm{V} : \mathrm{S})(\mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha)\,;$$

$$\gamma v^{-1} = (\mathrm{V} : \mathrm{S})(\mathrm{V}\alpha\gamma\,.\,\mathrm{V}\gamma\beta)\,;$$

by comparing which with the formula 281, XXVIII., after suppressing (291) the characteristic I., we find that the *three scalars*, t, u, v, are either Ist, the *cotangents of the angles opposite to the sides* a, b, c, *of the spherical triangle in which the three given unit-lines* α, β, γ *terminate*, or IInd, the *negatives of those cotangents*, the *angles themselves* of that triangle being as usual supposed to be *positive* (309, (10.)), according as the *rotation* round α from β to γ is *negative* or *positive*: that is (294, (3.)), according as $\mathrm{S}\alpha\beta\gamma >$ or < 0; or finally, by XVIII., according as the *fourth scalar*, s, is negative or positive, because the second member of that equation XVIII. is *always negative*, as being the product of three *squares of vectors* (282, 292).

(9.) In the Ist case, which is that of 309, (1.), we see then *anew*, by V. and VI., that we are *permitted to interpret the scalars* A, B, C, *in the exponential formula* 310, I., *as equal to the angles of the spherical triangle* (8.), which are usually denoted by the *same letters*. But we see also, that we may *add any even multiples of* π *to those three angles, without disturbing the exponential equation*; or any *one even*, and *two odd* multiples of π, in any order, so as to preserve a *positive product of cosecants*, because s is, for *this* case, *negative* in VI., by (8.).

(10.) In the IInd case, which is that of 309, (11.), we may, for similar reasons, *interpret the scalars* A, B, C, in the formula 310, I., as equal to the *negatives of the angles of the triangle*; and as thus having, what VI. *now* requires, because s is now *positive* (8.), a *negative product of cosecants*, while their *cotangents* have the values required. But we may also *add*, as in (9.), *any multiples of* π, to the scalars thus found for the formula, provided that the *number* of the *odd* multiples, so added, is itself *even* (0 or 2).

(11.) The conclusions of 309, or 310, respecting the *interpretation of the exponential formula*, are therefore confirmed, and might have been anticipated, by the present *new analysis*: in conducting which it is evident that we have been dealing with *real scalars*, and with *real vectors*, only.

(12.) If this last *restriction* were *removed*, and *imaginary values admitted*, in the solution of the *quaternion equation* VII., we might have begun by operating, as in II., on that equation, by the *four characteristics*,

$$\text{XXI.} \ldots \mathrm{S}, \quad \mathrm{S}\,.\,\alpha, \quad \mathrm{S}\,.\,\beta, \quad \text{and} \quad \mathrm{S}\,.\,\gamma;$$

which would have given, with the significations 297, (1.), (3.), of l, m, n, and e, and therefore with the following *relation* between those *four scalar data*,

$$\text{XXII.} \ldots e^2 = 1 - l^2 - m^2 - n^2 + 2lmn,$$

a system of *four scalar equations*, involving the *four sought scalars*, s, t, u, v; from which it might have been required to deduce the (real or imaginary) values of those four scalars, by the ordinary processes of *algebra*.

(13.) The four scalar equations, so obtained, are the following:

$$\text{XXIII.} \ldots \begin{cases} 0 = e + lt + mu + nv - tuv + s; \\ 0 = et + mtu + ntv + uv - l; \\ 0 = -eu + ltu + tv + nuv + m - 2ln; \\ 0 = ev + tu + ltv + muv - n; \end{cases}$$

eliminating uv and u between the three last of which, we find, with the help of XXII., the determinant,

$$\text{XXIV.} \ldots 0 = \begin{vmatrix} 1, & mt, & ntv + et - l \\ m, & t, & ltv + ev - n \\ n, & lt - e, & tv + m - 2ln \end{vmatrix} = e\,(t^2 + 1)(ev - n + lm);$$

and analogous eliminations give,

$$\text{XXV.} \ldots 0 = e\,(t^2 + 1)\,(eu - m + nl),$$

and $$\text{XXVI.} \ldots 0 = (t^2 + 1)\,\{e^2 uv - (m - nl)(n - lm) + (1 - l^2)\,(et - l + mn)\}.$$

(14.) Rejecting then the factor $t^2 + 1$ we find, as the *only real solution* of the problem (12.), the following system of values:

$$\text{XXVII.} \ldots \; et = l - mn; \quad eu = m - nl; \quad ev = n - lm;$$

and

$$\text{XXVIII.} \ldots \; e^3 s = -(1 - l^2)(1 - m^2)(1 - n^2);$$

which correspond precisely to those otherwise found before, in (4.) (5.) (6.), and might therefore serve to *reproduce the interpretation* of the *exponential formula* (310).

(15.) But on the purely *algebraic side*, it is found, by a similar analysis, that the *four equations* XXIII. are *satisfied also* by a system of *four imaginary solutions*, represented by the following formulæ:

$$\text{XXIX.} \ldots \begin{cases} t^2 + 1 = 0; \quad v^2 + 1 = 0; \\ s = tuv - lt - mu - nv - e = 0; \end{cases}$$

which it may be sufficient to have mentioned in passing, since they do not appear to have any such *geometrical* interest, as to deserve to be dwelt on here: though, as regards the *consistency* of the different processes employed, it may be remembered that in passing (2.) from the equation VII. to IX., after certain preliminary multiplications, we *divided by* $t^2 + 1$, as we were entitled to do, when seeking only for *real* solutions, because t was supposed to be a *scalar*.

(16.) This seems to be a natural occasion for remarking that the following *general transformation* exists, *whatever three vectors* may be denoted by α, β, γ:

$$\text{XXX.} \ldots \; S(V\beta\gamma \,.\, V\gamma\alpha \,.\, V\alpha\beta) = -(S\alpha\beta\gamma)^2;$$

which proves in a new way (comp. 180), that *the rotation round the line* $V\beta\gamma$, *from* $V\gamma\alpha$ *to* $V\alpha\beta$, *is always positive*; or is directed in the *same sense* (281, (3.)), as the rotation round $V\alpha\beta$ from α to β, &c.

(17.) In like manner we have generally,

$$\text{XXXI.} \ldots \; S(V\alpha\beta \,.\, V\gamma\alpha \,.\, V\beta\gamma) = +(S\alpha\beta\gamma)^2,$$

and

$$\text{XXXII.} \ldots \; S(V\gamma\beta \,.\, V\alpha\gamma \,.\, V\beta\alpha) = +(S\alpha\beta\gamma)^2;$$

so that *the rotation round* $V\gamma\beta$ *from* $V\alpha\gamma$ *to* $V\beta\alpha$ *is negative*, whatever arrangement the three diplanar vectors α, β, γ may have among themselves.

(18.) If then A″, B″, C″ be the *negative poles* of the *three successive sides*, BC, CA, AB, of *any spherical triangle*, the *rotation round* A″ *from* B″ *to* C″ *is* *negative*: which is entirely consistent with the *opposite result* (180), respecting the system of the three *positive poles* A′, B′, C′.

(19.) A *quantitative interpretation* of the equation XXX. may also be easily assigned: for we may infer from it (by 281, (4.), and 294, (3.)) that *if* OABC *be any pyramid, and if normals* OA′, OB′, OC′ *to the three faces* BOC, COA, AOB *have their lengths numerically equal to the areas of those faces* (as bearing the same ratios to *units*, &c.), then (with a similar reference to units) *the volume of the new pyramid*, OA′B′C′, *will be three quarters of the square of the volume of the old pyramid*, OABC.

313. But an allusion was made, in 310, to an *extension* of the *exponential formula* which has lately been under discussion; and in fact, that formula admits of being easily extended, from *triangles* to *polygons* upon the sphere: for we may write, generally,

$$\text{I.} \ldots \alpha_n^{\frac{2A_n}{\pi}} \alpha_{n-1}^{\frac{2A_{n-1}}{\pi}} \ldots \alpha_2^{\frac{2A_2}{\pi}} \alpha_1^{\frac{2A_1}{\pi}} = (-1)^n,$$

if $A_1A_2 \ldots A_{n-1} A_n$ be *any spherical polygon*, and if the scalars $A_1, A_2, \ldots$ in the exponents denote the positive or negative *angles* of that polygon, considered as the *rotations* $A_nA_1A_2$, $A_1A_2A_3$, ... namely those from A_1A_n to A_1A_2, &c.; while n is any positive whole number* > 2.

(1.) One mode of proving this extended formula is the following. Let $OC = \gamma$ be the unit-vector of an arbitrary point C on the spheric surface; and conceive that arcs of great circles are drawn from this point C to the n successive corners of the polygon. We shall thus have a system of n spherical triangles, and each angle of the polygon will (generally) be decomposed into two (positive or negative) *partial angles*, which may be thus denoted:

$$\text{II.} \ldots CA_1A_2 = A'_1, \quad CA_2A_3 = A'_2, \ldots;$$

$$\text{III.} \ldots A_nA_1C = A''_1, \quad A_1A_2C = A''_2, \ldots;$$

so that, with attention to signs of angles in the additions,

$$\text{IV.} \ldots A_1 = A'_1 + A''_1, \quad A_2 = A'_2 + A''_2, \text{ \&c.}$$

Also let

$$\text{V.} \ldots A_2CA_1 = C_1, \quad A_3CA_2 = C_2, \text{ \&c.};$$

and therefore

$$\text{VI.} \ldots C_1 + C_2 + \ldots + C_n = \text{an even multiple of } \pi,$$

which reduces itself to 2π in the simple case of a polygon with no re-entrant angles, and with the point C in its interior.

* The formula admits of interpretation, even for the case $n = 2$.

(2.) Then, for the triangle CA_1A_2, of which the angles are C_1, A'_1, A''_2, we have, by 310, III., the equation,

$$\text{VII.} \ldots a_2^{\frac{2A''_2}{\pi}} a_1^{\frac{2A'_1}{\pi}} = -\gamma^{-\frac{2C_1}{\pi}};$$

and in like manner, for the triangle CA_2A_3, we have

$$\text{VIII.} \ldots a_3^{\frac{2A''_3}{\pi}} a_2^{\frac{2A'_2}{\pi}} = -\gamma^{-\frac{2C_2}{\pi}}, \text{ \&c.}$$

But, when we multiply VII. by VIII., we obtain, by IV., the product,

$$\text{IX.} \ldots a_3^{\frac{2A''_3}{\pi}} a_2^{\frac{2A_2}{\pi}} a_1^{\frac{2A'_1}{\pi}} = +\gamma^{-\frac{2(C_1+C_2)}{\pi}};$$

and so proceeding, we have at last, by VI., a product of the form,

$$\text{X.} \ldots a_1^{\frac{2A''_1}{\pi}} a_n^{\frac{2A_n}{\pi}} \ldots a_2^{\frac{2A_2}{\pi}} a_1^{\frac{2A'_1}{\pi}} = (-1)^n;$$

which reduces itself to I., when it is multiplied *by* $a^{-\frac{2A''_1}{\pi}}$, and *into* $a^{\frac{2A''_1}{\pi}}$ (comp. 310, (3.)). The theorem is therefore proved.

(3.) In words (comp. 310, (8.)), "*the spherical sum of the successive angles of any spherical polygon, if taken in a suitable order, is equal to a multiple of two right angles, which is odd or even, according as the number of the sides* (or corners) of the polygon *is itself odd or even*": the *definition* formerly given (180, (4.)), of a *Spherical Sum of Angles*, being of course retained. And the *reasoning* may be briefly stated thus. When an arbitrary point C is taken on the spherical surface, as in (1.), the *spherical sum* of the *two partial angles*, at the ends of *any one side*, is the *supplement* of the angle which *that side subtends, at the point* C; but the *sum* of all *such* subtended angles is either *four right angles*, or some whole *multiple* thereof: therefore the *sum* of their *supplements* can differ only by some such multiple from $n\pi$, if n be the number of the sides.

(4.) Whatever that *number* may be, if we denote by p_n the *exponential product* in the formula I., we have for *every vector* ρ, and for *every quaternion* q, the equations:

$$\text{XI.} \ldots p_n \rho p_n^{-1} = \rho; \qquad \text{XII.} \ldots p_n q p_n^{-1} = q;$$

whereof the former may (by 308, (8.)) be thus interpreted:

"*If any line* OP, *drawn from the centre* O *of a sphere, be made to revolve conically round any n radii,* $OA_1, \ldots OA_n$, *as n successive axes of rotation, through* *angles equal respectively to the doubles of the angles of the spherical polygon* $A_1 \ldots A_n$, *the line will be brought back to its initial position, by the composition of these n rotations.*"

(5.) Another way of proving the extended formula I., for any spherical *polygon*, is analogous to that which was employed in 311 for the case of a *triangle* on a sphere, and may be stated as follows. Let $A'_1, A'_2, \ldots A'_n$ be the positive poles of the arcs $A_1A_2, A_2A_3, \ldots A_nA_1$; and let $\alpha'_1, \alpha'_2, \ldots \alpha'_n$ be the unit-vectors of those n poles. Then the point A_1 is the positive pole of the new arc $A'_1A'_n$, and the angle A_1 of the polygon at that point is measured by the supplement of that arc; with similar results for other corners of the polygon. Thus we have the system of expressions (comp. 311, VI.):

$$\text{XIII.} \ldots \alpha_1^{\frac{2A_1}{\pi}} = \alpha'_1\alpha'_n; \ \ldots \ \alpha_n^{\frac{2A_n}{\pi}} = \alpha'_n\alpha'_{n-1};$$

so that the *product of powers* in I. is equal to the following *product of n squares of unit-lines*, and therefore to the n^{th} *power of negative unity*.

$$\text{XIV.} \ldots \alpha'_n\alpha'_{n-1} \,.\, \alpha'_{n-1}\alpha'_{n-2} \ldots \alpha'_2\alpha'_1 \,.\, \alpha'_1\alpha'_n = (-1)^n;$$

and thus the extended theorem is proved anew.

(6.) This latter process may be translated into another *theorem of rotation*, on which it is possible that we may briefly return,* in the Second and last Chapter of this Third Book, but upon which we cannot here delay.

(7.) It may be remarked however here (comp. 309, XII.), that the *extended exponential formula* I. may be thus written:

$$\text{XV.} \ldots c\alpha_n S\, A_n \,.\, c\alpha_{n-1} S\, A_{n-1} \ldots c\alpha_2 S\, A_2 \,.\, c\alpha_1 S\, A_1 = (-1)^n.$$

(8.) For example, if ABCD be any *spherical quadrilateral*, of which the *angles* (suitably measured) are denoted by A, .. D, so that A represents the positive or negative *rotation* from AD to AB, &c., while α, β, γ, δ are the unit vectors of its corners, then

$$\text{XVI.} \ldots c\delta S\, D \,.\, c\gamma S\, C \,.\, c\beta S\, B \,.\, c\alpha S\, A = +1.$$

(9.) Hence (comp. 309, XIII.), we may write also,

$$\text{XVII.} \ldots (\cos C - \gamma \sin C)(\cos D - \delta \sin D) = (\cos B + \beta \sin B)(\cos A + \alpha \sin A);$$

and therefore, by taking scalars on both sides, and changing signs,

$$\text{XVIII.} \ldots -\cos C \cos D + \sin C \sin D \cos CD = -\cos B \cos A + \sin B \sin A \cos BA;$$

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* Compare 297, (24.).

in fact, each member of this last formula is equal (by 309, XIV.) to the cosine of the angle AEB, or CED, if the opposite sides AD, BC of the quadrilateral intersect in E.

(10.) Let $\rho = \text{OP}$ be the unit vector of *any fifth point*, P, upon the spheric surface; then operating by $\text{S} . \rho$ on XVII., we obtain this other general formula,

$$\text{XIX.} \ldots \begin{cases} 0 = \sin \text{A} \cos \text{B} \cos \text{AP} + \sin \text{B} \cos \text{A} \cos \text{BP} + \sin \text{A} \sin \text{B} \sin \text{AB} \sin \text{PQ} \\ \quad + \sin \text{C} \cos \text{D} \cos \text{CP} + \sin \text{D} \cos \text{C} \cos \text{DP} + \sin \text{C} \sin \text{D} \sin \text{CD} \sin \text{PR}; \end{cases}$$

in which the *sines of the sides* AB, CD are treated as *always positive*; but the *sines of the perpendiculars* PQ and PR, on those two sides, are regarded as *positive* or *negative*, according as the *rotations* round P, from A to B and from C to D, are *negative* or *positive*: and hence, by assigning particular positions to P, several other but less general equations of *spherical tetragonometry* can be derived.

(11.) For example, if we place P at the *intersection*, say F, of the *opposite sides* AB, CD, the two last perpendiculars will vanish, and *two* of the *six terms* will *disappear*, from the general formula XIX.; and a similar *reduction to four terms* will occur, if we make the arbitrary point P the *pole* of a *side*, or of a *diagonal*.

314. The definition of the *power* a^t, which was assigned in 308, enables us to form some useful expressions, by quaternions, for *circular*, *elliptic*, and *spiral loci*, in a given plane, or in space, a few of which may be mentioned here.

(1.) Let α be any given unit-vector OA, and β any other given line OB, perpendicular to it; then, by the definition (308), if we write,

$$\text{I.} \ldots \text{OP} = \rho = \alpha^t \beta, \quad \text{T}\alpha = 1, \quad \text{S}\alpha\beta = 0,$$

the *locus of the point* P will be the *circumference of a circle*, with O for *centre*, and OB for *radius*, and in a *plane perpendicular to* OA.

(2.) If we *retain* the condition $\text{T}\alpha = 1$, but *not* the condition $\text{S}\alpha\beta = 0$, then the *product* $\alpha^t\beta$ will be in general a *quaternion*, and not merely a *vector*; but if we take its *vector-part* (292), we can form this *new* vector-expression,

$$\text{II.} \ldots \text{OP} = \rho = \text{V} . \alpha^t\beta = \beta \cos x + \gamma \sin x,$$

where

$$\text{III.} \ldots 2x = t\pi, \quad \text{and} \quad \text{IV.} \ldots \gamma = \text{OC} = \text{V}\alpha\beta;$$

and *now* the *locus* of P is a *plane ellipse*, with its *centre* at O, and with OB and OC for its *major* and *minor semiaxes*: while the angular quantity, x, is what is often called the *excentric anomaly*.

(3.) If we write, under the same conditions (2.),

$$\text{V.} \ldots \text{OB}' = \beta' = \text{V}\beta\alpha : \alpha = \alpha^{-1}\gamma, \quad \text{and} \quad \text{VI.} \ldots \text{OP}' = \rho' = \text{V}\rho\alpha : \alpha = \alpha\text{V}\rho\alpha,$$

so that B′ and P′ are the *projections* (203) of B and P *on a plane drawn through* O, *at right angles to the unit-line* OA, we have then, by II., the equation,

$$\text{VII.} \ldots \rho' = \beta' \cos x + \gamma \sin x = \alpha^t \beta';$$

so that the *locus of this projected point* P′ is a *circle*, with OB′ and OC for two rectangular radii.

(4.) Under the same conditions, the *elliptic locus* (2.), of the point P *itself*, is the *section* of the *right cylinder* (compare 203, (5.)),

$$\text{VIII.} \ldots \text{TV}\alpha\rho = \text{TV}\alpha\beta = \text{T}\gamma,$$

made by the *plane*,

$$\text{IX.} \ldots 0 = \text{S}\gamma\beta\rho, \quad \text{or} \quad \text{IX}'. \ldots \beta^2 \text{S}\alpha\rho = \text{S}\alpha\beta\,\text{S}\beta\rho \text{ (comp. 298, XXVI.)};$$

as a confirmation of which last form we have, by II. and IV.,

$$\text{X.} \ldots \text{S}\alpha\rho = \text{S}\alpha\beta \cos x, \quad \text{S}\beta\rho = \beta^2 \cos x.$$

(5.) If we *retain* the condition $\text{S}\alpha\beta = 0$ (1.), but *not now* the condition $\text{T}\alpha = 1$, we may again write the equation I. for ρ; but the *locus of* P will now be a *logarithmic spiral*, with O for its *pole*, in the plane perpendicular to OA; because *equal angular motions*, of the *turning line* OP, correspond now to *equal multiplications of the length* of that line ρ.

(6.) For example, when the scalar exponent t is increased by 4, so that the *revolving unit line*,

$$\text{XI.} \ldots \text{U}\rho = \text{U}\alpha^t . \text{U}\beta$$

returns (comp. 309, XXIV.) to the *direction* which it had before the increase of t was made, the *length* $\text{T}\rho$ of the *turning line* ρ *itself*, or of the *radius vector of the locus*, is *multiplied by* $\text{T}\alpha^4$; which constant and positive scalar is *not now* equal to *unity*.

(7.) If we *reject both* the conditions (1.),

$$\text{T}\alpha = 1, \quad \text{and} \quad \text{S}\alpha\beta = 0,$$

so that the *line* α, or the *base* of the *power* α^t, is now *neither an unit-line, nor perpendicular to* β, namely to the line on which that power *operates*, as a *factor*, we must *again* take *vector parts*, but we have now this *new* expression:

$$\text{XII.} \ldots \text{OP} = \rho = \text{V}.\, \alpha^t \beta = a^t (\beta \cos x + \gamma \sin x);$$

in which we have written, for abridgment,

$$\text{XIII.} \ldots a = \text{T}\alpha, \quad \gamma = \text{V}(\text{U}\alpha . \beta).$$

(8.) In this more complex case, the *locus* of P is *still* a *plane curve*, and may be said to be now an *elliptic** *logarithmic spiral*; for if we *suppress* the *scalar factor*, a^t, we fall back on the *form* II., and have again an *ellipse* as the locus: but when we *take account* of that factor, we find (comp. (2.)) that *equal increments of excentric anomaly* (x), in the *auxiliary ellipse* so determined, correspond to *equal multiplications of the length* ($T\rho$), of the vector of the *new spiral.*

(9.) We may also project B and P, as in (3.), into points B′ and P′, on the plane through O perpendicular to OA, which plane still contains the extremity C of the auxiliary vector γ; and then, since it is easily proved that $\gamma = U\alpha . \beta'$, the equation of the *projected spiral* becomes (with $T\alpha >$ or < 1),

$$\text{XIV.} \ldots \rho' = a^t(\beta' \cos x + \gamma \sin x) = \alpha^t\beta';$$

so that we are brought back to the case (5.), and the *projected* curve is seen to be a logarithmic spiral, of the known and *ordinary* kind.

(10.) Several *spirals of double curvature* are easily represented, on the same general plan, by merely introducing a *vector-term proportional to* t, combined or not with a *constant* vector-term, in each of the expressions above given, for the *variable* vector ρ. For example, the equation,

$$\text{XV.} \ldots \rho = ct\alpha + \alpha^t\beta, \quad \text{with} \quad T\alpha = 1, \quad \text{and} \quad S\alpha\beta = 0,$$

while c is *any constant scalar* different from zero, represents a *helix*, on the right circular cylinder VIII.

(11.) And if we introduce a new and variable scalar, u, as a *factor* in the right-hand term, and so write,

$$\text{XVI.} \ldots \rho = ct\alpha + u\alpha^t\beta,$$

we shall have an expression for a *variable vector* ρ, considered as depending on *two variable scalars* (t and u), which thus becomes (99) the expression for a *vector of a surface*: namely of that important *Screw Surface*, which is the *locus of the perpendiculars*, let fall from the various points of a *given helix, on the axis* of the cylinder of revolution, on which that helix, or spiral curve, is traced.

* The *usual* logarithmic spiral might perhaps be called, by contrast to *this* one, a *circular logarithmic spiral.* Compare the following sub-article (9.), respecting the *projection* of what is here called an *elliptic* logarithmic spiral.

315. Without at present pursuing farther the study of these *loci* by quaternions, it may be remarked that the definition (308) of the power α^t, especially for the case when $T\alpha = 1$, combined with the laws (182) of i, j, k, and with the identification (295) of those three important right versors with their own indices, enables us to establish the following among other *transformations*, which will be found useful on several occasions.

(1.) Let α be *any unit-vector*, and let t be *any scalar*; then,

$$\text{I.} \ldots \mathrm{S}\,.\,\alpha^{-t} = \mathrm{S}\,.\,\alpha^{t}; \quad \text{II.} \ldots \mathrm{S}\,.\,\alpha^{-t-1} = \mathrm{S}\,.\,\alpha^{t+1} = -\,\mathrm{S}\,.\,\alpha^{t-1};$$

$$\text{III.} \ldots \alpha^{t} = \mathrm{S}\,.\,\alpha^{t} + \alpha\mathrm{S}\,.\,\alpha^{t-1}; \quad \text{IV.} \ldots \alpha^{-t} = \mathrm{S}\,.\,\alpha^{t} - \alpha\mathrm{S}\,.\,\alpha^{t-1};$$

$$\text{V.} \ldots (\mathrm{S}\,.\,\alpha^{t})^2 + (\mathrm{S}\,.\,\alpha^{t-1})^2 = \alpha^{t}\alpha^{-t} = 1.$$

(2.) Let α and ι *be any two unit-vectors*, and let t be still *any scalar*; then

$$\text{VI.} \ldots \mathrm{S}\,.\,\alpha^{t} = \mathrm{S}\,.\,\iota^{t}; \quad \text{VII.} \ldots \mathrm{V}\,.\,\alpha^{t} = \alpha\mathrm{S}\,.\,\alpha^{t-1};$$

$$\text{VIII.} \ldots \alpha\mathrm{V}\,.\,\alpha^{t} = \alpha^2\mathrm{S}\,.\,\alpha^{t-1} = \mathrm{S}\,.\,\alpha^{t+1}.$$

(3.) Hence, by the laws of i, j, k,

$$\text{IX.} \ldots i\mathrm{V}\,.\,i^{t} = j\mathrm{V}\,.\,j^{t} = k\mathrm{V}\,.\,k^{t} = \mathrm{S}\,.\,\alpha^{t+1}.$$

(4.) We have also, by the same principles and laws,

$$\text{X.} \ldots i\mathrm{V}\,.\,j^{t} = \mathrm{V}\,.\,k^{t}; \quad j\mathrm{V}\,.\,k^{t} = \mathrm{V}\,.\,i^{t}; \quad k\mathrm{V}\,.\,i^{t} = \mathrm{V}\,.\,j^{t};$$

$$\text{XI.} \ldots j\mathrm{V}\,.\,i^{t} = -\,\mathrm{V}\,.\,k^{t}; \quad k\mathrm{V}\,.\,j^{t} = -\,\mathrm{V}\,.\,i^{t}; \quad i\mathrm{V}\,.\,k^{t} = -\,\mathrm{V}\,.\,j^{t}.$$

(5.) The expression 308, (10.), for an *arbitrary vector* ρ, may be put under the following form:

$$\text{XII.} \ldots \rho = r\mathrm{V}\,.\,k^{2s+1} + rk^{2t}\mathrm{V}\,.\,i^{2s}.*$$

(6.) And it may be expanded as follows:

$$\text{XIII.} \ldots \rho = r\,\{(i\cos t\pi + j\sin t\pi)\sin s\pi + k\cos s\pi\}.$$

(7.) We shall return, briefly, in the Second Chapter of this Book [337], on some of these last expressions, in connexion with *differentials* and *derivatives* of *powers of vectors*; but, for the purposes of the present section, they may suffice.

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* [Since $j^{2s} = jSi^{2s-1} + Sk^{2s}$ this follows at once from $\rho = rk^t j^{2s} k^{1-t}$, remembering that $jk^{1-t} = k^{t-1}j = k^t i$.]

SECTION 11.

On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulæ.

316. We shall conclude the present Chapter with a short Supplementary Section, in which the recent definition (308) of a *power of a vector*, with a *scalar exponent*, shall be extended so as to include the *general case*, of a *Power of a Quaternion*, with a *Quaternion Exponent*, even when the two quaternions so combined are *diplanar*: and a connected *definition* shall be given (consistent with the less general one of the same kind, which was assigned in the Second Chapter of the Second Book), for the *Logarithm of a Quaternion in an arbitrary Plane*:* together with a few additional Formulæ, which could not be so conveniently introduced before.

(1.) We propose, then, to write, *generally*,

$$\text{I.} \ldots \varepsilon^q = 1 + \frac{q}{1} + \frac{q^2}{1.2} + \frac{q^3}{1.2.3} + \&\text{c.};$$

q being *any quaternion*, and ε being the real and known base of the natural (or Napierian) system of logarithms, of real and positive scalars: so that (as usual),

$$\text{II.} \ldots \varepsilon = \varepsilon^1 = 1 + \frac{1}{1} + \frac{1^2}{1.2} + \&\text{c.} = 2{\cdot}71828\ldots$$

(Compare 240, (1.) and (2.).)

(2.) We shall also write, for any quaternion q, the following expression for what we shall call its *principal logarithm*, or simply its *Logarithm*:

$$\text{III.} \ldots \mathrm{l}q = \mathrm{lT}q + \angle q \,.\, \mathrm{UV}q;$$

and thus shall have (comp. 243) the equation,

$$\text{IV.} \ldots \varepsilon^{\mathrm{l}q} = q.$$

(3.) When q is any *actual* quaternion (144), which does not *degenerate* (131) into a *negative scalar*, the formula III. assigns a *definite value* for the *logarithm*, $\mathrm{l}q$; which is such (comp. again 243) that

$$\text{V.} \ldots \mathrm{Sl}q = \mathrm{lT}q; \qquad \text{VI.} \ldots \mathrm{Vl}q = \angle q \,.\, \mathrm{UV}q;$$

$$\text{VII.} \ldots \mathrm{UVl}q = \mathrm{UV}q; \qquad \text{VIII.} \ldots \mathrm{TVl}q = \angle q;$$

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* The quaternions considered, in the Chapter referred to, were all supposed to be in the plane of the right versor i. But see the Second Note to page 277.

the *scalar part of the logarithm* being thus the (natural) *logarithm of the tensor*; and the *vector part* of the same logarithm lq being constructed by a *line* in the direction of the *axis* $\mathrm{Ax}.q$, of which the *length* bears, to the assumed *unit of length*, the same *ratio* as that which the *angle* $\angle q$ bears, to the usual *unit of angle* (comp. 241, (2.), (4.)).

(4.) If it were merely required to *satisfy the equation*,

$$\text{IX.} \ldots \varepsilon^{q'} = q,$$

in which q is supposed to be a *given* and *actual* quaternion, which is not equal to any negative scalar (3.), we might do this by writing (compare again 243),

$$\text{X.} \ldots q' = (\log q)_n = lq + 2n\pi \mathrm{UV}q,$$

where n is *any whole number*, positive or negative or null; and in this view, what we have called *the logarithm*, lq, of the quaternion q, is only what may be considered as the *simplest solution* of the *exponential equation* IX., and may, as such, be thus denoted:

$$\text{XI.} \ldots lq = (\log q)_0.$$

(5.) The *excepted case* (3.), where q is a *negative scalar*, becomes on this plan a case of *indetermination*, but *not* of *impossibility*: since we have, for example, by the definition III., the following expression for the *logarithm of negative unity*,

$$\text{XII.} \ldots l(-1) = \pi\sqrt{-1};$$

which in its *form* agrees with old and well-known results, but is here *interpreted* as signifying *any unit-vector*, of which the *length* bears to the *unit* of length the ratio of π to 1 (comp. 243, VII.).

(6.) We propose also to write, generally, for *any two quaternions*, q and q', *even if diplanar*, the following expression (comp. 243, (4.)) for what may be called the *principal value* of the *power*, or simply *the Power*, in which the former quaternion q is the *base*, while the latter quaternion q' is the *exponent*:

$$\text{XIII.} \ldots q^{q'} = \varepsilon^{q'lq};$$

and thus this *quaternion power* receives, *in general*, with the help of the definitions I. and III., a perfectly *definite signification*.

(7.) When the *base*, q, becomes a *vector*, ρ, its *angle* becomes a *right angle*; the definition III. gives therefore, for this case,

$$\text{XIV.} \ldots l\rho = l\mathrm{T}\rho + \frac{\pi}{2}\mathrm{U}\rho;$$

and this is the quaternion which is to be multiplied by q', in the expression,

$$\text{XV.} \ldots \rho^{q'} = \varepsilon^{q'l\rho}.$$

(8.) When, for the same *vector-base*, the *exponent* q' becomes a *scalar*, t, the last formula becomes:

$$\text{XVI.} \ldots \rho^t = \varepsilon^{t l \rho} = T\rho^t . \varepsilon^{x U\rho}, \quad \text{if} \quad 2x = t\pi;$$

and because, by I., the relation $(U\rho)^2 = -1$ gives,

$$\text{XVII.} \ldots \varepsilon^{x U\rho} = \cos x + U\rho \sin x, \quad \text{or briefly,} \quad \text{XVII}' \ldots \varepsilon^{x U\rho} = c\rho s x,$$

we see that the former definition, 308, I., of the power α^t, is in this way *reproduced*, as one which is *included* in the *more general definition* XIII., of the *power* $q^{q'}$; for we may write, by the last mentioned definition,

$$\text{XVIII.} \ldots (U\rho)^t = \varepsilon^{x U\rho} = c\rho s \frac{t\pi}{2} \text{ (comp. 234, VIII.)}$$

with the recent values XVI. and XVII., of x and $\varepsilon^{x U}\rho$.

(9.) In the present theory of *diplanar quaternions*, we cannot expect to find that the *sum of the logarithms* of any two proposed *factors*, shall be *generally equal* to the *logarithm of the product*; but for the simpler and earlier case of *complanar* quaternions, that *algebraic property* may be considered to exist, with due modifications for *multiplicity of value.**

(10.) The definition III. enables us, however, to establish *generally* the very simple formula (comp. 243, II. III.):

$$\text{XIX.} \ldots lq = l(Tq . Uq) = lTq + lUq;$$

in which (comp. (3.)),

$$\text{XX.} \ldots lUq = \angle q . UVq = Vlq; \quad \text{XXI.} \ldots TlUq = \angle q; \quad \text{XXII.} \ldots UlUq = UVq.$$

(11.) We have also generally, by XIII., for *any scalar exponent*, t, and *any quaternion base*, q, the *power*,

$$\text{XXIII.} \ldots q^t = \varepsilon^{t l q} = (Tq)^t . (\cos t \angle q + UVq . \sin t \angle q);$$

or briefly,

$$\text{XXIII}' \ldots q^t = Tq^t . c\upsilon s\, t \angle q, \quad \text{if} \quad \upsilon = UVq;$$

in which the parentheses about Tq may be omitted, because

$$\text{XXIV.} \ldots T(q^t) = (Tq)^t = Tq^t \text{ (comp. 237, II.)}.$$

* In 243, (3), it might have been observed, that *every value* of *each member* of the formula IX., there given, is *one of the values* of the *other member*; and a similar remark applies to the formulæ I. and II. of 236.

(12.) When the base and exponent of a power are *two rectangular vectors*, ρ and ρ', then, whatever their *lengths* may be, the product $\rho' \mathrm{l} \rho$ is, by XIV., a *vector*; but ε^α is always a *versor*,

$$\text{XXV.} \ldots \varepsilon^\alpha = \cos \mathrm{T}\alpha + \mathrm{U}\alpha \sin \mathrm{T}\alpha, \text{ if } \alpha \text{ be any vector};$$

we have therefore,

$$\text{XXVI.} \ldots \mathrm{T} \,.\, \rho^{\rho'} = 1, \quad \text{if} \quad \mathrm{S} \,.\, \rho\rho' = 0;$$

or in words, the *power* $\rho^{\rho'}$ is *a versor*, under this condition of *rectangularity*.

(13.) For example (comp. 242, (7.),* and the shortly following formula XXVIII.),

$$\text{XXVII.} \ldots i^j = \varepsilon^{j \mathrm{l} i} = -k; \quad j^i = \varepsilon^{i \mathrm{l} j} = +k;$$

and generally if the *base* be an *unit-line*, and the *exponent* a line of *any length*, but *perpendicular to the base*, the *axis of the power* is a line *perpendicular to both*; unless the *direction* of that *axis* becomes *indeterminate*, by the *power* reducing itself to a *scalar*, which in certain cases may happen.

(14.) Thus whatever scalar c may be, we may write,

$$\text{XXVIII.} \ldots i^{cj} = \varepsilon^{cj \mathrm{l} i} = \varepsilon^{-\frac{1}{2}ck\pi} = \cos \frac{c\pi}{2} - k \sin \frac{c\pi}{2};$$

this *power*, then, is a *versor* (12.), and its *axis* is *generally* the line $\mp k$; but in the *case* when c is any *whole* and *even number*, this versor *degenerates* into positive or negative *unity* (153), and the axis becomes *indeterminate* (131).

(15.) If, for any *real* quaternion q, we write again,

$$\text{XXIX.} \ldots \mathrm{UV}q = v, \quad \text{and therefore} \quad \text{XXX.} \ldots vq = qv, \quad \text{and} \quad \text{XXXI.} \ldots v^2 = -1,$$

the process of 239 will hold good, when we change i to v; the *series*, denoted in I. by ε^q, is therefore *always at last convergent*,† *however great* (but finite) *the tensor* $\mathrm{T}q$ *may be*; and in like manner the two following other series, derived from it, which represent (comp. 242, (3.)) what we shall *call, generally*, by analogy to known expressions, the *cosine* and *sine* of the *quaternion* q, are always *ultimately convergent*:

$$\text{XXXII.} \ldots \cos q = \tfrac{1}{2}(\varepsilon^{vq} + \varepsilon^{-vq}) = 1 - \frac{q^2}{1.2} + \frac{q^4}{1.2.3.4} - \&c.;$$

$$\text{XXXIII.} \ldots \sin q = \frac{1}{2v}(\varepsilon^{vq} - \varepsilon^{-vq}) = \frac{q}{1} - \frac{q^3}{1.2.3} + \frac{q^5}{1.2.3.4.5} - \&c.$$

* In the theory of *complanar quaternions*, it was found convenient to admit a certain *multiplicity of value* for a *power*, when the *exponent* was not a *whole number*; and therefore a *notation* for the *principal value of a power* was employed, with which the conventions of the present section enable us now to dispense.

† In fact, it can be proved that this final convergence exists, even when the quaternion is *imaginary*, or when it is replaced by a *biquaternion* (214, (8.)); but we have no occasion here to consider any but *real* quaternions.

(16.) We shall also *define* that the *secant, cosecant, tangent,* and *cotangent* of a *quaternion,* supposed still to be *real,* are the functions:

$$\text{XXXIV}\ldots \sec q = \frac{2}{\varepsilon^{vq}+\varepsilon^{-vq}}; \quad \operatorname{cosec} q = \frac{2v}{\varepsilon^{vq}-\varepsilon^{-vq}};$$

$$\text{XXXV}\ldots \tan q = \frac{v^{-1}(\varepsilon^{vq}-\varepsilon^{-vq})}{\varepsilon^{vq}+\varepsilon^{-vq}}; \quad \cot q = \frac{v(\varepsilon^{vq}+\varepsilon^{-vq})}{\varepsilon^{vq}-\varepsilon^{-vq}};$$

and thus shall have the usual relations, $\sec q = 1 : \cos q$, &c.

(17.) We shall also have,

$$\text{XXXVI}\ldots \varepsilon^{vq} = \cos q + v \sin q, \quad \varepsilon^{-vq} = \cos q - v \sin q;$$

and therefore, as in trigonometry (comp. 315, (1.)),

$$\text{XXXVII}\ldots (\cos q)^2 + (\sin q)^2 = \varepsilon^{vq} . \varepsilon^{-vq} = \varepsilon^0 = 1,$$

whatever quaternion q may be.

(18.) And *all the formulæ of trigonometry,* for *cosines* and *sines* of *sums* of *two* or *more arcs,* &c., will thus hold good *for quaternions also, provided* that the quaternions to be combined are *in any common plane*; for example,

$$\text{XXXVIII}\ldots \cos(q'+q) = \cos q' \cos q - \sin q' \sin q, \quad \text{if} \quad q' \parallel\!| q.$$

(19.) This *condition of complanarity* is here a *necessary* one; because (comp. (9.)) it is necessary for the establishment of the *exponential relation* between *sums* and *powers.*

(20.) Thus, we may indeed write,

$$\text{XXXIX}\ldots \varepsilon^{q'+q} = \varepsilon^{q'} . \varepsilon^{q}, \quad \textit{if} \quad q' \parallel\!| q;$$

but, *in general,* the developments of these two expressions give the difference,

$$\text{XL}\ldots \varepsilon^{q'+q} - \varepsilon^{q'}\varepsilon^{q} = \frac{qq'-q'q}{2} + \text{terms of third and higher dimensions};$$

and $$\text{XLI}\ldots \tfrac{1}{2}(qq'-q'q) = \mathrm{V}(\mathrm{V}q . \mathrm{V}q'),$$

an expression which does not vanish, when the quaternions q and q' are *diplanar.*

(21.) A few supplementary formulæ, connected with the present Chapter, may be appended here, as was mentioned at the commencement of this Article (316). And first it may be remarked, as connected with the theory of *powers of vectors,* that if α, β, γ be *any three unit-lines,* OA, OB, OC, and if σ denote the *area* of the spherical triangle ABC, then the formula 298, XX. may be thus written :

$$\text{XLII.}\ldots \frac{\alpha+\beta}{\beta+\gamma}\cdot\frac{\gamma+\alpha}{\alpha+\beta}\cdot\frac{\beta+\gamma}{\gamma+\alpha}=\alpha^{\frac{2\sigma}{\pi}};$$

the exponent being here a scalar.

(22.) The immediately preceding formula, 298, XIX., gives for *any three vectors*, the relation :

$$\text{XLIII.}\ldots (\mathrm{U}\alpha\beta\gamma)^2+(\mathrm{U}\beta\gamma)^2+(\mathrm{U}\alpha\gamma)^2+(\mathrm{U}\alpha\beta)^2+4\mathrm{U}\alpha\gamma\,.\,\mathrm{SU}\alpha\beta\,.\,\mathrm{SU}\beta\gamma=-2;$$

for example, if α, β, γ be made equal to i, j, k, the first member of this equation becomes, $1-1-1-1+0=-2$.

(23.) The following is a much more complex identity, involving as it does not only *three arbitrary vectors* α, β, γ, but also *four arbitrary scalars*, a, b, c, and r; but it has some geometrical applications, and a student would find it a good exercise in *transformations*, to investigate a proof of it for himself. To abridge notation, the three vectors α, β, γ, and the three scalars a, b, c, are considered as each composing a *cycle*, with respect to which are formed *sums* Σ, and *products* Π, on a plan which may be thus exemplified :

$$\text{XLIV.}\ldots \Sigma a\mathrm{V}\beta\gamma=a\mathrm{V}\beta\gamma+b\mathrm{V}\gamma\alpha+c\mathrm{V}\alpha\beta;\qquad \Pi a^2=a^2b^2c^2.$$

This being understood, the formula to be proved is the following :

$$\begin{aligned}\text{XLV.}\ldots &(\mathrm{S}\alpha\beta\gamma)^2+(\Sigma a\mathrm{V}\beta\gamma)^2+r^2(\Sigma\mathrm{V}\beta\gamma)^2-r^2(\Sigma a(\beta-\gamma))^2\\ &+2\Pi(r^2+\mathrm{S}\beta\gamma+bc)=2\Pi(r^2+\alpha^2)+2\Pi a^2\\ &+\Sigma(r^2+\alpha^2+a^2)\{(\mathrm{V}\beta\gamma)^2+2bc(r^2+\mathrm{S}\beta\gamma)-r^2(\beta-\gamma)^2\};\end{aligned}$$

the sign of summation in the last line governing all that follows it.

(24.) For example, by making the *four scalars* a, b, c, r each $=0$, this formula gives, for *any three vectors* α, β, γ, the relation,

$$\text{XLVI.}\ldots (\mathrm{S}\alpha\beta\gamma)^2+2\Pi\mathrm{S}\beta\gamma=2\Pi\alpha^2+\Sigma\,.\,\alpha^2(\mathrm{V}\beta\gamma)^2;$$

which agrees with the very useful equation 294, LIII., because

$$\text{XLVII.}\ldots \alpha^2(\mathrm{V}\beta\gamma)^2=\alpha^2\{(\mathrm{S}\beta\gamma)^2-\beta^2\gamma^2\}=(\alpha\mathrm{S}\beta\gamma)^2-\Pi\alpha^2.$$

(25.) Let α, β, γ be the *vectors of three points* A, B, C, which are *exterior to a given sphere*, of which the *radius* is r, and the *equation* is,

$$\text{XLVIII.}\ldots \rho^2+r^2=0 \text{ (comp. 282, XIII.)};$$

and let a, b, c denote the *lengths of the tangents* to that sphere, which are drawn from those three points respectively. We shall then have the relations:

$$\text{XLIX.} \ldots \alpha^2 + a^2 = \beta^2 + b^2 = \gamma^2 + c^2 = -r^2;$$

thus $r^2 + \alpha^2 = -a^2$, &c., and the second member of the formula XLV. vanishes; the first member of that formula is therefore *also* equal to zero, for these significations of the letters: and thus a *theorem* is obtained, which is found to be extremely useful, in the investigation by quaternions of the system of the *eight* (real or imaginary) *small circles, which touch a given set of three small circles on a sphere.*

(26.) We cannot enter upon *that* investigation here; but may remark that because the vector ρ of the foot P, of the perpendicular OP let fall the origin O on the right line AB, is given by the expression,

$$\text{L.} \ldots \rho = \alpha \mathrm{S} \frac{\beta}{\beta - \alpha} + \beta \mathrm{S} \frac{\alpha}{\alpha - \beta} = \frac{\mathrm{V}\beta\alpha}{\alpha - \beta},$$

as may be proved in various ways, the *condition of contact* of that *right line* AB with the *sphere* XLVIII. is expressed by the equation,

$$\text{LI.} \ldots \mathrm{TV}\beta\alpha = r\mathrm{T}(\alpha - \beta); \quad \text{or} \quad \text{LII.} \ldots (\mathrm{V}\beta\alpha)^2 = r^2(\alpha - \beta)^2;$$

or by another easy transformation, with the help of XLIX.,

$$\text{LIII.} \ldots (r^2 + \mathrm{S}\alpha\beta)^2 = (r^2 + \alpha^2)(r^2 + \beta^2) = a^2b^2.$$

(27.) This last equation evidently admits of decomposition into *two factors*, representing *two alternative conditions*, namely,

$$\text{LIV.} \ldots r^2 + \mathrm{S}\alpha\beta - ab = 0; \quad \text{LV.} \ldots r^2 + \mathrm{S}\alpha\beta + ab = 0;$$

and if we still consider the *tangents* a and b (25.) as *positive*, it is easy to prove, in several different ways, that the *first* or the *second* factor is to be selected, according as the *point* P, at which the *line* AB *touches the sphere, does* or *does not* fall *between the points* A and B; or in other words, according as the *length* of that line is equal to the *sum*, or to the *difference*, of those two tangents.

(28.) In fact we have, for the first case,

$$\text{LVI.} \ldots \mathrm{T}(\beta - \alpha) = b + a, \quad \text{or} \quad 0 = (\beta - \alpha)^2 + (b + a)^2 = -2(r^2 + \mathrm{S}\alpha\beta - ab),$$

in virtue of the relations XLIX.; but, for the second case,

$$\text{LVII.} \ldots \mathrm{T}(\beta - \alpha) = \pm(b - a), \quad \text{or} \quad 0 = (\beta - \alpha)^2 + (b - a)^2 = -2(r^2 + \mathrm{S}\alpha\beta + ab);$$

and it may be remarked, that we might in this way have been led to find the system of the *two conditions* (27.) and thence the equation LIII., or its transformations, LII. and LI.

(29.) We may conceive a *cone of tangents* from A, *circumscribing* the *sphere* XLVIII., and touching it *along a small circle*, of which the *plane*, or the *polar plane of the point* A, is easily found to have for its equation,

$$\text{LVIII.} \ldots \mathrm{S}\alpha\rho + r^2 = 0 \text{ (comp. 294, (28.), and 215, (10.));}$$

and in like manner the equation,

$$\text{LIX.} \ldots \mathrm{S}\beta\rho + r^2 = 0,$$

represents the polar plane of the point B, which plane cuts the sphere in a *second small circle*: and *these two circles touch each other*, when *either* of the two conditions (27.) is satisfied; such *contact* being *external* for the case LIV., but *internal* for the case LV.

(30.) The *condition of contact* (26.), of the *line and sphere*, might have been otherwise found, as the condition of *equality of roots* in the *quadratic equation* (comp. 216, (2.)),

$$\text{LX.} \ldots 0 = (x\alpha + y\beta)^2 + (x + y)^2 r^2,$$

or

$$\text{LXI.} \ldots 0 = x^2(r^2 + \alpha^2) + 2xy(r^2 + \mathrm{S}\alpha\beta) + y^2(r^2 + \beta^2);$$

the *contact* being thus considered here as a case of *coincidence of intersections*.

(31.) The *equation of conjugation* (comp. 215, (13.)), which expresses that each of the two points A and B is in the polar plane of the other, is (with the present notations),

$$\text{LXII.} \ldots r^2 + \mathrm{S}\alpha\beta = 0;$$

the *equal* but *opposite roots* of LXI., which then exist if the line cuts the sphere, answering here to the well-known *harmonic division* of the *secant line* AB (comp. 215, (16.)), which thus connects *two conjugate points*.

(32.) In like manner, from the quadratic equation 216, III., we get this analogous equation,

$$\text{LXIII.} \ldots \mathrm{S}\frac{\lambda}{\alpha}\,\mathrm{S}\frac{\mu}{\alpha} - \mathrm{S}\left(\mathrm{V}\frac{\lambda}{\beta}.\mathrm{V}\frac{\mu}{\beta}\right) = 1,$$

connecting the vectors λ, μ of any two points L, M, which are *conjugate relatively to the ellipsoid* 216, II.; and if we place the point L, *on the surface*, the equation LXIII. will represent the *tangent plane at that point* L, considered as the *locus of the conjugate point* M; whence it is easy to deduce the *normal*, at any point of the ellipsoid. But all researches respecting *normals to surfaces* can be better conducted, in connexion with the *Differential Calculus of Quaternions*, to which we shall next proceed.

(33.) It may however be added here, as regards *Powers of Quaternions* with *scalar exponents* (11.), that the symbol $q^t r q^{-t}$ represents a quaternion formed from r, by a conical rotation of its axis round that of q, through an angle $= 2t \angle q$; and that both members of the equation,

$$\text{LXIV.} \ldots (qrq^{-1})^t = qr^t q^{-1},$$

are symbols of one common quaternion.

[Some care must be taken in the interpretation of the expressions $q^{q'q''}$ and $(q^{q'})^{q''}$. By the definition XIII.,

$$q^{q'q''} = \varepsilon^{q'q''lq} = \varepsilon^{q'lq^{q''}} = (q^{q''})^{q'} \text{ and } (q^{q'})^{q''} = \varepsilon^{q''lq^{q'}} = \varepsilon^{q''q'lq} = q^{q''q'}.$$

This is quite consistent with the rule that in an operating product the factor to the right operates first on the operand. If the expression $lq^{q'}$ had been interpreted as equal to $lq \,.\, q'$ instead of $q' \,.\, l\,q$, then indeed the equality $(q^{q'})^{q''} = q^{q'q''} = \varepsilon^{lq \cdot q'q''}$ would have held good, but the general rule would have been disobeyed.]

CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS, TO GEOMETRICAL AND PHYSICAL QUESTIONS.

SECTION 1.

On the Definition of Simultaneous Differentials.

317. In the foregoing Chapter of the present Book, and in several parts of the Book preceding it, we have taken occasion to exhibit, as we went along, a considerable variety of *Examples*, of the *Geometrical Application of Quaternions*: but these have been given, chiefly as assisting to impress on the reader the *meanings of new notations*, or of *new combinations of symbols*, when such presented themselves in turn to our notice. In this concluding Chapter, we desire to offer a few *additional examples*, of the same *geometrical* kind, but dealing, more freely than before, with *tangents* and *normals* to *curves* and *surfaces*; and to give at least some *specimens*, of the application of quaternions to *Physical Inquiries*. But it seems necessary that we should first establish here some *Principles*, and some *Notations*, respecting *Differentials of Quaternions*, and of their *Functions*, generally.

318. The *usual definitions*, of *differential coefficients*, and of *derived functions*, are found to be inapplicable generally to the present Calculus, on account of the (generally) *non-commutative* character of quaternion-multiplication (168, 191). It becomes, therefore, necessary to have recourse to a *new Definition of Differentials*, which yet ought to be so framed, as to be *consistent with*, and to *include*, the *usual Rules of Differentiation*: because *scalars* (131), as well as *vectors* (292), have been seen to be *included*, under the general *Conception of Quaternions*.

319. In seeking for such a new definition, it is natural to go back to the first principles of the whole subject of Differentials: and to consider how the great Inventor of *Fluxions* might be supposed to have dealt with the question, if he had been *deprived* of that powerful resource of *common calculation*, which is supplied by the *commutative property* of *algebraic multiplication*; or by the familiar equation,

$$xy = yx,$$

considered as a *general* one, or as subsisting for *every pair of factors*, x and y; while *limits* should still be *allowed*, but *infinitesimals* be still *excluded*: and indeed the *fluxions themselves* should be regarded as *generally finite*,* according to what seems to have been the ultimate *view* of NEWTON.

320. The answer to this question, which a study of the Principia appears to suggest, is contained in the following *Definition*, which we believe to be a perfectly general one, as regards the *older* Calculus, and which we propose to *adopt* for Quaternions:—

"*Simultaneous Differentials* (or *Corresponding Fluxions*) are *Limits of Equimultiples*† *of Simultaneous and Decreasing Differences.*"

And conversely, whenever any *simultaneous differences*, of any system of variables, all *tend* to *vanish together*, according to any *law*, or system of laws; then, if any *equimultiples* of those decreasing differences all *tend* together to any system of finite *limits*, those *Limits* are *said* to be *Simultaneous Differentials* of the related *Variables* of the *System*; and are *denoted*, as such, by prefixing the *letter* d, as a *characteristic of differentiation*, to the *Symbol* of each such *variable*.

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* Compare the remarks annexed to the Second Lemma of the Second Book of the Principia (Third Edition, London, 1726); and especially the following passage (page 244):

"Neque enim spectatur in hoc Lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes et fluxiones quantitatum nominare licet) vel finitæ quævis quantitates velocitatibus hisce proportionales."

† As regards the notion of *multiplying* such *differences*, or generally any quantities which all *diminish together*, in order to render their *ultimate relations* more evident, it may be suggested by various parts of the *Principia* of Sir Isaac Newton; but especially by the First Section of the First Book. See for example the Seventh Lemma (p. 31), under which such expressions as the following occur: "intelligantur semper AB et AD ad puncta longinqua b et d produci," . . . . "ideoque rectæ semper finitæ Ab, Ad, . . ." The direction, "ad puncta longinqua produci," is repeated in connexion with the Eighth and Ninth Lemmas of the same Book and Section; while under the former of those two Lemmas we meet the expression, "triangula semper finita," applied to the *magnified representations* of *three triangles*, which all *diminish indefinitely together*: and under the latter Lemma the words occur, "manente longitudinæ Ae," where Ae is a *finite* and *constant line*, obtained by a *constantly increasing multiplication* of a *constantly diminishing line* AE (page 33 of the edition cited).

321. More fully and symbolically, let

$$\text{I.} \ldots q, r, s, \ldots$$

denote *any system of connected variables* (quaternions or others); and let

$$\text{II.} \ldots \Delta q, \Delta r, \Delta s, \ldots$$

denote, as usual, a system of their *connected* (or *simultaneous*) *differences*; in such a manner that the sums,

$$\text{III.} \ldots q + \Delta q, \quad r + \Delta r, \quad s + \Delta s, \ldots$$

shall be a *new system of variables*, satisfying the *same laws of connexion*, whatever they may be, as those which are satisfied by the *old system* I. Then, in *returning gradually* from the new system to the old one, or in proceeding gradually from the old to the new, the simultaneous *differences* II. can all be made (in general) to *approach together to zero*, since it is evident that they may *all vanish together*. But *if*, while the *differences themselves* are thus supposed to *decrease** *indefinitely together*, we *multiply them all* by some one *common* but *increasing number*, n, the *system of their equimultiples*,

$$\text{IV.} \ldots n\Delta q, \quad n\Delta r, \quad n\Delta s, \ldots$$

may tend to become *equal* to some determined *system of finite limits*. And *when this happens*, as in all ordinary cases it may be *made* to do, by a suitable *adjustment* of the *increase* of n to the *decrease* of Δq, &c., the *limits* thus obtained are *said* to be *simultaneous differentials* of the related variables, q, r, s; and are *denoted*, as such, by the symbols,

$$\text{V.} \ldots \mathrm{d}q, \quad \mathrm{d}r, \quad \mathrm{d}s, \ldots$$

SECTION 2.

Elementary Illustrations of the Definition, from Algebra and Geometry.

322. To leave no possible doubt, or obscurity, on the *import* of the foregoing *Definition*, we shall here apply it to determine the *differential of a square*, in *algebra*, and that of a *rectangle*, in *geometry*; in doing which we shall show, that while for such cases the *old rules* are *reproduced*, the *differentials* treated of *need not be small*; and that it would be a *vitiation*, and *not* a

* A *quaternion* may be said to *decrease*, when its *tensor* decreases; and to decrease *indefinitely*, when that tensor *tends to zero*.

correction, of the results, if any additional *terms* were introduced into their expressions, for the purpose of rendering *all the differentials equal* to the corresponding *differences*: though *some* of them may be *assumed* to be so, namely, in the first Example, *one*, and in the second Example, *two*.

(1.) In Algebra, then, let us consider the equation,

$$\text{I.} \ldots y = x^2,$$

which gives,

$$\text{II.} \ldots y + \Delta y = (x + \Delta x)^2,$$

and therefore, as usual,*

$$\text{III.} \ldots \Delta y = 2x\Delta x + \Delta x^2;$$

or what comes to the same thing,

$$\text{IV.} \ldots n\Delta y = 2xn\Delta x + n^{-1}(n\Delta x)^2,$$

where n is an *arbitrary multiplier*, which may be supposed, for simplicity, to be a positive whole number.

(2.) Conceive now that while the *differences* Δx and Δy, remaining always connected with each other and with x by the equation III., *decrease*, and *tend together to zero*, the *number* n *increases*, in the transformed equation IV., and *tends to infinity*, in such a manner that the *product*, or *multiple*, $n\Delta x$, tends to some *finite limit* a; which may happen, for example, by our obliging Δx to satisfy always the condition,

$$\text{V.} \ldots \Delta x = n^{-1}a, \quad \text{or} \quad n\Delta x = a,$$

after a previous *selection* of some *given* and *finite value* for a.

(3.) We shall then have, with this last condition V., the following expression by IV., for the *equimultiple* $n\Delta y$, of the *other difference*, Δy:

$$\text{VI.} \ldots n\Delta y = 2xa + n^{-1}a^2 = b + n^{-1}a^2, \quad \text{if} \quad b = 2xa.$$

But because a, and therefore a^2, is *given* and *finite*, (2.), while the number n increases indefinitely, the *term* $n^{-1}a^2$, in this expression VI. for $n\Delta y$, indefinitely *tends* to *zero*, and its *limit* is *rigorously null*. Hence the *two finite quantities*, a and b (since x is supposed to be finite), are *two simultaneous limits*, to which, under the supposed conditions, the *two equimultiples*, $n\Delta x$ and $n\Delta y$,

* We write here, as is common, Δx^2 to denote $(\Delta x)^2$; while $\Delta \,.\, x^2$ would be written, on the same known plan, for $\Delta(x^2)$, or Δy. In like manner we shall write dx^2, as usual, for $(dx)^2$; and shall denote $d(x^2)$ by $d \,.\, x^2$. Compare the notations Sq^2, $S\,.\,q^2$, and Vq^2, $V\,.\,q^2$, in 199 and 204.

tend;* they are, therefore, by the *definition* (320), *simultaneous differentials* of x and y: and we may *write* accordingly (321),

$$\text{VII.}\ldots \mathrm{d}x = a, \quad \mathrm{d}y = b = 2xa;$$

or, as usual, after elimination of a,

$$\text{VIII.}\ldots \mathrm{d}y = \mathrm{d}\,.\,x^2 = 2x\mathrm{d}x.$$

(4.) And it would *not improve, but vitiate*, according to the adopted *definition* (320), this usual expression for the *differential of the square* of a variable x in algebra, if we were to *add* to it the *term* $\mathrm{d}x^2$, in imitation of the formula III. for the *difference* $\Delta\,.\,x^2$. For this would come to supposing that, for a *given* and *finite value*, a, of dx, or of $n\Delta x$, the *term* $n^{-1}a^2$, or $n^{-1}\mathrm{d}x^2$, in the expression VI. for $n\Delta y$, *could fail to tend to zero*, while the *number*, n, by which the *square* of $\mathrm{d}x$ is *divided*, *increases without limit*, or *tends* (as above) *to infinity*.

(5.) As an *arithmetical example*, let there be the *given values*,

$$\text{IX.}\ldots x = 2, \quad y = x^2 = 4, \quad \mathrm{d}x = 1000;$$

and let it be required to compute, as a consequence of the *definition* (320), the *arithmetical value* of the *simultaneous differential*, $\mathrm{d}y$. We have now the following *equimultiples* of *simultaneous differences*,

$$\text{X.}\ldots n\Delta x = \mathrm{d}x = 1000; \quad n\Delta y = 4000 + 1000000n^{-1};$$

but the limit of the n^{th} *part* of a *million* (or of *any greater*, but *given* and *finite number*) is *exactly zero*, if n increase *without limit*; the required *value* of $\mathrm{d}y$ is, therefore, *rigorously*, in this example,

$$\text{XI.}\ldots \mathrm{d}y = 4000.$$

(6.) And we see that these two *simultaneous differentials*,

$$\text{XII.}\ldots \mathrm{d}x = 1000, \quad \mathrm{d}y = 4000,$$

are *not*, in *this* example, even *approximately equal* to the two *simultaneous differences*,

$$\text{XIII.}\ldots \Delta x = \mathrm{d}x = 1000, \quad \Delta y = 1002^2 - 2^2 = 1004000,$$

which answer to the value $n = 1$; although, no doubt, from the very *conception*

* In this case, indeed, the multiple $n\Delta x$ has by V. a *constant value*, namely a; but it is found convenient to extend the use of the word, *limit*, so as to *include* the case of constants: or to say, generally, that a *constant* is *its own limit*.

of simultaneous *differentials*, as embodied in the *definition* (320), they must admit of having such *equisubmultiples* of themselves taken,

$$\text{XIV.} \ldots n^{-1}\,\mathrm{d}x \quad \text{and} \quad n^{-1}\,\mathrm{d}y,$$

as to be *nearly equal*, for *large values of the number n*, to *some system* of simultaneous and *decreasing differences*,

$$\text{XV.} \ldots \Delta x \quad \text{and} \quad \Delta y;$$

and *more and more nearly equal* to such a *system*, even in the way of *ratio*, as they *all* become *smaller and smaller together*, and *tend* together to *vanish*.

(7.) For example, while the *differentials themselves* retain the *constant values* XII., their *millionth parts* are, respectively,

$$\text{XVI.} \ldots n^{-1}\,\mathrm{d}x = 0{\cdot}001, \quad \text{and} \quad n^{-1}\,\mathrm{d}y = 0{\cdot}004, \quad \text{if} \quad n = 1000000;$$

and the same value of the number n gives, by X., the equally rigorous values of two *simultaneous differences*, as follows,

$$\text{XVII.} \ldots \Delta x = 0{\cdot}001, \quad \text{and} \quad \Delta y = 0{\cdot}004001;$$

so that *these values* of the *decreasing differences* XV. may already be considered to be *nearly equal* to the two *equisubmultiples*, XIV. or XVI., of the two *simultaneous differentials*, XII. And it is evident that *this approximation* would be *improved*, by taking *higher values* of the *number*, n, without the *rigorous* and *constant values* XII., of $\mathrm{d}x$ and $\mathrm{d}y$, being *at all affected* thereby.

(8.) It is, however, evident also, that after assuming $y = x^2$, and $x = 2$, as in IX., we *might* have *assumed any other finite value for the differential* $\mathrm{d}x$, *instead* of the value 1000; and should then have *deduced a different* (but still *finite*) *value* for the *other differential*, $\mathrm{d}y$, and *not* the *formerly* deduced value, 4000: but there would always exist, *in this example*, or for *this form* of the *function*, y, and for *this value* of the *variable*, x, the *rigorous relation* between the *two simultaneous differentials*, $\mathrm{d}x$ and $\mathrm{d}y$,

$$\text{XVIII.} \ldots \mathrm{d}y = 4\mathrm{d}x,$$

which is obviously a *case* of the equation VIII., and can be proved by similar reasonings.

323. Proceeding to the promised *Example from Geometry* (322), we shall *again* see that differences and differentials are not in general to be *confounded* with each other, and that the latter (like the former) *need not be small*. But we shall *also* see that the *differentials* (*like the differences*), which enter into a

statement of relation, or into the enunciation of a proposition, respecting quantities which *vary together*, according to any *law* or *laws*, *need not even be homogeneous among themselves*: it being sufficient that *each separately* should be *homogeneous with the variable* to which it *corresponds*, and of which it *is* the *differential*, as *line* of *line*, or *area* of *area*. It will also be seen that the *definition* (320) enables us to *construct the differential of a rectangle*, as the *sum of two other* (*finite*) *rectangles*, *without any reference to units of length*, or of *area*, and without even the *thought* of employing *any numerical calculation* whatever.

(1.) Let, then, as in the annexed figure 74, ABCD be any given rectangle, and let BE and DG be any arbitrary but given and finite increments of its sides, AB and AD. Complete the increased rectangle GAEF, or briefly AF, which will thus exceed the given rectangle AC, or CA, by the sum of the three partial rectangles, CE, CF, CG; or by what we may call the *gnomon*,* CBEFGDC. On the diagonal CF take a point I, so that the line CI may be any arbitrarily selected submultiple of that diagonal; and draw through I, as in the figure, lines HM, KL, parallel to the sides AD, AB; and therefore intercepting, on the sides AB, AD prolonged, equisubmultiples BH, DK of the two given increments, BE, DG, of those two given sides.

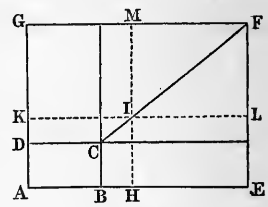

Fig. 74.

(2.) Conceive now that, in this construction, the *point* I *approaches to* C, or that we take a series of new points I, on the given diagonal CF, nearer and nearer to the given point C, by taking the *line* CI successively a *smaller and smaller part* of that diagonal. Then the *two new linear intervals*, BH, DK, and the *new gnomon*, CBHIKDC, or the *sum* of the *three new partial rectangles*, CH, CI, CK, will *all* indefinitely *decrease*, and will *tend* to *vanish together*: remaining, however, always a *system* of *three simultaneous differences* (or *increments*), of the two given *sides*, AB, AD, and of the given *area*, or *rectangle*, AC.

(3.) But the *given increments*, BE and DG, of the two given *sides*, are always (by the *construction*) *equimultiples* of the *two first* of the *three new* and *decreasing differences*; they *may*, therefore, by the *definition* (320), be *arbitrarily taken* as *two simultaneous differentials* of the *two sides*, AB and AD, *provided that* we *then treat*, as the corresponding or *simultaneous differential of the rectangle* AC, the

* The word, *gnomon*, is here used with a slightly more extended signification, than in the Second Book of Euclid.

limit of the equimultiple of the new gnomon (2.), or of the decreasing *difference* between the *two rectangles*, AC and AI, whereof the *first* is *given*.

(4.) We are then, *first*, to *increase* this *new gnomon*, or the *difference* of AC, AI, or the *sum* (2.) of the three partial rectangles, CH, CI, CK, in the *ratio* of BE to BH, or of DG to DK; and *secondly*, to seek the *limit of the area so increased*. For *this last limit* will, by the *definition* (320), be exactly and rigorously *equal* to the sought *differential of the rectangle* AC; *if* the *given* and finite *increments*, BE and DG, be *assumed* (as by (3.) they may) to be the *differentials of the sides*, AB, AD.

(5.) Now when we thus increase the *two new* partial rectangles, CH and CK, we get precisely the *two old* partial rectangles, CE and CG; which, as being *given* and *constant*, must be considered to be *their own limits*.* But when we *increase*, in the *same ratio*, the *other new partial rectangle* CI, we do *not recover* the *old partial rectangle* CF, corresponding to it; but obtain the *new* rectangle CL, or the equal rectangle CM, which is *not constant*, but *diminishes indefinitely* as the point I approaches to C; in such a manner that the *limit of the area*, of this new rectangle CL or CM, is *rigorously null*.

(6.) *If*, then, the *given increments*, BE, DG, be *still assumed* to be the *differtials of the given sides*, AB, AD (an *assumption* which has been seen to be *permitted*), the *differential of the given area*, or *rectangle*, AC, is *proved* (*not assumed*) to be, as a necessary *consequence of the definition* (320), exactly and *rigorously equal to the sum of the two partial rectangles* CE and CG; because *such* is the *limit* (5.) *of the multiple of the new gnomon* (2.), in the construction.

(7.) And if any one were to suppose that he could *improve* this known *value* for the *differential of a rectangle*, by *adding to it* the *rectangle* CF, as a new *term*, or *part*, so as to make it *equal* to the *old* or *given gnomon* (1.), he would (the *definition* being granted) *commit a geometrical error*, equivalent to that of supposing that the *two similar rectangles* CI and CF, bear to each other the *simple ratio*, instead of bearing (as they do) the *duplicate ratio*, of their *homologous sides*.

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* Compare the note to page 434.

SECTION 3.

On some general Consequences of the Definition.

324. Let there be any proposed equation of the form,

$$\text{I.} \ldots \quad Q = F(q, r, \ldots);$$

and let dq, dr, . . . be any *assumed* (but generally *finite*) and *simultaneous differentials* of the *variables*, $q, r, \ldots$ whether scalars, or vectors, or quaternions, on which Q is supposed to *depend*, by the equation I. Then the *corresponding* (or simultaneous) *differential* of their *function*, Q, is equal (by the definition 320, compare 321) to the following *limit*:

$$\text{II.} \ldots \quad \mathrm{d}Q = \lim_{n=\infty} . \, n \, \{F(q + n^{-1}\,\mathrm{d}q, \quad r + n^{-1}\,\mathrm{d}r, \ldots) - F(q, r, \ldots)\};$$

where n is any whole number (or other positive* scalar) which, as the formula expresses, is conceived to become indefinitely greater and greater, and so to tend to infinity. And if, in particular, we consider the function Q as involving only *one* variable q, so that

$$\text{III.} \ldots \quad Q = f(q) = fq,$$

then

$$\text{IV.} \ldots \quad \mathrm{d}Q = \mathrm{d}fq = \lim_{n=\infty} . \, n \, \{f(q + n^{-1}\,\mathrm{d}q) - fq\};$$

a *formula for the differential of a single explicit function of a single variable*, which agrees perfectly with those given, near the end of the First Book, for the differentials of a *vector*, and of a *scalar*, considered each as a function (100) of a *single scalar variable*, t: but which is now *extended*, as a consequence of the *general definition* (320), to the case when the connected *variables*, q, Q, and their *differentials*, dq, dQ, are *quaternions*: with an analogous application, of the still *more general Formula of Differentiation* II., to *Functions of several Quaternions*.

(1.) As an example of the use of the formula IV., let the function of q be its *square*, so that

$$\text{V.} \ldots \quad Q = fq = q^2.$$

Then, by the formula,

$$\text{VI.} \ldots \quad \mathrm{d}Q = \mathrm{d}fq = \lim_{n=\infty} . \, n \, \{(q + n^{-1}\,\mathrm{d}q)^2 - q^2\} = \lim_{n=\infty} . \, (q \, . \, \mathrm{d}q + \mathrm{d}q \, . \, q + n^{-1}\,\mathrm{d}q^2),$$

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* Except in some rare cases of *discontinuity*, not at present under our consideration, this *scalar* n may as well be conceived to tend to *negative infinity*.

where dq^2 signifies* the square of dq; that is,

$$\text{VII.} \ldots \mathrm{d}\,.\,q^2 = q\,.\,\mathrm{d}q + \mathrm{d}q\,.\,q;$$

or without the points† between q and $\mathrm{d}q$,

$$\text{VII}'. \ldots \mathrm{d}\,.\,q^2 = q\mathrm{d}q + \mathrm{d}q\,q;$$

an expression for the *differential of the square of a quaternion*, which does not *in general* admit of any further *reduction*: because q and $\mathrm{d}q$ are not generally *commutative*, as *factors* in multiplication. When, however, it *happens*, as in algebra, that $q\,.\,\mathrm{d}q = \mathrm{d}q\,.\,q$, by the two quaternions q and $\mathrm{d}q$ being *complanar*, the expression VII. then evidently reproduces the *usual form*, 322, VIII., or becomes,

$$\text{VIII.} \ldots \mathrm{d}\,.\,q^2 = 2q\mathrm{d}q, \quad \textit{if} \quad \mathrm{d}q \parallel\!| \, q \; (123).$$

(2.) As another example, let the function be the *reciprocal*,

$$\text{IX.} \ldots Q = fq = q^{-1}.$$

Then, because

$$\begin{aligned} \text{X.} \ldots f(q + n^{-1}\,\mathrm{d}q) - fq &= (q + n^{-1}\,\mathrm{d}q)^{-1} - q^{-1} \\ &= (q + n^{-1}\,\mathrm{d}q)^{-1}\,\{q - (q + n^{-1}\mathrm{d}q)\}\,q^{-1} \\ &= -\,n^{-1}\,(q + n^{-1}\,\mathrm{d}q)^{-1}\,.\,\mathrm{d}q\,.\,q^{-1}, \end{aligned}$$

of which, when multiplied by n, the limit is $-\,q^{-1}\,\mathrm{d}q\,.\,q^{-1}$, we have the following expression for the *differential of the reciprocal of a quaternion*,

$$\text{XI.} \ldots \mathrm{d}\,.\,q^{-1} = -\,q^{-1}\,.\,\mathrm{d}q\,.\,q^{-1};$$

or without the *points*‡ in the second member, $\mathrm{d}q$ being treated (as in VII′.) as a *whole symbol*,

$$\text{XI}'. \ldots \mathrm{d}\,.\,q^{-1} = -\,q^{-1}\,\mathrm{d}q\,q^{-1};$$

an expression which does not *generally* admit of being any farther *reduced*, but becomes, as in the ordinary calculus,

$$\text{XII.} \ldots \mathrm{d}\,.\,q^{-1} = -q^{-2}\,\mathrm{d}q, \quad \textit{if} \quad \mathrm{d}q \parallel\!| \, q,$$

that is, for the *case of complanarity*, of the quaternion and its differential.§

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* Compare the note to page 433.

† The *point* between d and q^2, in the first member of VII., is *indispensable*, to distinguish the *differential of the square* from the *square of the differential*. But just as this latter *square* is denoted briefly by $\mathrm{d}q^2$, so the *products*, $q\,.\,\mathrm{d}q$ and $\mathrm{d}q\,.\,q$, may be written as $q\mathrm{d}q$ and $\mathrm{d}q\,q$; the *symbol*, $\mathrm{d}q$, being thus treated as a *whole* one, or as if it were a *single letter*. Yet, for greater clearness of expression, we shall *retain the point* between q and $\mathrm{d}q$, in several (though not in all) of the subsequent formulæ, leaving it to the student to *omit* it, at his pleasure.

‡ Compare the note immediately preceding.

§ [See 329 (4.) for a result including XI.]

325. Other *Examples of Quaternion Differentiation* will be given in the following section; but the two foregoing may serve sufficiently to exhibit the nature of the operation, and to show the *analogy* of its results to those of the older calculus, while exemplifying also the *distinction* which generally exists between them. And we shall here proceed to explain a *notation*, which (at least in the *statement* of the present theory of differentials) appears to possess some advantages; and will enable us to offer a still more brief *symbolical definition*, of the *differential of a function* fq, than before.

(1.) We have *defined* (320, 324), that if $\mathrm{d}q$ be called the *differential* of a (quaternion or other) *variable*, q, then the *limit of the multiple*,

$$\text{I.} \ldots n\,\{f(q + n^{-1}\,\mathrm{d}q) - fq\},$$

of an *indefinitely decreasing difference* of the *function*, fq, of that (single) variable q, when taken relatively to an *indefinite increase* of the *multiplying number*, n, is the corresponding or simultaneous *differential of that function*, and is *denoted*, as such, by the *symbol* $\mathrm{d}fq$.

(2.) But *before* we thus pass to the *limit*, relatively to n, and while that *multiplier*, n, is still considered and treated as *finite*, the *multiple* I. is evidently a *function of that number*, n, *as well* as of the *two independent variables*, q and $\mathrm{d}q$. And we propose to *denote* (at least for the present) *this new function* of the *three variables*,

$$\text{II.} \ldots n,\ q,\ \text{and}\ \mathrm{d}q,$$

of which the *form depends*, according to the *law* expressed by the formula I., *on the form of the given function*, f, by the *new symbol*,

$$\text{III.} \ldots f_n(q,\ \mathrm{d}q);$$

in such a manner as to write, for *any two variables*, q and q', and *any number*, n, the *equation*,

$$\text{IV.} \ldots f_n(q,\ q') = n\,\{f(q + n^{-1}\,q') - fq\};$$

which may obviously be also written thus,

$$\text{V.} \ldots f(q + n^{-1}q') = fq + n^{-1}f_n(q,\ q'),$$

and is here regarded as *rigorously exact*, in virtue of the *definitions*, and without anything whatever being *neglected*, as *small*.

(3.) For example, it appears from the little calculation in 324, (1.), that,

$$\text{VI.} \ldots f_n(q, q') = qq' + q'q + n^{-1}q'^2, \quad \text{if } fq = q^2;$$

and from 324, (2.), that,

$$\text{VII.} \ldots f_n(q, q') = -(q + n^{-1}q')^{-1}q'q^{-1}, \quad \text{if } fq = q^{-1}.$$

(4.) And the definition of dfq may now be briefly thus expressed :

$$\text{VIII.} \ldots dfq = f_\infty\,(q, dq);$$

or, if the *sub-index* $_\infty$ be *understood*, we may write, still more simply,

$$\text{IX.} \ldots dfq = f(q, dq);$$

this last expression, $f(q, dq)$, or $f(q, q')$, denoting thus a *function of two independent variables* q and q', of which the *form* is *derived** or *deduced* (comp. (2.)), from the *given* or *proposed form* of the function fq, of a *single variable*, q, according to a *law* which it is one of the main objects of the *Differential Calculus* (at least as regards Quaternions) to study.

326. One of the most important *general properties*, of the *functions of this class* $f(q, q')$, is that they are all *distributive* with respect to the *second independent variable*, q', which is introduced in the foregoing process of what we have called *derivation*,† from some *given function* fq, of a *single variable*, q : a theorem which may be proved as follows, whether the two independent variables be, or be not, quaternions.

(1.) Let q'' be any *third* independent variable, and let n be *any number*; then the formula 325, V. gives the three following equations, resulting from the *law of derivation* of $f_n(q, q')$ from fq :

$$\text{I.} \ldots f(q + n^{-1}q'') = fq + n^{-1}f_n(q, q'');$$

$$\text{II.} \ldots f(q + n^{-1}q'' + n^{-1}q') = f(q + n^{-1}q'') + n^{-1}f_n(q + n^{-1}q'', q');$$

$$\text{III.} \ldots f(q + n^{-1}q' + n^{-1}q'') = fq + n^{-1}f_n(q, q' + q'');$$

* It was remarked, or hinted, in 318, that the *usual definition* of a *derived function*, namely, that given by Lagrange in the *Calcul des Fonctions*, cannot be taken as a *foundation* for a differential calculus of *quaternions*: although *such derived functions* of *scalars* present themselves occasionally in the applications of that calculus, as in 100, (3.) and (4.), and in some analogous but more general cases, which will be noticed soon. The *present Law of Derivation* is of an entirely different kind, since it conducts, as we see, from a *given* fuction of *one* variable, to a *derived* function of *two* variables, which are in general *independent* of each other. The function $f_n(q, q')$, of the *three* variables, n, q, q', may *also* be called a *derived function*, since it is *deduced*, by the *fixed law* IV., from the *same given function* fq, although it has in general a *less simple form* than *its own limit*, $f_\infty\,(q, q')$, or $f(q, q')$.

† Compare the note immediately preceding.

by comparing which we see at once that

$$\text{IV.} \ldots f_n(q, q' + q'') = f_n(q + n^{-1}q'', q') + f_n(q, q''),$$

the *form* of the *original function*, fq, and the *values* of the *four variables*, q, q', q'', and n, remaining altogether *arbitrary*: except that n is supposed to be a *number*, or at least a *scalar*, while q, q', q'' *may* (or may *not*) *be quaternions*.

(2.) For example, if we take the particular function $fq = q^2$, which gives the form 325, VI. of the derived function $f_n(q, q')$, we have

$$\text{V.} \ldots f_n(q, q'') = qq'' + q''q + n^{-1}q''^2;$$

$$\text{VI.} \ldots f_n(q, q' + q'') = q(q' + q'') + (q' + q'')\, q + n^{-1}(q' + q'')^2;$$

and therefore

$$\begin{aligned}\text{VII.} \ldots f_n(q, q' + q'') - f_n(q, q'') &= qq' + q'q + n^{-1}(q'^2 + q'q'' + q''q') \\ &= (q + n^{-1}q'')\, q' + q'\, (q + n^{-1}q'') + n^{-1}q'^2 \\ &= f_n(q + n^{-1}q'', q'),\end{aligned}$$

as required by the formula IV.

(3.) Admitting then that formula as proved, for *all* values of the number n, we have only to conceive that *number* (or *scalar*) to *tend to infinity*, in order to deduce this *limiting form* of the equation:

$$\text{VIII.} \ldots f_\infty\ (q, q' + q'') = f_\infty\ (q, q') + f_\infty\ (q, q'');$$

or simply, with the *abridged notation* of 325, (4.),

$$\text{IX.} \ldots f(q, q' + q'') = f(q, q') + f(q, q'');$$

which contains the expression of the *functional property*, above asserted to exist.

(4.) For example, by what has been already shown (comp. 325, (3.) and (4.)),

$$\text{X.} \ldots \text{if } fq = q^2, \quad \text{then} \quad f(q, q') = qq' + q'q;$$

$$\text{and XI.} \ldots \text{if } fq = q^{-1}, \quad \text{then} \quad f(q, q') = -\, q^{-1}q'q^{-1};$$

in *each* of which instances we see that the *derived function* $f(q, q')$ is *distributive relatively to* q', although it is only in the *first* of them that it happens to be distributive with respect to q *also*.

(5.) It follows at once from the formula IX. that we have generally*

$$\text{XII.} \ldots f(q, 0) = 0;$$

* We abstract here from some exceptional cases of *discontinuity*, &c.

and it is not difficult to prove, as a result including this, that

$$\text{XIII.} \ldots f(q, xq') = xf(q, q'), \text{ if } x \text{ be any } \textit{scalar}.$$

(6.) As a confirmation of this last result, we may observe that the *definition* of $f(q, q')$ may be expressed by the following formula (comp. 324, IV., and 325, IX.):

$$\text{XIV.} \ldots f(q, q') = \lim_{n=\infty} . n\{f(q + n^{-1}q') - fq\};$$

we have therefore, if x be any finite scalar, and $m = x^{-1}n$,

$$\text{XV.} \ldots f(q, xq') = x . \lim_{m=\infty} . m\{f(q + m^{-1}q') - fq\};$$

a transformation which gives the recent property XIII., since it is evident that the letter m may be written instead of n, in the formula of definition XIV.

327. Resuming then the general expression 325, IX., or writing anew,

$$\text{I.} \ldots \mathrm{d}fq = f(q, \mathrm{d}q),$$

we see (by 326, IX.) that *this derived function*, $\mathrm{d}fq$, of q and $\mathrm{d}q$, is always (as in the examples 324, VII. and XI.) *distributive with respect to that differential* $\mathrm{d}q$, considered as an *independent variable*, whatever the *form* of the *given function* fq may be. We see also (by 326, XIII.), that *if the differential* $\mathrm{d}q$ *of the variable*, q, *be multiplied by any scalar*, x, *the differential* $\mathrm{d}fq$, *of the function* fq, comes to be *multiplied*, at the same time, by the *same scalar*, or that

$$\text{II.} \ldots f(q, x\mathrm{d}q) = xf(q, \mathrm{d}q), \text{ if } x \text{ be any scalar.}$$

And in fact it is evident, from the very *conception* and *definition* (320) of *simultaneous differentials*, that every *system* of *such* differentials must admit of being *all changed together* to any system of *equimultiples*, or *equisubmultiples*, of themselves, *without ceasing* to be *simultaneous* differentials: or more generally, that it is *permitted to multiply all the differentials of a system*, by *any common scalar*.

(1.) It follows that the *quotient*,

$$\text{III.} \ldots \mathrm{d}fq : \mathrm{d}q = f(q, \mathrm{d}q) : \mathrm{d}q,$$

of the *two simultaneous differentials*, $\mathrm{d}fq$ and $\mathrm{d}q$, *does not change* when the differential $\mathrm{d}q$ is thus *multiplied by any scalar*; and consequently that this quotient III. is *independent of the tensor* $\mathrm{Td}q$, although it is *not generally independent of the versor* $\mathrm{Ud}q$, if q and $\mathrm{d}q$ be *quaternions*: except that it remains in general *unchanged*, when we merely change that *versor* to its own *opposite* (or negative), or to $-\mathrm{U}\mathrm{d}q$, because this comes to multiplying $\mathrm{d}q$ by -1, which is a scalar.

(2.) For example, the quotient,

$$\text{IV.} \ldots \mathrm{d} . q^2 : \mathrm{d}q = q + \mathrm{d}q . q . \mathrm{d}q^{-1} = q + \mathrm{U}\mathrm{d}q . q . \mathrm{U}\mathrm{d}q^{-1},$$

in which $\mathrm{d}q^{-1}$ and $\mathrm{U}\mathrm{d}q^{-1}$ denote the reciprocals of $\mathrm{d}q$ and $\mathrm{U}\mathrm{d}q$, is very far from being independent of $\mathrm{d}q$, or at least of $\mathrm{U}\mathrm{d}q$; since it represents, as we see, the *sum* of the *given* quaternion q, and of a certain *other* quaternion, which latter, in its *geometrical interpretation* (comp. 191, (5.)), may be considered as being *derived* from q, by a *conical rotation* of $\mathrm{Ax} . q$ round $\mathrm{Ax} . \mathrm{d}q$, through an *angle* $= 2\angle \mathrm{d}q$: so that both the *axis* and the *quantity* of this rotation *depend* on the versor $\mathrm{U}\mathrm{d}q$, and *vary with that versor*.

(3.) In general we may, if we please, say that the *quotient* III. is a *Differential Quotient*; but we ought not to call it a *Differential Coefficient* (comp. 318), becouse $\mathrm{d}fq$ does *not* generally admit of *decomposition into two factors*, whereof *one* shall be the *differential* $\mathrm{d}q$, and the *other* a *function of q alone*.

(4.) And for the same reason, we ought not to call that *Quotient* a *Derived Function* (comp. again 318), unless in so speaking we understand a *Function of Two* independent Variables*, namely of q and $\mathrm{U}\mathrm{d}q$, as before.

(5.) When, however, a *quaternion*, q, is considered as a *function of a scalar variable*, t, so that we have an equation of the form,

$$\text{V.} \ldots q = ft, \text{ where } t \text{ denotes a scalar,}$$

it is *then* permitted (comp. 100, (3.) and (4.)) to write,

$$\text{VI.} \ldots \mathrm{d}q : \mathrm{d}t = \mathrm{d}ft : \mathrm{d}t = \lim_{n=\infty} . \frac{n}{\mathrm{d}t}\left\{f\left(t + \frac{\mathrm{d}t}{n}\right) - ft\right\}$$
$$= \lim_{h=0} . h^{-1}\{f(t+h) - ft\}$$
$$= f't = \mathrm{D}_t ft = \mathrm{D}_t q;$$

and to *call this limit*, as usual, a *derived function of t*, because *it is* (in fact) a *function of that scalar variable, t, alone*, and is *independent of the scalar differential*, $\mathrm{d}t$.

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* Compare the note to 325, (4.).

(6.) We may also *write*, under these circumstances, the *differential equation*,

$$\text{VII.} \ldots \mathrm{d}q = \mathrm{D}_t q \,.\, \mathrm{d}t, \quad \text{or} \quad \text{VIII.} \ldots \mathrm{d}fq = f't \,.\, \mathrm{d}t,$$

and may *call the derived quaternion*, $\mathrm{D}_t q$, or $f't$, as usual, a *differential coefficient* in *this* formula, because the *scalar differential*, $\mathrm{d}t$, is (in fact) *multiplied by it*, in the expression thus found for the *quaternion differential*, $\mathrm{d}q$ or $\mathrm{d}ft$.

(7.) But as regards the *logic* of the question (comp. again 100, (3.)), it is important to remember that *we* regard this *derived function*, or *differential coefficient*,

$$\text{IX.} \ldots f't, \quad \text{or} \quad \mathrm{D}_t ft, \quad \text{or} \quad \mathrm{D}_t q,$$

as being an *actual quotient* VI., obtained by *dividing an actual quaternion*,

$$\text{X.} \ldots \mathrm{d}ft, \quad \text{or} \quad \mathrm{d}q,$$

by an actual scalar, $\mathrm{d}t$, of which the *value* is altogether *arbitrary*, and may (if we choose) be supposed to be *large* (comp. 322); while the *dividend quaternion* X. *depends, for its value, on the values of the two independent scalars*, t and $\mathrm{d}t$, *and on the form of the function* ft, according to the *law* which is expressed by the *general formula* 324, IV., for the *differentiation of explicit functions of any single variable*.

328. It is easy to conceive that similar remarks apply to *quaternion functions of more variables than one*; and that when the *differential* of *such* a *function* is expressed (comp. 324, II.) under the form,

$$\text{I.} \ldots \mathrm{d}Q = \mathrm{d}F(q, r, s, \ldots) = F(q, r, s, \ldots \mathrm{d}q, \mathrm{d}r, \mathrm{d}s, \ldots),$$

the *new function* F is always *distributive*, with respect to *each* separately of the *differentials* $\mathrm{d}q$, $\mathrm{d}r$, $\mathrm{d}s$, . . ; being also *homogeneous of the first dimension* (comp. 327), with respect to *all* those differentials, considered as a *system*; in such a manner that, whatever may be the *form* of the *given quaternion function*, Q, or F, the *derived** *function* F, or the *third member* of the *formula* I., must possess this general *functional property* (comp. 326, XIII., and 327, II.),

$$\text{II.} \ldots F(q, r, s, \ldots x\mathrm{d}q, x\mathrm{d}r, x\mathrm{d}s \ldots) = xF(q, r, s, \ldots \mathrm{d}q, \mathrm{d}r, \mathrm{d}s, \ldots),$$

where x may be *any scalar*: so that *products*, as well as *squares*, of the *differentials* $\mathrm{d}q$, $\mathrm{d}r$, &c., of q, r, &c. considered as so many *variables* on which Q *depends*, are *excluded* from the *expanded expression* of the *differential* $\mathrm{d}Q$ *of the function* Q.

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* Compare the note last referred to.

(1.) For example, if the function to be differentiated be a *product* of *two* quaternions,

$$\text{III.} \ldots Q = F(q, r) = qr,$$

then it is easily found from the general formula 324, II., that (because the *limit* of $n^{-1} . \mathrm{d}q . \mathrm{d}r$ is *null*, when the *number* n *increases without limit*) the differential of the function is,

$$\text{IV.} \ldots \mathrm{d}Q = \mathrm{d} . qr = \mathrm{d}F(q, r) = F(q, r, \mathrm{d}q, \mathrm{d}r) = q . \mathrm{d}r + \mathrm{d}q . r;$$

with analogous results, for differentials of products of more than two quaternions.

(2.) Again, if we take this other function,

$$\text{V.} \ldots Q = F(q, r) = q^{-1}r,$$

then, applying the same general formula 324, II., and observing that we have, for *all values* of the *number* (or other *scalar*), n, and of the *four quaternions*, q, r, q', r', the identical *transformation* (comp. 324, (2.)),

$$\text{VI.} \ldots n\{(q + n^{-1}q')^{-1}(r + n^{-1}r') - q^{-1}r\} = q^{-1}r' - (q + n^{-1}q')^{-1}q'q^{-1}(r + n^{-1}r'),$$

we find, as the required *limit*, when n tends to *infinity*, the following *differential of the function*:

$$\text{VII.} \ldots \mathrm{d}Q = \mathrm{d} . q^{-1}r = \mathrm{d}F(q, r) = F(q, r, \mathrm{d}q, \mathrm{d}r) = q^{-1} . \mathrm{d}r - q^{-1} . \mathrm{d}q . q^{-1}r;$$

which is again, like the expression IV., *distributive* with respect to *each* of the *differentials* $\mathrm{d}q$, $\mathrm{d}r$, of the *variables* q, r, and does *not* involve the *product* of those two differentials: although these two differential expressions, IV. and and VII., are both entirely *rigorous*, and are not *in any way* dependent on any supposition that the *tensors* of $\mathrm{d}q$ and $\mathrm{d}r$ are *small* (comp. again 322).

329. In thus differentiating a function of more variables than one, we are led to consider what may be called *Partial Differentials of Functions of two or more Quaternions*; which may be thus *denoted*,

$$\text{I.} \ldots \mathrm{d}_q Q,\ \mathrm{d}_r Q,\ \mathrm{d}_s Q, \ldots$$

if Q be a function, as above, of $q, r, s, \ldots$ which is here supposed to be differentiated with respect to *each* variable *separately*, *as if* the *others* were *constant*. And then, if $\mathrm{d}Q$ denote, as before, what may be called, by contrast, the *Total Differential* of the function Q, we shall have the *General Formula*,

$$\text{II.} \ldots \mathrm{d}Q = \mathrm{d}_q Q + \mathrm{d}_r Q + \mathrm{d}_s Q + \ldots;$$

or, briefly and symbolically,

$$\text{III.} \ldots \mathrm{d} = \mathrm{d}_q + \mathrm{d}_r + \mathrm{d}_s + \ldots,$$

if $q, r, s, \ldots$ denote the quaternion variables on which the *quaternion function* depends, of which the total differential is to be taken; whether those *variables* be all *independent*, or be *connected* with each other, by any *relation* or relations.

(1.) For example (comp. 328, (1.)),

$$\text{IV.} \ldots \text{if } Q = qr, \text{ then } \mathrm{d}_q Q = \mathrm{d}q \,.\, r, \text{ and } \mathrm{d}_r Q = q \,.\, \mathrm{d}r;$$

and the *sum* of these *two partial differentials* of Q makes up its *total differential* $\mathrm{d}Q$, as otherwise found above.

(2.) Again (comp. 328, (2.)),

$$\text{V.} \ldots \text{if } Q = q^{-1}r, \text{ then } \mathrm{d}_q Q = -q^{-1}\mathrm{d}q \,.\, q^{-1}r; \quad \mathrm{d}_r Q = q^{-1}\mathrm{d}r;$$

and $\mathrm{d}_q Q + \mathrm{d}_r Q$ = the same $\mathrm{d}Q$ as that which was otherwise found before, for this form of the function Q.

(3.) To exemplify the possibility of a *relation* existing between the *variables* q and r, let those variables be now supposed *equal* to each other in V.; we shall then have $Q = 1$, $\mathrm{d}Q = 0$; and accordingly we have here $\mathrm{d}_q Q = -q^{-1}\mathrm{d}q = -\mathrm{d}_r Q$.

(4.) Again, in IV., let $qr = c$ = any *constant* quaternion; we shall then again have $0 = \mathrm{d}Q = \mathrm{d}_q Q + \mathrm{d}_r Q$; and may infer that

$$\text{VI.} \ldots \mathrm{d}r = -q^{-1} \,.\, \mathrm{d}q \,.\, r, \quad \text{if } qr = c = \text{const.};$$

a result which evidently agrees with, and includes, the expression 324, XI., for the *differential of a reciprocal*.

(5.) A *quaternion*, q, may happen to be expressed as a *function of two or more scalar variables*, $t, u, \ldots$; and then it will have, as such, by the present Article, its *partial differentials*, $\mathrm{d}_t q$, $\mathrm{d}_u q$, &c. But because, by 327, VII., we may in *this* case write,

$$\text{VII.} \ldots \mathrm{d}_t q = \mathrm{D}_t q \,.\, \mathrm{d}t, \quad \mathrm{d}_u q = \mathrm{D}_u q \,.\, \mathrm{d}u, \ldots$$

where the *coefficients* are *independent* of the *differentials* (as in the ordinary calculus), we shall have (by II.) an expression for the *total differential* $\mathrm{d}q$, of the form,

$$\text{VIII.} \ldots \mathrm{d}q = \mathrm{d}_t q + \mathrm{d}_u q + \ldots = \mathrm{D}_t q \,.\, \mathrm{d}t + \mathrm{D}_u q \,.\, \mathrm{d}u + \ldots;$$

and may at pleasure say, *under the conditions here supposed*, that the *derived quaternions*,

$$\text{IX.} \ldots \mathrm{D}_t q, \quad \mathrm{D}_u q, \ldots$$

are either the *Partial Derivatives*, or the *Partial Differential Coefficients*, of the *Quaternion Function*,

$$\text{X.} \ldots q = F(t, u, \ldots);$$

with analogous remarks for the *case*, when the *quaternion*, q, *degenerates* (comp. 289) into a *vector*, ρ.

330. In general, it may be considered as evident, from the definition in 320, that the *differential of a constant is zero*; so that if Q be changed to *any constant quaternion*, c, in the equation 324, I., then $\mathrm{d}Q$ is to be *replaced by* 0, in the *differentiated equation*, 324, II. And if there be given any *system of equations*, connecting the *quaternion variables*, q, r, s, ... we may treat the corresponding *system of differentiated equations*, as holding good, for the *system of simultaneous differentials*, $\mathrm{d}q$, $\mathrm{d}r$, $\mathrm{d}s$, ...; and may therefore, legitimately in theory, whenever in practice it shall be found to be possible, *eliminate* any one or more of those *differentials*, between the equations of this system.

(1.) As an example, let there be the two equations,

$$\text{I.} \ldots qr = c, \quad \text{and} \quad \text{II.} \ldots s = r^2,$$

where c denotes a constant quaternion. Then (comp. 328, (1.), and 324, (1.)) we have the two differentiated equations corresponding,

$$\text{III.} \ldots q \,.\, \mathrm{d}r + \mathrm{d}q \,.\, r = 0; \qquad \text{IV.} \ldots \mathrm{d}s = r \,.\, \mathrm{d}r + \mathrm{d}r \,.\, r;$$

in which the *points** might be omitted. The former gives,

$$\text{V.} \ldots \mathrm{d}r = -q^{-1}\mathrm{d}q \,.\, r, \text{ as in 329, VI.};$$

and when we substitute this value in the latter, we thereby *eliminate the differential* $\mathrm{d}r$, and obtain this *new differential equation*,

$$\text{VI.} \ldots \mathrm{d}s = -rq^{-1} \,.\, \mathrm{d}q \,.\, r - q^{-1} \,.\, \mathrm{d}q \,.\, r^2.$$

(2.) The equation I. gives also the expression,

$$\text{VII.} \ldots r = q^{-1}c;$$

the equation II. gives therefore this other expression,

$$\text{VIII.} \ldots s = (q^{-1}c)^2 = q^{-1}cq^{-1}c,$$

* Compare the second note to 324, (1.).

by *elimination before differentiation.* And if, in the formula VI., we substitute the expressions VII. and VIII. for r and s, we get this *other* differential equation,

$$\text{IX.} \ldots \mathrm{d}\,.\,(q^{-1}c)^2 = -\,q^{-1}cq^{-1}\,.\,\mathrm{d}q\,.\,q^{-1}c - q^{-1}\,.\,\mathrm{d}q\,.\,q^{-1}cq^{-1}c\,;$$

which might have been otherwise obtained (comp. again 324, (1.) and (2.)), under the form,

$$\text{X.} \ldots \mathrm{d}\,.\,(q^{-1}c)^2 = q^{-1}c\,.\,\mathrm{d}(q^{-1}c) + \mathrm{d}(q^{-1}c)\,.\,q^{-1}c.$$

331. *No special rules* are required, for the *differentiation of functions of functions* of quaternions; but it may be instructive to show, briefly, how the consideration of *such* differentiation conducts (comp. 326) to a *general property of functions of the class* $f(q, q')$; and how that property can be *otherwise* established.

(1.) Let f, ϕ, and ψ denote any functional operators, such that

$$\text{I.} \ldots \psi q = \phi(fq)\,;$$

then writing

$$\text{II.} \ldots r = fq, \text{ and III.} \ldots s = \phi r, \text{ we have IV.} \ldots s = \psi q\,;$$

$$\text{whence V.} \ldots \mathrm{d}s = \mathrm{d}\psi q = \mathrm{d}\phi r.$$

That is, we may (as usual) *differentiate the compound function*, $\phi(fq)$, *as if* fq *were an independent variable*, r; and *then*, in the expression so found, *replace the differential* $\mathrm{d}fq$ *by its value*, obtained by *differentiating the simple function*, fq. For this comes virtually to the *elimination of the differential* $\mathrm{d}r$, or of the *symbol* $\mathrm{d}fq$, in a way which we have seen to be permitted (330).

(2.) But, by the *definitions* of $\mathrm{d}fq$ and $f_n(q, q')$, we saw (325, VIII. IX.) that the differential $\mathrm{d}fq$ might generally be denoted by $f_\infty(q, \mathrm{d}q)$, or briefly by $f(q, \mathrm{d}q)$; whence $\mathrm{d}\phi r$ and $\mathrm{d}\psi q$ may also, by an extension of the same notation, be represented by the analogous symbols, $\phi_\infty(r, \mathrm{d}r)$ and $\psi_\infty(q, \mathrm{d}q)$, or simply by $\phi(r, \mathrm{d}r)$ and $\psi(q, \mathrm{d}q)$.

(3.) We ought, therefore, to find that

$$\text{VI.} \ldots \psi_\infty(q, \mathrm{d}q) = \phi_\infty(fq, f_\infty(q, \mathrm{d}q)), \quad \text{if} \quad \psi q = \phi(fq)\,;$$

or briefly that

$$\text{VII.} \ldots \psi(q, q') = \phi(fq, f(q, q')), \quad \text{if} \quad \psi q = \phi fq,$$

for *any two quaternions*, q, q', and *any two functions*, f, ϕ; provided that the functions $f_n(q, q')$, $\phi_n(q, q')$, $\psi_n(q, q')$ are *deduced* (or *derived*) from the functions fq, ϕq, ψq, according to the *law* expressed by the formula 325, IV.;

and that then the *limits* to which these *derived functions* $f_n(q, q')$, &c. *tend*, when the *number* n tends to *infinity*, are denoted by these *other functional symbols*, $f(q, q')$, &c.

(4.) To prove this *otherwise*, or to establish this *general property* VII., *of functions of this class* $f(q, q')$, *without any use of differentials*, we may observe that the general and rigorous *transformation* 325, V., of the formula 325, IV. by which the functions $f_n(q, q')$ are *defined*, gives for *all* values of n the equation:

$$\text{VIII.} \ldots \phi f(q + n^{-1}q') = \phi\,(fq + n^{-1}f_n(q, q')\,) = \phi fq + n^{-1}\phi_n(fq, f_n(q, q')\,)\,;$$

but also, by the same general transformation,

$$\text{IX.} \ldots \psi\,(q + n^{-1}q') = \psi q + n^{-1}\psi_n(q, q')\,;$$

hence *generally*, for *all values of the number* n, as well as for *all values of the two independent quaternions*, q, q', and for *all forms of the two functions*, f, ϕ, we may write,

$$\text{X.} \ldots \psi_n(q, q') = \phi_n(fq, f_n(q, q')\,), \quad \text{if} \quad \psi q = \phi fq\,;$$

an equation of which the *limiting form*, for $n = \infty$, is (with the notations used) the equation VII. which was to be proved.

(5.) It is scarcely worth while to verify the general formula X., by any particular example: yet, merely as an exercise, it may be remarked that if we take the forms,

$$\text{XI.} \ldots fq = q^2, \quad \phi q = q^2, \quad \psi q = q^4,$$

of which the two first give, by 325, VI., the common derived form,

$$\text{XII.} \ldots f_n(q, q') = \phi_n(q, q') = qq' + q'q + n^{-1}q'^2,$$

the formula X. becomes,

$$\text{XIII.} \ldots \psi_n(q, q') = \phi_n(q^2, qq' + q'q + n^{-1}q'^2)$$
$$= q^2(qq' + q'q + n^{-1}q'^2) + (qq' + q'q + n^{-1}q'^2)\,q^2 + n^{-1}\,(qq' + q'q + n^{-1}q'^2)^2\,;$$

which agrees with the value deduced immediately from the function ψq or q^4, by the definition 325, IV., namely,

$$\text{XIV.} \ldots \psi_n(q, q') = n\{(q + n^{-1}q')^4 - q^4\} = n\{(q^2 + n^{-1}(qq' + q'q + n^{-1}q'^2))^2 - (q^2)^2\}.$$

(6.) In general, the *theorem*, or *rule*, for *differentiating* as in (1.) a *function of a function*, of a quaternion or other variable, may be briefly and symbolically expressed by the formula,

$$\text{XV.} \ldots \mathrm{d}(\phi f)q = \mathrm{d}\phi(fq);$$

and if we did not otherwise know it, a *proof* of its correctness would be supplied, by the recent proof of the correctness of the equivalent formula VII.

SECTION 4.

Examples of Quaternion Differentiation.

332. It will now be easy and useful to give a short collection of *Examples of Differentiation of Quaternion Functions and Equations*, additional to and inclusive of those which have incidentally occurred already, in treating of the *principles* of the subject.

(1.) If c be any *constant* quaternion (as in 330), then

$$\text{I.} \ldots \mathrm{d}c = 0; \qquad \text{II.} \ldots \mathrm{d}(fq + c) = \mathrm{d}fq;$$

$$\text{III.} \ldots \mathrm{d}\,.\,cfq = c\mathrm{d}fq; \quad \text{IV.} \ldots \mathrm{d}(fq\,.\,c) = \mathrm{d}fq\,.\,c.$$

(2.) In general,

$$\text{V.} \ldots \mathrm{d}(fq + \phi q + \ldots) = \mathrm{d}fq + \mathrm{d}\phi q + \ldots; \quad \text{or briefly,} \quad \text{VI.} \ldots \mathrm{d}\Sigma = \Sigma\mathrm{d},$$

if Σ be used as a mark of summation.

(3.) Also, $$\text{VII.} \ldots \mathrm{d}(fq\,.\,\phi q) = \mathrm{d}fq\,.\,\phi q + fq\,.\,\mathrm{d}\phi q;$$

and similarly for a product of more functions than two: the *rule* being simply, to *differentiate each factor separately, in its own place*, or without disturbing the *order* of the *factors* (comp. 318, 319); and then to *add together the partial results* (comp. 329).

(4.) In particular, if m be any positive whole number,

$$\text{VIII.} \ldots \mathrm{d}\,.\,q^m = q^{m-1}dq + q^{m-2}\mathrm{d}q\,.\,q \ldots + q\mathrm{d}q\,.\,q^{m-2} + \mathrm{d}q\,.\,q^{m-1};$$

and because we have seen (324, (2.)) that

$$\text{IX.} \ldots \mathrm{d}\,.\,q^{-1} = -\,q^{-1}\,.\,\mathrm{d}q\,.\,q^{-1},$$

we have this analogous expression for the differential of a *power* of a *quaternion*, with a *negative* but *whole exponent*,

$$\text{X.} \ldots \mathrm{d} \cdot q^{-m} = - q^{-m}\mathrm{d} \cdot q^{m} \cdot q^{-m}$$
$$= - q^{-1}\,\mathrm{d}q \cdot q^{-m} - q^{-2}\,\mathrm{d}q \cdot q^{1-m} - \ldots - q^{1-m}\,\mathrm{d}q \cdot q^{-2} - q^{-m}\,\mathrm{d}q \cdot q^{-1}.$$

(5.) To *differentiate a square root*, we are to *resolve the linear equation*,*

$$\text{XI.} \ldots q^{\frac{1}{2}} \cdot \mathrm{d} \cdot q^{\frac{1}{2}} + \mathrm{d} \cdot q^{\frac{1}{2}} \cdot q^{\frac{1}{2}} = \mathrm{d}q\,; \quad \text{or} \quad \text{XI}'. \ldots rr' + r'r = q',$$

if we write, for abridgment,

$$\text{XII.} \ldots r = q^{\frac{1}{2}}, \quad q' = \mathrm{d}q, \quad r' = \mathrm{d} \cdot q^{\frac{1}{2}} = \mathrm{d}r.$$

(6.) Writing also, for this purpose,

$$\text{XIII.} \ldots s = \mathrm{K}r = \mathrm{K} \cdot q^{\frac{1}{2}},$$

whence (by 190, 196) it will follow that

$$\text{XIV.} \ldots rs = \mathrm{N}r = \mathrm{T}r^2 = \mathrm{T}q, \quad \text{and} \quad \text{XV.} \ldots r + s = 2\mathrm{S}r = 2\mathrm{S} \cdot q^{\frac{1}{2}},$$

the *product* and *sum* of these two *conjugate quaternions*, r and s, being thus *scalars* (140, 145), we have, by XI′.,

$$\text{XVI.} \ldots r^{-1}\,q's = r's + sr'\,;$$

whence, by addition,

$$\text{XVII.} \ldots q' + r^{-1}q's = (r + s)\,r' + r'\,(r + s) = 2r'(r + s)\,;$$

and finally,

$$\text{XVIII.} \ldots r' = \frac{q' + r^{-1}q's}{2(r + s)}, \quad \text{or} \quad \text{XIX.} \ldots \mathrm{d} \cdot q^{\frac{1}{2}} = \frac{\mathrm{d}q + q^{-\frac{1}{2}}\mathrm{d}q \cdot \mathrm{K} \cdot q^{\frac{1}{2}}}{4\mathrm{S} \cdot q^{\frac{1}{2}}}\,;$$

an expression for the differential of the square-root of a quaternion, which will be found to admit of many transformations, not needful to be considered here.

(7.) In the three last sub-articles, as in the three preceding them, it has been supposed, for the sake of generality, that q and $\mathrm{d}q$ are two *diplanar*

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* Although such *solution of a linear equation*, or equation of the *first degree*, in quaternions, is easily enough accomplished in the present instance, yet in general the problem presents difficulties, without the consideration of which the theory of *differentiation of implicit functions of quaternions* would be entirely incomplete. But a *general method*, for the solution of *all such equations*, will be sketched in a subsequent Section.

quaternions; but if in any application they *happen*, on the contrary, to be *complanar*, the expressions are then *simplified*, and take *usual*, or *algebraic forms*, as follows:

$$\text{XX.} \ldots \mathrm{d} \,.\, q^m = mq^{m-1}\mathrm{d}q; \qquad \text{XXI.} \ldots \mathrm{d} \,.\, q^{-m} = -\,mq^{-m-1}\mathrm{d}q;$$

and

$$\text{XXII.} \ldots \mathrm{d} \,.\, q^{\frac{1}{2}} = \tfrac{1}{2}q^{-\frac{1}{2}}\mathrm{d}q, \quad \textit{if} \quad \text{XXIII.} \ldots \mathrm{d}q \,|||\, q (123);$$

because, when q' is complanar with q, and therefore with $q^{\frac{1}{2}}$, or with r, in the expression XVIII., the numerator of that expression may be written as $r^{-1}q'$ $'r + s$).

(8.) More generally, if x be *any scalar exponent*, we may write, as in the ordinary calculus, but still under the *condition of complanarity* XXIII.,

$$\text{XXIV.} \ldots \mathrm{d} \,.\, q^x = xq^{x-1}\mathrm{d}q; \quad \text{or} \quad \text{XXV.} \ldots q\mathrm{d} \,.\, q^x = xq^x\mathrm{d}q.$$

333. The *functions* of quaternions, which have been lately differentiated, may be said to be of *algebraic form*; the following are a few examples of differentials of what may be called, by contrast, *transcendental functions of quaternions*: the condition of *complanarity* ($\mathrm{d}q \,|||\, q$) being however *here* supposed to be satisfied, in order that the expressions may not become too complex. In fact, *with this simplification*, they will be found to assume, for the most part, the *known* and *usual forms*, of the *ordinary differential calculus*.

(1.) Admitting the definitions in 316, and supposing throughout that $\mathrm{d}q \,|||\, q$, we have the usual expressions for the differentials of ε^q and $\mathrm{l}q$, namely,

$$\text{I.} \ldots \mathrm{d} \,.\, \varepsilon^q = \varepsilon^q\mathrm{d}q; \quad \text{II.} \ldots \mathrm{dl}q = q^{-1}\mathrm{d}q.$$

(2.) We have also, by the same system of definitions (316),

$$\text{III.} \ldots \mathrm{d}\sin q = \cos q\mathrm{d}q; \quad \text{IV.} \ldots \mathrm{d}\cos q = -\sin q\mathrm{d}q; \ \&\text{c.}$$

(3.) Also, if r and $\mathrm{d}r$ be complanar with q and $\mathrm{d}q$, then, by 316,

$$\text{IV}'. \ldots \mathrm{d} \,.\, q^r = \mathrm{d} \,.\, \varepsilon^{r\mathrm{l}q} = q^r\mathrm{d} \,.\, r\mathrm{l}q = q^r(\mathrm{l}q\mathrm{d}r + q^{-1}r\mathrm{d}q);$$

or in the notation of partial differentials (329),

$$\text{V.} \ldots \mathrm{d}_q \,.\, q^r = rq^{r-1}\mathrm{d}q, \quad \text{and} \quad \text{VI.} \ldots \mathrm{d}_r \,.\, q^r = q^r\mathrm{l}q\mathrm{d}r.$$

(4.) In particular, if the *base* q be a *given* or *constant vector*, α, and if the *exponent* r be a *variable scalar*, t, then (by the value 316, XIV. of $\mathrm{l}\rho$) the recent formula IV. becomes,

$$\text{VII.} \ldots \mathrm{d} \,.\, \alpha^t = \left(\mathrm{lT}\alpha + \frac{\pi}{2}\mathrm{U}\alpha\right)\alpha^t\mathrm{d}t.$$

(5.) If then the base α be a *given unit line*, so that $\mathrm{lT}\alpha = 0$, and $\mathrm{U}\alpha = \alpha$, we may write simply,

$$\text{VIII.} \ldots \mathrm{d} \,.\, \alpha^t = \frac{\pi}{2} \alpha^{t+1} \mathrm{d}t, \quad \text{if} \quad \mathrm{d}\alpha = 0, \quad \text{and} \quad \mathrm{T}\alpha = 1.$$

(6.) This useful formula, for the differential of a power of a constant unit line, with a variable scalar exponent, may be obtained more rapidly from the equation 308, VII., which gives,

$$\text{IX.} \ldots \alpha^t = \cos \frac{t\pi}{2} + \alpha \sin \frac{t\pi}{2}, \quad \text{if} \quad \mathrm{T}\alpha = 1;$$

since it is evident that the differential of this expression is equal to the expression itself multiplied by $\frac{1}{2}\pi\alpha \mathrm{d}t$, because $\alpha^2 = -1$.

(7.) The formula VIII. admits also of a simple *geometrical interpretation*, connected with the *rotation* through *t right angles*, in a *plane perpendicular to* α, of which rotation, or *version*, the *power* α^t, or the *versor* $\mathrm{U}\alpha^t$, is considered (308) to be the *instrument*,* or *agent*, or *operator* (comp. 293).

334. Besides *algebraical* and *transcendental forms*, there are *other results of operation* on a quaternion, q, or on a function thereof, which may be regarded as forming a *new class* (or kind) of *functions*, arising out of the *principles* and *rules* of the *Quaternion Calculus itself*: namely those which we have denoted in former Chapters by the *symbols*,

$$\text{I.} \ldots \mathrm{K}q, \ \mathrm{S}q, \ \mathrm{V}q, \ \mathrm{N}q, \ \mathrm{T}q, \ \mathrm{U}q,$$

or by symbols formed through *combinations* of the same *signs of operation*, such as

$$\text{II.} \ldots \mathrm{SU}q, \ \mathrm{VU}q, \ \mathrm{UV}q, \ \&\text{c.}$$

And it is essential that we should know how to *differentiate* expressions *of these forms*, which can be done in the following manner, with the help of the principles of the present and former Chapters, and *without* now assuming the *complanarity*, $\mathrm{d}q \parallel\!| q$.

(1.) In general, let f represent, for a moment, *any distributive symbol*, so that for any two quaternions, q and q', we shall have the equation,

$$\text{III.} \ldots f(q + q') = fq + fq';$$

* Compare the first note to page 135.

and therefore also* (comp. 326, (5.)),

$$\text{IV.} \ldots f(xq) = xfq, \text{ if } x \text{ be any scalar.}$$

(2.) Then, with the notation 325, IV., we shall have

$$\text{V.} \ldots f_n(q, q') = n\{f(q + n^{-1}q') - fq\} = fq';$$

and therefore, by 325, VIII., for any *such* function fq, we shall have the differential expression,

$$\text{VI.} \ldots \mathrm{d}fq = f\mathrm{d}q.$$

(3.) But S, V, K have been seen to be *distributive symbols* (197, 207); we can therefore infer at once that

$$\text{VII.} \ldots \mathrm{dK}q = \mathrm{Kd}q; \quad \text{VIII.} \ldots \mathrm{dS}q = \mathrm{Sd}q; \quad \text{IX.} \ldots \mathrm{dV}q = \mathrm{Vd}q;$$

or in words, that *the differentials of the conjugate, the scalar, and the vector of a quaternion are*, respectively, *the conjugate, the scalar, and the vector of the differential of that quaternion.*

(4.) To find the *differential of the norm*, Nq, or to deduce an *expression for* dNq, we have (by VII. and 145) the equation,

$$\text{X.} \ldots \mathrm{dN}q = \mathrm{d}\,.\,q\mathrm{K}q = \mathrm{d}q\,.\,\mathrm{K}q + q\,.\,\mathrm{Kd}q;$$

but $\qquad q\mathrm{K}q' = \mathrm{K}\,.\,q'\mathrm{K}q$, by 145, and 192, II.;

and $\qquad (1 + \mathrm{K})\,.\,q'\mathrm{K}q = 2\mathrm{S}\,.\,q'\mathrm{K}q = 2\mathrm{S}(\mathrm{K}q\,.\,q')$, by 196, II., and 198, I.;

therefore

$$\text{XI.} \ldots \mathrm{dN}q = 2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q).$$

(5.) Or we might have deduced this expression XI. for dNq, more immediately, by the *general formula* 324, IV., from the earlier expression 200, VII., or 210, XX., for the *norm of a sum*, under the form,

$$\begin{aligned}\text{XI}'. \ldots \mathrm{dN}q &= \lim_{n=\infty} . \, n\{\mathrm{N}(q + n^{-1}\mathrm{d}q) - \mathrm{N}q\} \\ &= \lim_{n=\infty} . \{2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q) + n^{-1}\mathrm{Nd}q\} \\ &= 2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q),\end{aligned}$$

as before.

* In *quaternions* the equation III. is not a necessary consequence of IV., although the latter is so of the former; for example, the equation IV., but *not* the equation III., will be satisfied, if we assume $fq = qcq^{-1}c'q$, where c and c' are any two constant quaternions, which do not degenerate into scalars.

(6.) The *tensor*, Tq, is the *square-root* (190) *of the norm*, Nq; and because Tq and Nq are scalars, the formula 332, XXII. may be applied; which gives, for the *differential of the tensor* of a quaternion, the expression (comp. 158),

$$\text{XII.} \ldots \mathrm{dT}q = \frac{\mathrm{dN}q}{2\mathrm{T}q} = \mathrm{S}(\mathrm{KU}q \,.\, \mathrm{d}q) = \mathrm{S}\frac{\mathrm{d}q}{\mathrm{U}q},$$

a result which is more easily *remembered*, under the form,

$$\text{XIII.} \ldots \frac{\mathrm{dT}q}{\mathrm{T}q} = \mathrm{S}\frac{\mathrm{d}q}{q}.$$

(7.) The *versor* Uq is equal (by 188) to the *quotient*, q : Tq, of the quaternion q divided by its tensor Tq; hence the *differential of the versor* is,

$$\text{XIV.} \ldots \mathrm{dU}q = \mathrm{d}\frac{q}{\mathrm{T}q} = \left(\frac{\mathrm{d}q}{q} - \mathrm{S}\frac{\mathrm{d}q}{q}\right)\frac{q}{\mathrm{T}q} = \mathrm{V}\frac{\mathrm{d}q}{q} \,.\, \mathrm{U}q;$$

whence follows at once this formula, analogous to XIII., and like it easily remembered,

$$\text{XV.} \ldots \frac{\mathrm{dU}q}{\mathrm{U}q} = \mathrm{V}\frac{\mathrm{d}q}{q}.$$

(8.) We might also have observed that because (by 188) we have generally $q = \mathrm{T}q \,.\, \mathrm{U}q$, therefore (by 332, (3.)) we have also,

$$\text{XVI.} \ldots \mathrm{d}q = \mathrm{dT}q \,.\, \mathrm{U}q + \mathrm{T}q \,.\, \mathrm{dU}q,$$

and

$$\text{XVII.} \ldots \frac{\mathrm{d}q}{q} = \frac{\mathrm{dT}q}{\mathrm{T}q} + \frac{\mathrm{dU}q}{\mathrm{U}q};$$

if then we have in any manner established the equation XIII., we can immediately deduce XV.; and conversely, the former equation would follow at once from the latter.

(9.) It may be considered as remarkable, that we should thus have *generally*, or *for any two quaternions*, q and dq, the formula:*

$$\text{XVIII.} \ldots \mathrm{S}(\mathrm{dU}q : \mathrm{U}q) = 0; \quad \text{or} \quad \text{XVIII}'. \ldots \mathrm{dU}q : \mathrm{U}q = \mathrm{S}^{-1}0;$$

* When the connexion of the theory of *normals to surfaces*, with the *differential calculus of quaternions*, shall have been (even briefly) explained in a subsequent Section, the student will perhaps be able to perceive, in this formula XVIII., a recognition, though not a very direct one, of the geometrical principle, that the *radii* of a *sphere* are its *normals*.

but this *vector character* of the *quotient* $dUq : Uq$ can easily be confirmed, as follows. Taking the *conjugate* of that quotient, we have, by VII. (comp. 192, II.; 158; and 324, XI.),

$$\text{XIX.} \ldots K(dUq \,.\, Uq^{-1}) = KUq^{-1} \,.\, dKUq = Uq \,.\, d(Uq^{-1}) = - dUq \,.\, Uq^{-1};$$

whence

$$\text{XX.} \ldots (1 + K)\,(dUq \,.\, Uq^{-1}) = 0;$$

which agrees (by 196, II.) with XVIII.

(10.) The *scalar character* of the *tensor*, Tq, enables us always to write, as in the ordinary calculus,

$$\text{XXI.} \ldots dlTq = dTq : Tq;$$

but $lTq = Slq$, by 316, V.; the recent formula XIII. may therefore, by VIII., be thus written,

$$\text{XXII.} \ldots Sdlq = dSlq = dTq : dq = S(dq : q); \quad \text{or} \quad \text{XXII}'. \ldots dlq - q^{-1}dq = S^{-1}0.$$

(11.) When $dq \parallel\!| \, q$, this last difference vanishes, by 333, II.; and the equation XV. takes the form,

$$\text{XXIII.} \ldots dlUq = Vdlq = dVlq.$$

And in fact we have *generally*, $lUq = Vlq$, by 316, XX., although the *differentials* of these two equal expressions do not *separately* coincide with the members of the recent formula XV., when q and dq are *diplanar*. We may however write generally (comp. XXII.),

$$\text{XXIV.} \ldots dlUq - dUq : Uq = V(dlq - dq : q) = dlq - dq : q.$$

335. We have now differentiated the *six simple functions* 334, I., which are formed by the operation of the *six characteristics*,

$$K, S, V, N, T, U;$$

and as regards the differentiation of the *compound functions* 334, II., which are formed by *combinations* of those former operations, it is easy on the same principles to determine them, as may be seen in the few following examples.

(1.) The *axis* Ax. q of a quaternion has been seen (291) to admit of being represented by the *combination* UVq; the *differential* of this axis may therefore, by 334, IX. and XIV., be thus expressed:

$$\text{I.} \ldots \mathrm{d}(\mathrm{Ax}.\, q) = \mathrm{dUV}q = \mathrm{V}(\mathrm{Vd}q : \mathrm{V}q) \,.\, \mathrm{UV}q\,;$$

whence

$$\text{II.} \ldots \frac{\mathrm{d}(\mathrm{Ax}.\, q)}{\mathrm{Ax}.\, q} = \frac{\mathrm{dUV}q}{\mathrm{UV}q} = \mathrm{V}\frac{\mathrm{Vd}q}{\mathrm{V}q}.$$

The *differential of the axis* is therefore, *generally*, a *line perpendicular to that axis*, or situated *in the plane of the quaternion*; but it *vanishes*, when the *plane* (and therefore the *axis*) of that quaternion is *constant*; or when the quaternion and its differential are *complanar*.

(2.) Hence,

$$\text{III.} \ldots \mathrm{dUV}q = 0, \quad \text{if} \quad \text{IV.} \ldots \mathrm{d}q \parallel\!\mid q\,;$$

and conversely this *complanarity* IV. may be expressed by the *equation* III.

(3.) It is easy to prove, on similar principles, that

$$\text{V.} \ldots \mathrm{dVU}q = \mathrm{VdU}q = \mathrm{V}\left(\mathrm{V}\frac{\mathrm{d}q}{q} \,.\, \mathrm{U}q\right);$$

and

$$\text{VI.} \ldots \mathrm{dSU}q = \mathrm{SdU}q = \mathrm{S}\left(\mathrm{V}\frac{\mathrm{d}q}{q} \,.\, \mathrm{U}q\right).$$

(4.) But in general, for any two quaternions, q and q', we have (comp. 223, (5.)) the transformations,

$$\text{VII.} \ldots \mathrm{S}(\mathrm{V}q' . q) = \mathrm{S}(\mathrm{V}q' . \mathrm{V}q) = \mathrm{S} . q'\mathrm{V}q\,;$$

and when we thus suppress the characteristic V before $\mathrm{d}q : q$, and insert it before $\mathrm{U}q$, under the sign S in the last expression VI., we may replace the new factor $\mathrm{VU}q$ by $\mathrm{TVU}q . \mathrm{UVU}q$ (188), or by $\mathrm{TVU}q . \mathrm{UV}q$ (274, XIII.), or by $-\mathrm{TVU}q : \mathrm{UV}q$ (204, V.), where the scalar factor $\mathrm{TVU}q$ may be taken outside (by 196, VIII.); also for $q^{-1} : \mathrm{UV}q$ we may substitute $1 : (\mathrm{UV}q . q)$, or $1 : q\mathrm{UV}q$, because $\mathrm{UV}q \parallel\!\mid q$; the formula VI. may therefore be thus written,

$$\text{VIII.} \ldots \mathrm{dSU}q = -\,\mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q} \,.\, \mathrm{TVU}q.$$

(5.) Now it may be remembered, that among the earliest *connexions* of quaternions with *trigonometry*, the following formulæ occurred (196, XVI., and 204, XIX.),

$$\text{IX.} \ldots \mathrm{SU}q = \cos \angle q, \quad \mathrm{TVU}q = \sin \angle q\,;$$

we had also, in 316, these expressions for the *angle* of a quaternion,

$$\text{X.} \ldots \angle q = \mathrm{TVl}q = \mathrm{TlU}q\,;$$

we may therefore establish the following expression for the *differential of the angle* of a quaternion,

$$\text{XI.} \ldots \mathrm{d}\angle q = \mathrm{dTVl}q = \mathrm{dTlU}q = \mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q}.$$

(6.) The following is another way of arriving at the same result, through the differentiation of the *sine* instead of the *cosine* of the angle, or through the calculation of $\mathrm{dTVU}q$, instead of $\mathrm{dSU}q$. For this purpose, it is only necessary to remark that we have, by 334, XII. XIV., and by some easy transformations of the kind lately employed in (4.), the formula,

$$\text{XII.} \ldots \mathrm{dTVU}q = \mathrm{S}\frac{\mathrm{VdU}q}{\mathrm{UVU}q} = \mathrm{S}\frac{\mathrm{dU}q}{\mathrm{UV}q} = \mathrm{S}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\cdot\frac{\mathrm{U}q}{\mathrm{UV}q}\right) = \mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q}.\mathrm{SU}q;$$

dividing which by $\mathrm{SU}q$, and attending to IX. and X., we arrive again at the expression XI., for the differential of the angle of a quaternion.

(7.) Eliminating $\mathrm{S}(\mathrm{d}q : q\mathrm{UV}q)$ between VIII. and XII., we obtain the *differential equation*,

$$\text{XIII.} \ldots \mathrm{SU}q.\mathrm{dSU}q + \mathrm{TVU}q.\mathrm{dTVU}q = 0;$$

of which, on account of the *scalar* character of the differentiated variables, the *integral* is evidently of the *form*,

$$\text{XIV.} \ldots (\mathrm{SU}q)^2 + (\mathrm{TVU}q)^2 = \text{const.};$$

and accordingly we saw, in 204, XX., that the sum in the first member of this equation is constantly equal to positive unity.

(8.) The formula XI. may also be thus written,

$$\text{XV.} \ldots \mathrm{d}\angle q = \mathrm{S}(\mathrm{V}(\mathrm{d}q.q) : \mathrm{UV}q);$$

with the verification, that when we suppose $\mathrm{d}q \,|||\, q$, as in IV., and therefore $\mathrm{dUV}q = 0$ by III., the expression under the sign S becomes the differential of the quotient, $\mathrm{Vl}q : \mathrm{UV}q$, and therefore, by 316, VI., of the angle $\angle q$ itself.

336. An important application of the foregoing principles and rules consists in the *differentiation of scalar functions of vectors*, when those functions are defined and expressed according to the laws and notations of quaternions. It will be found, in fact, that *such* differentiations play a very extensive part, in the applications of quaternions to *geometry*; but, for the moment, we shall treat them *here*, as merely exercises of calculation. The following are a few examples.

(1.) Let ρ denote, in these sub-articles, a *variable vector*; and let the following equation be proposed,

$$\text{I.} \ldots r^2 + \rho^2 = 0, \quad \text{in which} \quad \mathrm{V}r = 0,$$

so that r is a (generally variable) *scalar*. Differentiating, and observing that, by 279, III., $\rho\rho' + \rho'\rho = 2\mathrm{S}\rho\rho'$, if ρ' be any *second vector*, such as we suppose $\mathrm{d}\rho$ to be, we have, by 322, VIII., and 324, VII., the equation,

$$\text{II.} \ldots r\mathrm{d}r + \mathrm{S}\rho\mathrm{d}\rho = 0; \quad \text{or} \quad \text{III.} \ldots \mathrm{d}r = -r^{-1}\mathrm{S}\rho\mathrm{d}\rho = r\mathrm{S}\rho^{-1}\mathrm{d}\rho.$$

In fact, if r be supposed *positive*, it is here, by 282, II., the *tensor* of ρ; so that this last expression III. for $\mathrm{d}r$ is included in the general formula, 334, XIII.

(2.) If this tensor, r, be *constant*, the differential equation II. becomes simply,

$$\text{IV.} \ldots \mathrm{S}\rho\mathrm{d}\rho = 0, \quad \text{if} \quad -\rho^2 = \text{const.}, \quad \text{or if} \quad \mathrm{dT}\rho = 0.$$

(3.) Again, let the proposed equation be (comp. 282, XIX.),

$$\text{V.} \ldots r^2 = \mathrm{T}(\iota\rho + \rho\kappa), \quad \text{with} \quad \mathrm{d}\iota = 0, \quad \mathrm{d}\kappa = 0,$$

so that ι and κ are here *two constant vectors*. Then, squaring and differentiating, we have (by 334, XI., because $\mathrm{K}\iota\rho = \rho\iota$, &c.),

$$\text{VI.} \ldots 2r^3\mathrm{d}r = \tfrac{1}{2}\mathrm{dN}(\iota\rho + \rho\kappa) = \mathrm{S}(\rho\iota + \kappa\rho)(\iota\mathrm{d}\rho + \mathrm{d}\rho\kappa) = (\iota^2 + \kappa^2)\,\mathrm{S}\rho\mathrm{d}\rho + 2\mathrm{S}\kappa\rho\iota\mathrm{d}\rho;$$

or more briefly,

$$\text{VII.} \ldots 2r^{-1}\mathrm{d}r = \mathrm{S}\nu\mathrm{d}\rho,$$

if ν be an *auxiliary vector*, determined by the equation,

$$\text{VIII.} \ldots r^4\nu = (\iota^2 + \kappa^2)\,\rho + 2\mathrm{V}\kappa\rho\iota;$$

which admits of several transformations.

(4.) For example we may write, by 295, VII.,

$$\text{IX.} \ldots r^4\nu = (\iota^2 + \kappa^2)\,\rho + \kappa\rho\iota + \iota\rho\kappa = \iota(\iota\rho + \rho\kappa) + \kappa(\rho\iota + \kappa\rho);$$

or, by 294, III., and 282, XII.,

$$\text{X.} \ldots r^4\nu = (\iota^2 + \kappa^2)\rho + 2(\kappa\mathrm{S}\iota\rho - \rho\mathrm{S}\iota\kappa + \iota\mathrm{S}\kappa\rho) = (\iota - \kappa)^2\rho + 2(\iota\mathrm{S}\kappa\rho + \kappa\mathrm{S}\iota\rho); \ \&\text{c.}$$

(5.) The equation V. gives (comp. 190, V.), when squared without differentiation,

$$\text{XI.} \ldots r^4 = \mathrm{N}(\iota\rho + \rho\kappa) = (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho)$$
$$= (\iota^2 + \kappa^2)\rho^2 + \iota\rho\kappa\rho + \rho\kappa\rho\iota$$
$$= (\iota^2 + \kappa^2)\rho^2 + 2\mathrm{S}\iota\rho\kappa\rho$$
$$= (\iota - \kappa)^2\rho^2 + 4\mathrm{S}\iota\rho\,\mathrm{S}\kappa\rho = \&\text{c.},$$

by transformations of the same kind as before; we have therefore, by the recent expressions for $r^4\nu$, the following remarkably simple *relation* between the *two variable vectors*, ρ and ν,

$$\text{XII.} \ldots \mathrm{S}\nu\rho = 1; \quad \text{or} \quad \text{XII}'. \ldots \mathrm{S}\rho\nu = 1.$$

(6.) When the *scalar*, r, is *constant*, we have, by VII., the *differential equation*,

$$\text{XIII.} \ldots \mathrm{S}\nu\mathrm{d}\rho = 0; \quad \text{whence also} \quad \text{XIV.} \ldots \mathrm{S}\rho\mathrm{d}\nu = 0, \text{ by XII.};$$

a *relation of reciprocity* thus existing, between the *two vectors* ρ and ν, of which the *geometrical signification* will soon be seen.

(7.) Meanwhile, supposing r again to *vary*, we see that the last expression VI. for $2r^3\mathrm{d}r$ may be otherwise obtained, by taking half the differential of either of the two last expanded expressions XI. for r^4; it being remembered, in all these little calculations, that *cyclical permutation of factors, under the sign* S, *is permitted* (223, (10.)), even if those factors be *quaternions*, and whatever their *number* may be: and that if they be *vectors*, and if their number be *odd*, it is then permitted, *under the sign* V, to *invert their order* (295, (9.)), and so to write, for instance, $\mathrm{V}\iota\rho\kappa$ instead of $\mathrm{V}\kappa\rho\iota$, in the formula VIII.

(8.) As another example of a *scalar function of a vector*, let p denote the *proximity* (or *nearness*) of a *variable point* P to the *origin* O; so that

$$\text{XV.} \ldots p = (-\rho^2)^{-\frac{1}{2}} = \mathrm{T}\rho^{-1}, \quad \text{or} \quad \text{XV}'. \ldots p^{-2} + \rho^2 = 0.$$

Then,

$$\text{XVI.} \ldots \mathrm{d}p = \mathrm{S}\nu\mathrm{d}\rho, \quad \text{if} \quad \text{XVII.} \ldots \nu = p^3\rho = p^2\mathrm{U}\rho;$$

ν being here a *new auxiliary vector*, distinct from the one lately considered (VIII.), and having (as we see) the *same versor* (or the *same direction*) as the vector ρ *itself*, but having its *tensor* equal to the *square of the proximity of* P *to* O; or equal to the *inverse square of the distance*, of one of those two points from the other.

337. On the other hand, we have often occasion, in the applications, to consider *vectors as functions of scalars*, as in 99, but now with *forms* arising

out of operations on *quaternions*, and therefore such as had not been considered in the First Book. And whenever we have thus an expression such as either of the two following,

$$\text{I.} \ldots \rho = \phi(t), \quad \text{or} \quad \text{II.} \ldots \rho = \phi(s, t),$$

for the *variable vector* of a *curve*, or of a *surface* (comp. again 99), s and t being *two variable scalars*, and $\phi(t)$ and $\phi(s, t)$ denoting *any functions of vector form*, whereof the *latter* is here supposed to be entirely *independent** of the *former*, we may then employ (comp. 100, (4.) and (9.) and the more recent sub-articles, 327, (5.), (6.), and 329, (5.)) the *notations of derivatives*, total or partial; and so may write, as the *differentiated equations*, resulting from the forms I. and II. respectively, the following:

$$\text{III.} \ldots \mathrm{d}\rho = \phi't \,.\, \mathrm{d}t = \rho'\mathrm{d}t = \mathrm{D}_t\rho \,.\, \mathrm{d}t;$$

$$\text{IV.} \ldots \mathrm{d}\rho = \mathrm{d}_s\rho + \mathrm{d}_t\rho = \mathrm{D}_s\rho \,.\, \mathrm{d}s + \mathrm{D}_t\rho \,.\, \mathrm{d}t;$$

of which the geometrical significations have been already partially seen, in the sub-articles to 100, and will soon be more fully developed.

(1.) Thus, for the *circular locus*, 314, (1.), for which

$$\text{V.} \ldots \rho = \alpha^t\beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta = 0,$$

we have, by 333, VIII., the following *derived vector*,

$$\text{VI.} \ldots \rho' = \mathrm{D}_t\rho = \frac{\pi}{2}\alpha^{t+1}\beta = \frac{\pi}{2}\,\alpha\rho.$$

(2.) And for the *elliptic locus*, 314, (2.), for which

$$\text{VII.} \ldots \rho = \mathrm{V}.\,\alpha^t\beta, \quad \mathrm{T}\alpha = 1, \quad \text{but } not \ \mathrm{S}\alpha\beta = 0,$$

we have, in like manner, this other derived vector,

$$\text{VIII.} \ldots \rho' = \mathrm{D}_t\rho = \frac{\pi}{2}\mathrm{V}.\,\alpha^{t+1}\beta.$$

(3.) As an example of a *vector-function* of *more scalars than one*, let us resume the expression (308, XVIII.),

$$\text{IX.} \ldots \rho = rk^tj^skj^{-s}k^{-t};$$

---

* We are therefore not employing *here* the temporary notation of some recent Articles, according to which we should have had, $\mathrm{d}\phi q = \phi(q, \mathrm{d}q)$.

in which we shall now suppose that the tensor r is *given*, so that ρ is the *variable vector* of a point upon a *given spheric surface*, of which the *radius* is r, and the *centre* is at the origin; while s and t are *two independent scalar variables*, with respect to which the *two partial derivatives* of the vector ρ are to be determined.

(4.) The derivation relatively to t is easy; for, since ijk are *vector-units* (295), and since we have generally, by 333, VIII.,

$$\text{X.} \ldots \mathrm{d} \,.\, a^x = \frac{\pi}{2}\, a^{x+1}\mathrm{d}x, \quad \text{and therefore} \quad \text{XI.} \ldots \mathrm{D}_t \,.\, a^x = \frac{\pi}{2} a^{x+1}\mathrm{D}_t x,$$

if $\mathrm{T}a = 1$, and if x be any scalar function of t, we may write, at once, by 279, IV.,

$$\text{XII.} \ldots \mathrm{D}_t\rho = \frac{\pi}{2}\,(k\rho - \rho k) = \pi \mathrm{V} k\rho\,;$$

and we see that

$$\text{XIII.} \ldots \mathrm{S}\rho\mathrm{D}_t\rho = 0,$$

a result which was to be expected, on account of the equation,

$$\text{XIV.} \ldots \rho^2 + r^2 = 0,$$

which follows, by 308, XXIV., from the recent expression IX. for ρ.

(5.) To form an expression of about the same degree of simplicity, for the *other* partial derivative of ρ, we may observe that $j^{s+1}kj^{-s}$ is equal to its own vector part (its scalar vanishing); hence*

$$\text{XV.} \ldots \mathrm{D}_s\rho = \pi k^t j k^{-t}\rho\,; \quad \text{or} \quad \text{XVI.} \ldots \mathrm{D}_s\rho = \pi k^{2t} j\rho = \pi j k^{-2t}\rho,$$

by the transformation 308, (11.). And because the scalar of $k^t j k^{-t}$ is zero, we have thus the equation,

$$\text{XVII.} \ldots \mathrm{S}\rho\mathrm{D}_s\rho = 0,$$

which is analogous to XIII., and might have been otherwise obtained, by taking the derivative of XIV. with respect to the variable scalar s.

(6.) The partial derivative $\mathrm{D}_s\rho$ must be a *vector*; hence, by XV. or XVI., ρ must be *perpendicular* to the vector $k^t j k^{-t}$, or $k^{2t}j$, or jk^{-2t}; a result which, under the last form, is easily confirmed by the expression 315, XII. for ρ. In fact that expression gives, by 315, (3.) and (4.), and by the recent values

* [Thus $\mathrm{D}_s(j^s k j^{-s}) = \frac{\pi}{2}(j^{s+1}kj^{-s} + j^s kj^{-s-1}) = \pi j^{s+1}kj^{-s}$, and therefore $\mathrm{D}_s\rho = \pi\, r k^t j k^{-t} k^t j^s k j^{-s} k^{-t}$ which is equivalent to XV.]

XII. XVI., these *other forms* for the two partial derivatives of ρ, which have been above considered:

$$\text{XVIII.} \ldots \mathrm{D}_t\rho = \pi r k^{2t}\mathrm{V}.j^{2s}; \quad \text{XIX.} \ldots \mathrm{D}_s\rho = \pi r(k^{2t}\mathrm{V}.i^{2s+1} - \mathrm{V}.k^{2s});$$

which might have been immediately obtained, by partial derivations, from the expression 315, XII. itself, and of which *both* are *vector-forms*.

(7.) And hence, or immediately by *derivating* the expanded expression 315, XIII., we obtain these new forms for the partial derivatives of ρ:

$$\text{XX.} \ldots \mathrm{D}_t\rho = \pi r(j\cos t\pi - i\sin t\pi)\sin s\pi;$$

$$\text{XXI.} \ldots \mathrm{D}_s\rho = \pi r\{(i\cos t\pi + j\sin t\pi)\cos s\pi - k\sin s\pi\}.$$

(8.) We may add that not only is the variable vector ρ *perpendicular to each* of the *two derived vectors*, $\mathrm{D}_s\rho$ and $\mathrm{D}_t\rho$, but also *they* are perpendicular to *each other*; for we may write, by XII. and XVI.,

$$\text{XXII.} \ldots \mathrm{S}(\mathrm{D}_s\rho \,.\, \mathrm{D}_t\rho) = -\pi^2\mathrm{S}\,.\,k^{2t}j\rho^2 k = \pi^2 r^2\mathrm{S}\,.\,k^{2t}i = 0;$$

and the same conclusion may be drawn from the expressions XX. and XXI.

(9.) A *vector* may be considered as a function of *three independent scalar variables*, such as r, s, t; or rather it *must* be so considered, if it is to admit of being the vector of an *arbitrary point of space*: and then it will have a *total differential* (329) of the *trinomial form*,

$$\text{XXIII.} \ldots \mathrm{d}\rho = \mathrm{d}_r\rho + \mathrm{d}_s\rho + \mathrm{d}_t\rho = \mathrm{D}_r\rho\,.\,\mathrm{d}r + \mathrm{D}_s\rho\,.\,\mathrm{d}s + \mathrm{D}_t\rho\,.\,\mathrm{d}t;$$

and will thus have *three* partial derivatives*.

(10.) For example, when ρ has the expression IX., we have this *third* partial derivative,

$$\text{XXIV.} \ldots \mathrm{D}_r\rho = r^{-1}\rho = \mathrm{U}\rho,$$

which may also be thus more fully written (comp. again 315, XIII.),

$$\text{XXV.} \ldots \mathrm{D}_r\rho = k^t j^s k j^{-s} k^{-t} = (i\cos t\pi + j\sin t\pi)\sin s\pi + k\cos s\pi;$$

and we see that the *three derived vectors*,

$$\text{XXVI.} \ldots \mathrm{D}_r\rho,\ \mathrm{D}_s\rho,\ \mathrm{D}_t\rho,$$

compose here a *rectangular system*.

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* That is to say, *three of the first order*; for we shall soon have occasion to consider *successive differentials*, of functions of one or more variables, and so shall be conducted to the consideration of *orders* of differentials and derivatives, *higher than the first*.

SECTION 5.

On Successive Differentials, and Developments, of Functions of Quaternions.

338. There will now be no difficulty in the *successive differentiation*, total or partial, of functions of one or more quaternions; and *such* differentiation will be found to be *useful*, as in the ordinary calculus, in connexion with *developments of functions*: besides that it is *necessary* for many of those *geometrical and physical applications* of differentials of quaternions, on which we have not entered yet. A few *examples* of successive differentiation may serve to show, more easily than any general precepts, the nature and effects of the operation; and we shall begin, for simplicity, with *explicit functions of one quaternion variable.*

(1.) Take then the *square*, q^2, of a quaternion, as a function fq, which is to be *twice* differentiated. We saw, in 324, VII., that a *first* differentiation gave the equation,

$$\text{I.} \ldots \mathrm{d}fq = \mathrm{d}\,.\,q^2 = q\,.\,\mathrm{d}q + \mathrm{d}q\,.\,q\,;$$

but we are now to differentiate *again*, in order to form the *second differential* d^2fq of the *function* q^2, treating the differential of the variable q as *still* equal to $\mathrm{d}q$, and *in general* writing $\mathrm{dd}q = \mathrm{d}^2q$, where d^2q is a *new arbitrary quaternion*, of which the *tensor*, Td^2q, *need not be small* (comp. 322). And thus we get, *in general*, this *twice differentiated expression*, or *differential of the second order*,

$$\text{II.} \ldots \mathrm{d}^2fq = \mathrm{d}^2\,.\,q^2 = q\,.\,\mathrm{d}^2q + 2\mathrm{d}q^2 + \mathrm{d}^2q\,.\,q.$$

(2.) The *second differential of the reciprocal* of a quaternion is *generally* (comp. 324, XI.),

$$\text{III.} \ldots \mathrm{d}^2\,.\,q^{-1} = 2(q^{-1}\mathrm{d}q)^2\,q^{-1} - q^{-1}\mathrm{d}^2q\,.\,q^{-1}.$$

(3.) If ρ be a *variable vector*, then (comp. 336, (1.)) we have, for the first and second differentials of its square, the expressions:

$$\text{IV.} \ldots \mathrm{d}\,.\,\rho^2 = 2\mathrm{S}\rho\mathrm{d}\rho\,; \quad \text{V.} \ldots \mathrm{d}^2\,.\,\rho^2 = 2\mathrm{S}\rho\mathrm{d}^2\rho + 2\mathrm{d}\rho^2.$$

(4.) If $f\rho$ be any *other scalar function* of a variable vector ρ, and if (comp. again the sub-articles to 336) its *first* differential be put under the *form*,

$$\text{VI.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho, \text{ when } \nu \text{ is another variable vector,}$$

then the *second* differential of the same function may be expressed as follows:

$$\text{VII.} \ldots \mathrm{d}^2 f\rho = 2\mathrm{S}\nu \mathrm{d}^2\rho + 2\mathrm{S}\mathrm{d}\nu \mathrm{d}\rho;$$

in which we have written, briefly, $\mathrm{Sd}\nu\mathrm{d}\rho$, instead of $\mathrm{S}(\mathrm{d}\nu \,.\, \mathrm{d}\rho)$.

(5.) The following very simple equation will be found useful, in the theory of *motions*, performed under the influence of *central forces*:

$$\text{VIII.} \ldots \mathrm{dV}\rho\mathrm{d}\rho = \mathrm{V}\rho\mathrm{d}^2\rho; \quad \text{because} \quad \mathrm{V}\,.\,\mathrm{d}\rho^2 = 0.$$

(6.) As an example of the second differential of a *quaternion*, considered as a *function of a scalar variable* (comp. 333, VIII., and 337, (1.)), the following may be assigned, in which α denotes a given unit line, so that $\alpha^2 = -1$, $\mathrm{d}\alpha = 0$, but x is a variable scalar:

$$\text{IX.} \ldots \mathrm{d}^2\,.\,\alpha^x = \mathrm{d}\left(\frac{\pi}{2}\alpha^{x+1}\mathrm{d}x\right) = \frac{\pi}{2}\,\alpha^{x+1}\mathrm{d}^2x - \left(\frac{\pi}{2}\right)^2\alpha^x\mathrm{d}x^2.$$

(7.) The second differential of the *product* of *any two functions* of a *quaternion* q may be expressed as follows (comp. II.):

$$\text{X.} \ldots \mathrm{d}^2(fq\,.\,\phi q) = \mathrm{d}^2fq\,.\,\phi q + 2\mathrm{d}fq\,.\,\mathrm{d}\phi q + fq\,.\,\mathrm{d}^2\phi q.$$

339. The second differential, d^2q, of the variable quaternion q, enters *generally* (as has been seen) into the expression of the second differential d^2fq, of the function fq, as a *new* and *arbitrary* quaternion: but, for that very reason, it is *permitted*, and it is frequently found to be *convenient*, to *assume* that this *second* differential d^2q is equal to *zero*: or, what comes to the same thing, that the *first* differential $\mathrm{d}q$ is *constant*. And when we make this *new supposition*,

$$\text{I.} \ldots \mathrm{d}q = \textit{constant}, \quad \text{or} \quad \text{I}' \ldots \mathrm{d}^2q = 0,$$

the expressions for d^2fq become of course more simple, as in the following examples.

(1.) With this last supposition, I. or I′., we have the following *second* differentials, of the *square* and the *reciprocal* of a quaternion:

$$\text{II.} \ldots \mathrm{d}^2\,.\,q^2 = 2\mathrm{d}q^2; \quad \text{III.} \ldots \mathrm{d}^2\,.\,q^{-1} = 2(q^{-1}\mathrm{d}q)^2q^{-1} = 2q^{-1}(\mathrm{d}q\,.\,q^{-1})^2.$$

(2.) Again, if we suppose that c_0, c_1, c_2 are any three *constant* quaternions, and take the function,

$$\text{IV.} \ldots fq = c_0qc_1qc_2,$$

we find, under the same condition I. or I′., that its first and second differentials are,

$$\text{V.} \ldots \mathrm{d}fq = c_0\mathrm{d}q \,.\, c_1qc_2 + c_0qc_1\mathrm{d}q \,.\, c_2; \quad \text{VI.} \ldots \mathrm{d}^2fq = 2c_0\mathrm{d}q \,.\, c_1\mathrm{d}q \,.\, c_2;$$

in writing which, the *points** may be omitted.

(3.) The *first* differential, dq, remaining still entirely *arbitrary* (comp. 322, (8.), and 325, (2.)), so that no supposition is made that its *tensor* T$\mathrm{d}q$ is *small*, although we *now* suppose this differential dq to be *constant* (I.) we have *rigorously*,

$$\text{VII.} \ldots (q + \mathrm{d}q)^2 = q^2 + \mathrm{d} \,.\, q^2 + \tfrac{1}{2}\mathrm{d}^2 \,.\, q^2;$$

an equation which may be also written thus,

$$\text{VIII.} \ldots (q + \mathrm{d}q)^2 = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2) \,.\, q^2.$$

(4.) And in like manner we shall have, more generally, under the same condition of *constancy* of dq, the equation,

$$\text{IX.} \ldots f(q + \mathrm{d}q) = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2)\, fq,$$

if the function fq be the *sum* of *any number of monomes, each* separately of the *form* IV., and therefore each *rational, integral,* and *homogeneous* of the *second dimension,* with respect to the variable quaternion, q; or of *such* monomes, combined with others of the *first* dimension, and with *constant terms*: that is, if a_0, b_0, b_1, b'_0, b'_1, . . and c_0, c_1, c_2, c'_0, c'_1, c'_2, . . be *any constant quaternions,* and

$$\text{X.} \ldots fq = a_0 + \Sigma b_0qb_1 + \Sigma c_0qc_1qc_2.$$

340. It is easy to carry on the operation of differentiating, to the *third* and *higher orders*; remembering only that if, *in any former stage,* we have *denoted* the *first* differentials of q, dq, . . by dq, d^2q, . . we then *continue* so to denote them, in every *subsequent* stage of the *successive differentiation*: and that *if* we find it convenient to treat any *one* differential as *constant,* we must then treat *all* its *successive* differentials as *vanishing.* A few examples may be given, chiefly with a view to the *extension* of the recent formula 339, IX., for the *function* $f(q + \mathrm{d}q)$ *of a sum,* of *any two quaternions,* q and dq, to *polynomial forms,* of *dimensions* higher than the *second.*

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* Compare the second Note to page 439.

(1.) The *third differential of a square* is *generally* (comp. 338, II.),

$$\text{I.} \ldots \mathrm{d}^3 . q^2 = q . \mathrm{d}^3 q + \mathrm{d}^3 q . q + 3(\mathrm{d}q . \mathrm{d}^2 q + \mathrm{d}^2 q . \mathrm{d}q).$$

(2.) More generally, the *third* differential of a *product of two quaternion functions* (comp. 338, X.) may be thus expressed:

$$\text{II.} \ldots \mathrm{d}^3(fq . \phi q) = \mathrm{d}^3 fq . \phi q + 3\mathrm{d}^2 fq . \mathrm{d}\phi q + 3\mathrm{d}fq . \mathrm{d}^2\phi q + fq . \mathrm{d}^3\phi q.$$

(3.) More generally still, the n^{th} differential of a *product* is, as in the ordinary calculus,

$$\text{III.} \ldots \mathrm{d}^n(fq . \phi q) = \mathrm{d}^n fq . \phi q + n\mathrm{d}^{n-1} fq . \mathrm{d}\phi q + n_2 \mathrm{d}^{n-2} fq . \mathrm{d}^2\phi q + \ldots + fq . \mathrm{d}^n \phi q,$$

if $$n_2 = \frac{n(n-1)}{2}, \quad n_3 = \frac{n(n-1)(n-2)}{2.3}, \quad \&c.;$$

the only thing *peculiar* to quaternions being, that we are obliged to *retain* (generally) the *order of the factors*, in each term of this expansion III.

(4.) Hence, in particular, denoting briefly the function fq by r, and changing ϕq to q,

$$\text{IV.} \ldots \mathrm{d}^n . rq = \mathrm{d}^n r . q + n\mathrm{d}^{n-1} r . \mathrm{d}q, \quad \text{if} \quad \mathrm{d}^2 q = 0.$$

(5.) Hence also, under this *condition* that $\mathrm{d}q$ is constant, if c be any *other* constant quaternion, we have the transformation,

$$\text{V.} \ldots \left(1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \ldots + \frac{1}{2.3\ldots n}\mathrm{d}^n\right) . rqc =$$

$$\left(1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \ldots + \frac{1}{2.3\ldots(n-1)}\mathrm{d}^{n-1}\right) r . (q + \mathrm{d}q)c, \quad \text{if} \quad \mathrm{d}^n r = 0.$$

(6.) Hence, by 339, (4.), it is easy to infer that if we *interpret* the *symbol* ε^{d} by the *equation* (comp. 316, I.),

$$\text{VI.} \ldots \varepsilon^{\mathrm{d}} = 1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \&c.,$$

that is, if we interpret this *other* symbol $\varepsilon^{\mathrm{d}} fq$, as concisely denoting the *series* which is formed from fq, by *operating* on it with this symbolic development; and if the *function* fq, thus operated on, be *any finite polynome*, involving (like the expression 339, X.) *no fractional nor negative exponents*; we may then write, as an *extension* of a recent equation (339, IX.) the formula:

$$\text{VII.} \ldots \varepsilon^{\mathrm{d}} fq = f(q + \mathrm{d}q), \quad \text{if} \quad \mathrm{d}^2 q = 0;$$

which is here a perfectly *rigorous* one, *all* the *terms* of this *expansion* for a *function of a sum* of two quaternions, q and dq, becoming separately equal to *zero*, as soon as the symbolic *exponent* of d becomes greater than the *dimension* of the polynome.

(7.) We shall soon [342] see that there is a *sense*, in which this *exponential transformation* VII. may be *extended*, to *other functional forms* which are *not* composed as above: and that thus an *analogue of Taylor's Theorem* can be established *for Quaternions*. Meanwhile it may be observed that by changing dq to Δq, in the *finite expansion* obtained as above, we may write the formula as follows:

$$\text{VIII.} \ldots \varepsilon^{d} fq = f(q + \Delta q) = (1 + \Delta) fq, \quad \text{or briefly,} \quad \text{IX.} \ldots \varepsilon^{d} = 1 + \Delta;$$

which last *symbolical equation* may be *operated on*, or *transformed*, as in the *usual* calculus of *differences* and *differentials*. For instance, it being understood that we treat $\Delta^2 q$ as well as $d^2 q$ as vanishing, we have thus (for any positive and whole exponent m), the two following transformations of IX.,

$$\text{X.} \ldots \Delta^m = (\varepsilon^{d} - 1)^m, \quad \text{and} \quad \text{XI.} \ldots d^m = (\log (1 + \Delta))^m;$$

the *results* of *operating*, with the *symbols* thus *equated*, on any *polynomial function* fq, of the kind above described, being always *finite expansions*, which are *rigorously equal* to each other.

341. Let Fx and ϕx be *any two functions* of a *scalar variable*, of which both *vanish with that variable*; so that they satisfy the two conditions,

$$\text{I.} \ldots F0 = 0, \quad \phi 0 = 0.$$

Then the *three simultaneous values*,

$$\text{II.} \ldots x, \quad Fx, \quad \phi x,$$

of the variable and the two functions, are at the same time (comp. 320, 321) *three simultaneous differences*, as compared with this *other system* of three simultaneous values,

$$\text{III.} \ldots 0, \quad F0, \quad \phi 0.$$

If, then, any *equimultiples*,

$$\text{IV.} \ldots nx, \quad nFx, \quad n\phi x,$$

of the three values II., can be made, by any suitable *increase* of the *number*, n, combined with a *decrease* of the *variable*, x, to *tend together* to any *system of* *limits*, those *limits* must (by the *definition* in 320, compare again 321) admit of being considered as a *system of simultaneous differentials*,

$$\text{V.} \ldots \mathrm{d}x, \quad \mathrm{d}Fx, \quad \mathrm{d}\phi x,$$

answering to the *system of initial values* III.; and must be *proportional* to the *ultimate values* of the connected *system of derivatives*,

$$\text{VI.} \ldots 1, \quad F'x, \quad \phi' x, \quad \text{when } x \text{ tends to zero.}$$

We may therefore write, as *expressions* for those *ultimate values* of the two last *derived functions*,

$$\text{VII.} \ldots F'0 = \lim_{n=\infty} . nF\frac{1}{n}, \quad \phi'0 = \lim_{n=\infty} . n\phi\frac{1}{n}, \quad \text{if} \quad F0 = \phi 0 = 0.$$

And *even if these* last values *vanish*, or if the *two new conditions*

$$\text{VIII.} \ldots F'0 = 0, \quad \phi'0 = 0,$$

are satisfied, so that x, $F'x$, and $\phi'x$ are now (comp. II.) a *new system* of *simultaneous differences*, we may *still* establish the following *equation of limits of quotients*, which is *independent* of these *last* conditions VIII.,

$$\text{IX.} \ldots \lim_{x=0} (Fx : \phi x) = \lim_{x=0} (F'x : \phi' x), \quad \text{if} \quad F0 = \phi 0 = 0;$$

it being understood that, in certain *cases*, these *two quotients* may *both vanish with* x; or may *tend together to infinity*, when x *tends*, as before, *to zero*.

(1.) This theorem is so important, that it will not be useless to confirm it by a *geometrical illustration*, which may at the same time serve for a *geometrical proof*; at least for the extensive case where *both* the *functions* fx and ϕx are of *scalar forms*, and consequently may be *represented*, or *constructed*, by the *corresponding ordinates*, XY and XZ (or ordinates answering to one *common abscissa* OX), of *two curves* OyY and OzZ, which are *in one plane*, and set out from (or pass through) one *common origin* O, as in the annexed figure 75. We shall afterwards see that the result, so obtained, can be *extended to quaternion functions*.

(2.) Suppose then, first, that the ordinates of these two curves are *proportional*, or that they bear to each other one *fixed* and *constant ratio*; so that the equation,

$$\text{X.} \ldots \text{XY} : \text{XZ} = xy : xz,$$

is satisfied for *every pair* of *abscissæ*, OX and Ox, however *great* or *small* the corresponding *ordinates* may be. Prolonging then (if necessary) the *chord*

Yy of the *first curve*, to meet the *axis* of abscissæ in some point t, and so to determine a *subsecant* tX, we see at once (by similar triangles) that the *corresponding chord* Zz of the *second curve* will meet the same axis in the *same point*, t; and therefore that it will determine (*rigorously*) the *same subsecant*, tX.

(3.) Hence, if the point x be conceived to approach to X, so that the *secant* Yyt of the *first* curve tends to coincide with the *tangent* YT to *that* curve at the point Y, the *secant* Zzt of the *second* curve must tend to coincide with the line ZT, which line therefore must be the *tangent* to that second curve: or in other words, *corresponding subtangents coincide*, and of course are *equal*, under the supposed *condition* X., of a constant *proportionality of ordinates.*

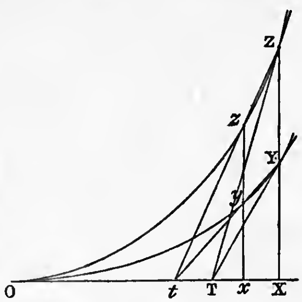

Fig. 75.

(4.) Suppose next that corresponding ordinates only *tend* to bear a *given* or *constant ratio* to each other; or that their (now) *variable* ratio tends to a given or fixed *limit*, when the common abscissa is indefinitely diminished, or when the *point* X *tends* to O; and let T be still the variable point in which the tangent to the *first* curve at Y meets the axis, so that the line TX is still the *first subtangent.* Then the corresponding tangent to the *second* curve at Z will *not* in general pass through the point T, but will meet the axis in some *different* point U. But the *ratio* of the two *corresponding subtangents*, TX and UX, which *had* been a ratio of *equality*, when the condition of *proportionality* X. was satisfied *rigorously*, will now at least *tend* to *such* a ratio; so that we shall have, under this *new condition*, of *tendency* to proportionality of ordinates, the *limiting equation*,

$$\text{XI.} \ldots \lim(\text{TX} : \text{UX}) = 1;$$

whence the equation IX. results, under the *geometrical form*,

$$\text{XII.} \ldots \lim(\tan \text{XTY} : \tan \text{XUZ}) = \lim(\text{XY} : \text{XZ}).$$

(5.) We might also have observed that, when the *proportion* X. is *rigorous*, corresponding *areas** (such as xXYy and xXZz) of the two curves are then *exactly* in the *given* ratio of the *ordinates*; so that this *other* equation, or proportion,

$$\text{XIII.} \ldots \text{OXY}y\text{O} : \text{OXZ}z\text{O} = \text{XY} : \text{XZ},$$

* Compare the Fourth Lemma of the First Book of the Principia; and see especially its Corollary, in which the reasoning of the present sub-article is virtually anticipated.

is then *also* rigorous. Hence if we only suppose, as in (4.), that the ordinates *tend* to some fixed *limiting ratio*, the *areas* must tend to the *same*; so that *if* the *second member* of the equation IX. have *any definite value*, as a *limit*, the *first* member must have the *same*: whereas the recent proof, by *subtangents*, served rather to show that *if* the *first* (or left hand) *limit* in IX. *existed*, then the *second* limit in that equation existed *also*, and was *equal* to the first.

(6.) If the *function* Fx be a *quaternion*, we may (by 221) express it as follows,

$$\text{XIV.} \ldots Fx = W + iX + jY + kZ,$$

where W, X, Y, Z are *four scalar functions* of x, of which *each* separately can be *constructed*, as the ordinate of a plane curve; and the recent *geometrical** reasoning will thus apply to each of them, and therefore to their *linear combination* Fx: which *quaternion function* reduces itself to a *vector function* of x, when $W = 0$.

(7.) And if ψx were *another* quaternion or vector function, we might first *substitute* it for Fx, and then *eliminate* the *scalar* function ϕx; so that a limiting equation of the *form* IX. may thus be proved to hold good, when *both* the functions compared are *vectors*, or *quaternions*, supposed still to *vanish* with x.

(8.) The general considerations, however, on which the equation IX., was lately established, appear to be more simple and direct; and it is evident that they give, in like manner, this other but analogous equation, in which $F''x$ and $\phi''x$ are *second derivatives*, and the conditions VIII. are now supposed to be satisfied:

$$\text{XV.} \ldots \lim_{x=0}(F'x : \phi'x) = \lim_{x=0}(F''x : \phi''x), \quad \text{if} \quad F'0 = 0,\ \phi'0 = 0.$$

* Instead of the equation IX., it has become usual, in modern works on the Differential Calculus, to give one of the following form (deduced from principles of Lagrange):

$$\frac{F(x)}{\phi(x)} = \frac{F'(\theta x)}{\phi'(\theta x)}, \quad \text{if} \quad F(0) = \phi(0) = 0;$$

θ denoting *some proper fraction*, or quantity between 0 and 1. And a *geometrical illustration*, which is also a *geometrical proof*, when the *functions* Fx and ϕx can be *constructed* (or conceived to be constructed) as the *ordinates of two plane curves*, is sometimes derived from the *axiom* (or *geometrical intuition*), that the *chord* of any finite and *plane arc* must be *parallel to the tangent*, drawn *at some point* of that *finite arc*. But this *parallelism* no longer *exists*, in general, when the *curve* is one of *double curvature*; and accordingly the *equation* in this note is *not generally true*, when the *functions* are *quaternions*; or even when *one* of them is a quaternion, or a *vector*.

And so we might proceed, as long as *successive derivatives, of higher orders*, continue to vanish together.

(9.) Hence, in particular, if we take this *scalar form*,

$$\text{XVI.} \ldots \phi x = \frac{x^m}{2.3\ldots m},$$

which evidently gives the values,

$$\text{XVII.} \ldots \phi 0 = 0, \quad \phi' 0 = 0, \quad \phi'' 0 = 0, \ldots \phi^{(m-1)} 0 = 0, \quad \phi^{(m)} 0 = 1,$$

and if we suppose that the function Fx is such that

$$\text{XVIII.} \ldots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots F^{(m-1)}0 = 0,$$

while $F^{(m)}0$ has any finite value, we may then establish this limiting equation:

$$\text{XIX.} \ldots \lim. (Fx : \phi x) = F^{(m)}0;$$

in which the *function* Fx, and the *value* $F^{(m)}0$, are *here* supposed to be *generally quaternions*; although they may *happen*, in particular cases, to *reduce* themselves (292) to *vectors*, or to *scalars*.

342. It will now be easy to *extend* the *Exponential Transformation* 340, VII.; and to show that there is a *sense* in which that very important *Formula*,

$$\text{I.} \ldots \epsilon^{\mathrm{d}} fq = f(q + \mathrm{d}q), \quad \text{if} \quad \mathrm{d}^2 q = 0,$$

which is, in fact, a known* mode of expressing the *Series* or *Theorem* of *Taylor*, holds good for *Quaternion Functions generally*, and not merely for those functions of *finite* and *polynomial form*, with *positive* and *whole exponents*, for which it was lately deduced, in 340, (6.). For let fq and $f(q + \mathrm{d}q)$ denote *any two states*, or *values*, of which *neither* is *infinite*, of *any function of a quaternion*; and of the m first differentials,

$$\text{II.} \ldots \mathrm{d}fq, \quad \mathrm{d}^2 fq, \ldots \quad \mathrm{d}^m fq, \quad \text{in which} \quad \mathrm{d}q = \text{const.},$$

let it be supposed that *no one* is *infinite*, and that the *last* of them is different from *zero*; while all that precede it, and the functions fq and $f(q + \mathrm{d}q)$ themselves, may or may not happen to vanish. Let the first m *terms*, of the

* Lacroix, for instance, in page 168 of the First Volume of his larger Treatise on the Differential and Integral Calculus (Paris, 1810), presents the Theorem of Taylor under the form,

$$u' = u + \frac{\mathrm{d}u}{1} + \frac{\mathrm{d}^2 u}{1.2} + \frac{\mathrm{d}^3 u}{1.2.3} + \frac{\mathrm{d}^4 u}{1.2.3.4} + \&c.;$$

where u' denotes the value which the function u receives, when the variable x receives the *arbitrary increment* $\mathrm{d}x$ (l'accroissement quelconque $\mathrm{d}x$).

exponential development of the *symbol* $(\epsilon^{\mathrm{d}} - 1)fq$, be denoted briefly by $q_1, q_2, \ldots q_m$; and let r_m denote what may be called the *remainder of the series*, or the *correction* which must be conceived to be added to the *sum* of these m terms, in order to produce the *exact value* of the *difference*,

$$\text{III.} \ldots \Delta fq = f(q + \Delta q) - fq = f(q + \mathrm{d}q) - fq;$$

in such a manner that we shall have *rigorously*, by the *notations* employed, the equation,

$$\text{IV.} \ldots f(q + \mathrm{d}q) = fq + q_1 + q_2 + \ldots + q_m + r_m, \quad \text{where} \quad q_m = \frac{\mathrm{d}^m fq}{2.3\ldots m};$$

this term q_m being different from *zero*, but *no one* of the terms being *infinite*, by what has been above supposed. Then we shall prove, *as a Theorem*, that

$$\text{V.} \ldots \lim. (\mathrm{T}r_m : \mathrm{T}q_m) = 0, \quad \text{if} \quad \lim. \mathrm{Td}q = 0;$$

or in words, that *the tensor of the remainder may be made to bear as small a ratio as we please, to the tensor of the last term retained, by diminishing the tensor, without changing the versor, of the differential* (or *difference*) $\mathrm{d}q$. And *this* very general result, which will soon be seen to extend to functions of *several* quaternions, is in the present Calculus that *analogue* of Taylor's theorem to which we lately alluded (in 340, (7.)); and it may be called, for the sake of reference, "*Taylor's Theorem adapted to Quaternions.*"

(1.) Writing

$$\text{VI.} \ldots Fx = f(q + x\mathrm{d}q) - fq - x\mathrm{d}fq - \frac{x^2}{2}\mathrm{d}^2fq - \ldots - \frac{x^{m-1}}{2.3\ldots(m-1)}\mathrm{d}^{m-1}fq,$$

we shall have the following successive derivatives with respect to x,

$$\text{VII.} \ldots \begin{cases} F'x = \mathrm{d}f(q + x\mathrm{d}q) - \mathrm{d}fq - x\mathrm{d}^2fq - \ldots - \dfrac{x^{m-2}}{2.3\ldots(m-2)}\mathrm{d}^{m-1}fq; \\ F''x = \mathrm{d}^2f(q + x\mathrm{d}q) - \mathrm{d}^2fq - \ldots - \dfrac{x^{m-3}}{2.3\ldots(m-3)}\mathrm{d}^{m-1}fq; \ldots \\ F^{(m-1)}x = \mathrm{d}^{m-1}f(q + x\mathrm{d}q) - \mathrm{d}^{m-1}fq; \text{ and finally,} \\ F^{(m)}x = \mathrm{d}^m f(q + x\mathrm{d}q); \end{cases}$$

because, by 327, VI., and 324, IV.,

$$\text{VIII.} \ldots \mathrm{D}f(q + x\mathrm{d}q) = \lim_{n=\infty} . n\{f(q + x\mathrm{d}q + n^{-1}\mathrm{d}q) - f(q + x\mathrm{d}q)\} = \mathrm{d}f(q + x\mathrm{d}q),$$

and in like manner,

$$\text{IX.} \ldots \mathrm{D}^2f(q + x\mathrm{d}q) = \mathrm{d}^2f(q + x\mathrm{d}q), \text{ \&c.};$$

the mark of derivation D referring to the scalar variable x, while d operates on q alone, and not here on x, nor on dq.

(2.) We have therefore, by VI. and VII., the values,

$$\text{X.} \ldots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots F^{(m-1)}0 = 0, \quad F^{(m)}0 = \mathrm{d}^m fq\,;$$

whence, by 341, XIX., we have this limiting equation,

$$\text{XI.} \ldots \lim_{x=0} \left(Fx : \frac{x^m}{2.3\ldots m} \right) = \mathrm{d}^m fq\,;$$

or

$$\text{XII.} \ldots \lim_{x=0} (Fx : \psi x) = 1, \quad \text{if} \quad \psi x = \left(\frac{x^m \mathrm{d}^m fq}{2.3\ldots m} \right).$$

(3.) But these two functions, Fx and ψx, are formed by IV. from $q_m + r_m$ and q_m, by changing dq to xdq; and instead of thus *multiplying* dq by a *decreasing scalar*, x, we may *diminish* its *tensor* Tdq, without changing its *versor* Udq. We may therefore say that, when this is done, the *quotient* $(q_m + r_m) : q_m$ *tends* to *unity*, or this other quotient $r_m : q_m$ to *zero*, as its *limit*; or in other words, the *limiting equation* V. holds good.

(4.) As an *example*, let the *function* fq be the *reciprocal*, q^{-1}; then (comp. 339, III.) its m^{th} differential is (for dq = const.),

$$\text{XIII.} \ldots \mathrm{d}^m fq = \mathrm{d}^m . q^{-1} = 2 . 3 \ldots m . q^{-1} (-r)^m, \quad \text{if} \quad r = \mathrm{d}q . q^{-1}\,;$$

and it is easy to prove, *without differentials*, that

$$\text{XIV.} \ldots (q + rq)^{-1} = q^{-1} (1 + r)^{-1} = q^{-1} \{1 - r + r^2 - \ldots + (-r)^m + (-r)^{m+1} (1 + r)^{-1}\}\,;$$

we have therefore here

$$\text{XV.} \ldots q_m = q^{-1} (-r)^m, \quad r_m = -q_m r (1 + r)^{-1}, \quad \mathrm{T}(r_m : q_m) = \mathrm{T}r . \mathrm{T}(1 + r)^{-1}\,;$$

and this last tensor indefinitely diminishes with Tdq, the quaternion q being supposed to have some given value different from zero.

(5.) In general, if we establish the following equation,

$$\text{XVI.} \ldots f(q + n^{-1}\mathrm{d}q) = fq + n^{-1}\mathrm{d}fq + \frac{n^{-2}}{2} \mathrm{d}^2 fq + \ldots + \frac{n^{1-m}}{2 . 3 \ldots (m-1)} \mathrm{d}^{m-1} fq$$
$$+ \frac{n^{-m}}{2 . 3 \ldots m} f_n^{(m)} (q, \mathrm{d}q),$$

as a *definitional extension* of the equation 325, V.; and if we suppose that neither the function fq itself, nor any one of its differentials as far as $\mathrm{d}^{m-1} fq$ is infinite; the result contained in the limiting equation XI. may then be expressed by the formula,

$$\text{XVII.} \ldots f_\infty^{(m)} (q, \mathrm{d}q) = \mathrm{d}^m fq\,;$$

which for the particular value $m = 1$, if we suppress the upper index, coincides with the form 325, VIII. of the definition $\mathrm{d}fx$, but for *higher* values of m contains a *theorem*: namely (when $\mathrm{d}^m fq$ is supposed *neither* to *vanish*, *nor* to become *infinite*), what we have called *Taylor's Theorem adapted to Quaternions.*

343. That very important theorem may be applied to cases, in which a *quaternion* (as in 327, (5.)), or a *vector* (as in 337), is expressed as a *function of a scalar*; also to *transcendental forms* (333), whenever the differentiations can be effected; and to those *new forms* (334), which result from the *peculiar operations* of the present Calculus itself. A few such applications may here be given.

(1.) Taking first this transcendental and quaternion function of a variable scalar,

$$\text{I.} \ldots q = ft = a^t, \quad \text{with} \quad \mathrm{T}a = 1, \quad \mathrm{d}a = 0, \quad \mathrm{d}t = \text{const.},$$

we have, by 333, VIII., the general term,

$$\text{II.} \ldots q_m = \frac{\mathrm{d}^m \,.\, a^t}{2\,.\,3\,.\,.\,m} = \frac{a^t}{2\,.\,3\,.\,.\,m}\left(\frac{\pi a \mathrm{d}t}{2}\right)^m = \frac{a^t\,(xa)^m}{2\,.\,3\,.\,.\,m}, \quad \text{if} \quad 2x = \pi \mathrm{d}t;$$

dividing then $\varepsilon^{\mathrm{d}} \,.\, a^t$ by a^t, we obtain an *infinite series*, which is found to be *correct*, and *convergent*; namely (comp. 308, (4.)),

$$\text{III.} \ldots a^{\mathrm{d}t} = 1 + xa + \frac{(xa)^2}{2} + \,.\,.+ \frac{(xa)^m}{2\,.\,3\,.\,.\,m} + \,.\,. = \varepsilon^{xa} = \cos\frac{\pi \mathrm{d}t}{2} + a \sin\frac{\pi \mathrm{d}t}{2}.$$

(2.) Correct and *finite expansions*, for $\mathrm{S}\,(q + \mathrm{d}q)$, $\mathrm{V}\,(q + \mathrm{d}q)$, $\mathrm{K}\,(q + \mathrm{d}q)$, and $\mathrm{N}\,(q + \mathrm{d}q)$, are obtained when we operate with ε^{d} on $\mathrm{S}q$, $\mathrm{V}q$, $\mathrm{K}q$, and $\mathrm{N}q$; for example ($\mathrm{d}q$ being still constant), the third and higher differentials of $\mathrm{N}q$ vanish by 334, XI., and we have

$$\text{IV.} \ldots \varepsilon^{\mathrm{d}}\mathrm{N}q = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2)\,\mathrm{N}q = \mathrm{N}q + 2\mathrm{S}\,(\mathrm{K}q \,.\, \mathrm{d}q) + \mathrm{N}\mathrm{d}q = \mathrm{N}\,(q + \mathrm{d}q);$$

an expression for the *norm of a sum*, which agrees with 210, XX., and with 200, VII.

(3.) To develop, on like principles, the *tensor and versor of a sum*, let us again write r for $\mathrm{d}q : q$, and denote the scalar and vector parts of this quotient by s and v; so that, by 334, XIII. and XV.,

$$\text{V.} \ldots s = \mathrm{S}r = \mathrm{S}\frac{\mathrm{d}q}{q} = \frac{\mathrm{d}\mathrm{T}q}{\mathrm{T}q}; \qquad \text{VI.} \ldots v = \mathrm{V}r = \mathrm{V}\frac{\mathrm{d}q}{q} = \frac{\mathrm{d}\mathrm{U}q}{\mathrm{U}q}.$$

(4.) Then writing also, for abridgment, as in a known notation of *factorials*,

$$\text{VII.} \ldots [-1]^{m} = (-1)\,.\,(-2)\,.\,(-3)\,.\,.\,.\,.\,(-m),$$

we shall have, by 342, XIII., dq being still treated as constant, the equation,

$$\text{VIII.}\ldots \mathrm{d}^m(s+v) = \mathrm{d}^m r = [-\overset{m}{1}]\, r^{m+1} = [-\overset{m}{1}]\, (s+v)^{m+1},$$

of which it is easy to separate the scalar and vector parts; for example,

$$\text{IX.}\ldots \mathrm{d}s = -\mathrm{S}\,.\,(s+v)^2 = -(s^2+v^2); \quad \mathrm{d}v = -\mathrm{V}\,.\,(s+v)^2 = -2sv.$$

(5.) We have also, by V. and VI.,

$$\text{X.}\ldots \frac{\mathrm{d}^m \mathrm{T}q}{\mathrm{T}q} = (s+\mathrm{d})\,\frac{\mathrm{d}^{m-1}\mathrm{T}q}{\mathrm{T}q} = \ldots = (s+\mathrm{d})^m 1;$$

$$\text{XI.}\ldots \frac{\mathrm{d}^m \mathrm{U}q}{\mathrm{U}q} = (v+\mathrm{d})\,\frac{\mathrm{d}^{m-1}\mathrm{U}q}{\mathrm{U}q} = \ldots = (v+\mathrm{d})^m 1;$$

the notation being such that we have, for instance, by IX.,

$$\text{XII.}\ldots (s+\mathrm{d})\,1 = s; \quad (s+\mathrm{d})^2\,1 = (s+\mathrm{d})\,s = s^2 + \mathrm{d}s = -v^2;$$

$$\text{XIII.}\ldots (v+\mathrm{d})\,1 = v; \quad (v+\mathrm{d})^2\,1 = (v+\mathrm{d})\,v = v^2 + \mathrm{d}v = v^2 - 2sv.$$

(6.) The *exponential formula* 342, I., gives, therefore,

$$\text{XIV.}\ldots \mathrm{T}(q+\mathrm{d}q) = \varepsilon^{\mathrm{d}}\mathrm{T}q = \varepsilon^{s+\mathrm{d}}1\,.\,\mathrm{T}q;$$

$$\text{XV.}\ldots \mathrm{U}(q+\mathrm{d}q) = \varepsilon^{\mathrm{d}}\mathrm{U}q = \varepsilon^{v+\mathrm{d}}1\,.\,\mathrm{U}q;$$

or, dividing and substituting,

$$\text{XVI.}\ldots \mathrm{T}(1+s+v) = \varepsilon^{s+\mathrm{d}}1; \quad \text{XVII.}\ldots \mathrm{U}(1+s+v) = \varepsilon^{v+\mathrm{d}}1;$$

s and v being here a *scalar* and a *vector*, which are entirely *independent* of each other; but of which, in the applications, the *tensors* must not be taken *too large*, in order that the *series* may *converge*.

(7.) The *symbolical expressions*, XVI. and XVII., for those two *series*, may be *developed* by (4.) and (5.); thus, if we only write down the terms which do not exceed the *second dimension*, with respect to s and v, we have by XII. and XIII. the development,

$$\text{XVIII.}\ldots \mathrm{T}(1+s+v) = 1 + s - \tfrac{1}{2}v^2 + \ldots,$$

$$\text{XIX.}\ldots \mathrm{U}(1+s+v) = 1 + v + (\tfrac{1}{2}v^2 - sv) + \ldots;$$

of which accordingly the *product* is $1 + s + v$, to the same order of approximation.

(8.) A *function of a sum* of two quaternions can sometimes be developed, *without differentials*, by processes of a more *algebraical character*; and when this happens, we may *compare* the result with the *form* given by *Taylor's Series*, as adapted to quaternions in 342, and so *deduce* the *values* of the *successive differentials* of the function; for example, we can *infer* the expression 342, XIII. for $d^m . q^{-1}$, from the *series* 342, XIV., for the *reciprocal of a sum.*

(9.) And not only may we *verify* the recent developments, XVIII. and XIX., by comparing them with the more *algebraical forms*,*

$$\text{XX.} \ldots \mathrm{T}(1+s+v) = (1+s+v)^{\frac{1}{2}}(1+s-v)^{\frac{1}{2}},$$

$$\text{XXI.} \ldots \mathrm{U}(1+s+v) = (1+s+v)^{\frac{1}{2}}(1+s-v)^{-\frac{1}{2}},$$

but also, if the first of these, for example (when expanded by *ordinary processes*, which are in *this* case applicable), have given us, *without* differentials,

$$\text{XXII.} \ldots \mathrm{T}(q+q') = (1+s-\tfrac{1}{2}v^2 \ldots)\mathrm{T}q, \text{ where } s = \mathrm{S}q'q^{-1}, \text{ and } v = \mathrm{V}q'q^{-1},$$

we can then *infer* the values of the *first* and *second differentials* of the *tensor* of a quaternion, as follows:

$$\text{XXIII.} \ldots \mathrm{dT}q = \mathrm{S}\frac{\mathrm{d}q}{q}.\mathrm{T}q; \quad \mathrm{d}^2\mathrm{T}q = -\left(\mathrm{V}\frac{\mathrm{d}q}{q}\right)^2\mathrm{T}q;$$

whereof the first agrees with 334, XII. or XIII., and the second can be deduced from it, under the form,

$$\text{XXIV.} \ldots \mathrm{d}^2\mathrm{T}q = \mathrm{d}\left(\mathrm{S}\frac{\mathrm{d}q}{q}.\mathrm{T}q\right) = \left(\left(\mathrm{S}\frac{\mathrm{d}q}{q}\right)^2 - \mathrm{S}.\left(\frac{\mathrm{d}q}{q}\right)^2\right)\mathrm{T}q.$$

(10.) In general, if we can only develop a function $f(q+q')$ as far as the term or terms which are of the *first dimension* relatively to q', we shall still obtain thus an expression for the *first differential* $\mathrm{d}fq$, by merely writing $\mathrm{d}q$ in the place of q'. But we have not chosen (comp. 100, (14.)) to regard *this property of the differential of a function* as the *fundamental one*, or to adopt it as the *definition* of $\mathrm{d}fq$; because we have not chosen to *postulate* the *general possibility* of such *developments of functions of quaternion sums*, of which in fact it is in many cases *difficult to discover the laws*, or even to *prove the existence*, except in some such way as that above explained.

(11.) This opportunity may be taken to observe, that (with recent notations) we have, by VIII., the symbolical expression,

$$\text{XXV.} \ldots \varepsilon^{s+v+\mathrm{d}}1 = 1+s+v; \quad \text{or} \quad \text{XXVI.} \ldots \varepsilon^{r+\mathrm{d}}1 = 1+r.\dagger$$

* [These are equivalent to the transformations

$$\sqrt{q}\sqrt{\mathrm{K}q} = \frac{\mathrm{T}q}{\sqrt{\mathrm{K}q}}\sqrt{\mathrm{K}q} = \mathrm{T}q \quad \text{and} \quad \frac{\sqrt{q}}{\sqrt{\mathrm{K}q}} = \frac{q}{\mathrm{T}q} = \mathrm{U}q.]$$

† [In fact by VIII., $\mathrm{d}r = -r^2$ and $(r+\mathrm{d})^2.1 = (r+\mathrm{d}).r = r^2 - r^2 = 0.$]

344. *Successive differentials* are also connected with *successive differences*, by laws which it is easy to investigate, and on which only a few words need here be said.

(1.) We can easily prove, from the definition 324, IV. of $\mathrm{d}fq$, that if $\mathrm{d}q$ be constant,

$$\text{I.} \ldots \mathrm{d}^2 fq = \lim_{n=\infty} n^2 \{f(q + 2n^{-1}\,\mathrm{d}q) - 2f(q + n^{-1}\,\mathrm{d}q) + fq\};$$

with analogous expressions for differentials of higher orders.

(2.) Hence we may say (comp. 340, X.) that the *successive differentials*,

$$\text{II.} \ldots \mathrm{d}fq, \quad \mathrm{d}^2fq, \quad \mathrm{d}^3fq, \ldots \quad \text{for} \quad \mathrm{d}^2q = 0,$$

are *limits* to which the following *multiples of successive differences*,

$$\text{III.} \ldots n\Delta fq, \quad n^2\Delta^2 fq, \quad n^3\Delta^3 fq, \ldots \quad \text{for} \quad \Delta^2 q = 0,$$

all simultaneously *tend*, when the multiple $n\Delta q$ is either constantly *equal* to $\mathrm{d}q$, or at least *tends* to become equal thereto, while the number n increases indefinitely.

(3.) And hence we might prove, in a new way, that *if the function* $f(q + \mathrm{d}q)$ *can be developed*, in a *series* proceeding according to *ascending* and *whole dimensions* with respect to $\mathrm{d}q$, the *parts* of this series, which are of those *successive* dimensions, must follow the *law* expressed by *Taylor's Theorem* adapted to Quaternions* (342).

345. It is easy to conceive that the foregoing results may be extended (comp. 338), to the successive differentiations of functions of *several quaternions*; and that thus there arises, in each such case, a *system of successive differentials, total and partial*: as also a system of *partial derivatives*, of orders higher than the first, when a *quaternion*, or a *vector*, is regarded (comp. 337) as a function of *several scalars*.

(1.) The *general* expression for the *second total differential*,

$$\text{I.} \ldots \mathrm{d}^2 Q = \mathrm{d}^2 F(q, r, \ldots),$$

involves d^2q, $\mathrm{d}^2r, \ldots$; but it is often convenient to suppose that all *these* second differentials *vanish*, or that the *first* differentials $\mathrm{d}q$, $\mathrm{d}r, \ldots$ are *constant*; and *then* $\mathrm{d}^m Q$, or $\mathrm{d}^m F(q, r, \ldots)$, becomes a rational, integral, and homogeneous function of the m^{th} dimension, of those first differentials $\mathrm{d}q$, $\mathrm{d}r, \ldots$, which may (comp. 329, III.) be thus denoted,

$$\text{II.} \ldots \mathrm{d}^m Q = (\mathrm{d}_q + \mathrm{d}_r + \ldots)^m Q; \quad \text{or briefly,} \quad \text{III.} \ldots \mathrm{d}^m = (\mathrm{d}_q + \mathrm{d}_r + \ldots)^m,$$

* Some remarks on the adaptation and proof of this important theorem will be found in the *Lectures*, pages 589, &c.

in developing which *symbolical power*, the *multinomial theorem* of *algebra* may be employed: because we have generally, for quaternions as in the ordinary calculus,

$$\text{IV.} \ldots \mathrm{d}_r \mathrm{d}_q = \mathrm{d}_q \mathrm{d}_r.$$

(2.) For example, if we denote dq and dr by q' and r', and suppose

$$\text{V.} \ldots Q = rqr, \quad \text{then} \quad \text{VI.} \ldots \mathrm{d}_q Q = rq'r; \quad \text{VII.} \ldots \mathrm{d}_r Q = r'qr + rqr';$$

and $$\text{VIII.} \ldots \mathrm{d}_r \mathrm{d}_q Q = \mathrm{d}_q \mathrm{d}_r Q = r'q'r + rq'r'.$$

And in general, *each* of the two equated symbols IV. gives, by its operation on $F(q, r)$, the *limit* of this other function, or product (comp. 344, I.),

$$\text{IX.} \ldots nn'\{F(q + n^{-1}\mathrm{d}q,\ r + n'^{-1}\mathrm{d}r) - F(q,\ r + n'^{-1}\mathrm{d}r) - F(q + n^{-1}\,\mathrm{d}q,\ r) + F(q, r)\};$$

in which the numbers n and n' are supposed to tend to infinity.

(3.) We may also write, for functions of *several* quaternions,

$$\text{X.} \ldots Q + \Delta Q = F(q + \mathrm{d}q,\ r + \mathrm{d}r, \ldots) = \varepsilon^{\mathrm{d}_q + \mathrm{d}_r + \cdots} F(q, r);$$

or briefly, $$\text{XI.} \ldots 1 + \Delta = \varepsilon^{\mathrm{d}_q + \mathrm{d}_r + \cdots} = \varepsilon^{\mathrm{d}};$$

with *interpretations* and *transformations* analogous to those which have occurred already, for functions of a *single* quaternion.

(4.) Finally, as an example of *successive* and *partial derivation*, if we resume the vector expression 308, XVIII. (comp. 315, XII. and XIII.), namely,

$$\text{XII.} \ldots \rho = rk^t j^s k j^{-s} k^{-t},$$

which has been seen to be capable of representing the vector of *any point of space*, we may observe that it gives, *without trigonometry*, by the principle mentioned in 308, (11.), and by the sub-articles to 315, not only the form

$$\text{XIII.} \ldots \rho = rk^t j^{2s} k^{1-t}, \text{ as in 308, XIX.},$$

but also, if α be *any vector unit*,

$$\text{XIV.} \ldots \rho = rk^{t+1} j^{-2s} k^{-t} = rk^t (k\mathrm{S} \,.\, \alpha^{2s} + i\mathrm{S} \,.\, \alpha^{2s-1}) \,.\, k^{-t};$$

whence

$$\text{XV.} \ldots \rho = r\mathrm{V} \,.\, k^{2s+1} + rk^{2t}\mathrm{V} \,.\, i^{2s}, \text{ as in 315, XII.}$$

(5.) We have therefore the following new expressions (compare the sub-articles to 337), for the two *partial derivatives* of the *first order*, of this variable vector ρ, taken with respect to s and t:

$$\text{XVI.} \ldots \mathrm{D}_s \rho = \pi rk^t j^s ij^{-s} k^{-t} = -\pi \rho k^t jk^{-t},$$

with the verification, that

$$\text{XVII.} \ldots \rho \mathrm{D}_s \rho = \pi r^2 \,.\, k^t j^s kj^{-s} k^{-t} \,.\, k^t j^s ij^{-s} k^{-t} = \pi r^2 k^t jk^{-t}; \quad \text{and}$$

$$\text{XVIII.} \ldots \mathrm{D}_t\rho = \pi r k^{2t}\mathrm{V}\,.\,j^{2s} = \pi r k^{2t} j\mathrm{S}\,.\,\alpha^{2s-1} = r^{-1}\rho\mathrm{D}_s\rho\,.\,\mathrm{S}\,.\,\alpha^{2s-1}$$

whence

$$\text{XIX.} \ldots \rho\mathrm{D}_t\rho = -r\mathrm{D}_s\rho\,.\,\mathrm{S}\,.\,\alpha^{2s-1}, \quad \text{and} \quad \text{XX.} \ldots \mathrm{D}_s\rho\,.\,\mathrm{D}_t\rho = \pi^2 r\rho\mathrm{S}\,.\,\alpha^{2s-1};$$

while

$$\text{XXI.} \ldots \mathrm{D}_r\rho = r^{-1}\rho = k^t j^s k j^{-s} k^{-t}, \text{ as in 337, XXV.};$$

so that we have the following *ternary product* of these *derived vectors* of the first order,

$$\text{XXII.} \ldots \mathrm{D}_r\rho\,.\,\mathrm{D}_s\rho\,.\,\mathrm{D}_t\rho = \pi^2\rho^2\mathrm{S}\,.\,\alpha^{2s-1} = \pi r^2\mathrm{D}_s\mathrm{S}\,.\,\alpha^{2s};$$

the *scalar character* of which product depends (comp. 299, (9.)) on the circumstance, that the vectors thus multiplied compose (337, (10.)) a *rectangular system*.

(6.) It is easy then to infer, for the *six partial derivatives* of ρ, of the *second order*, taken with respect to the same three scalar variables, r, s, t, the expressions:

$$\text{XXIII.} \ldots \mathrm{D}_r^2\rho = 0; \quad \mathrm{D}_r\mathrm{D}_s\rho = \mathrm{D}_s\mathrm{D}_r\rho = r^{-1}\mathrm{D}_s\rho; \quad \mathrm{D}_r\mathrm{D}_t\rho = \mathrm{D}_t\mathrm{D}_r\rho = r^{-1}\mathrm{D}_t\rho;$$

$$\text{XXIV.} \ldots \mathrm{D}_s^2\rho = -\pi^2\rho; \quad \mathrm{D}_s\mathrm{D}_t\rho = \mathrm{D}_t\mathrm{D}_s\rho = \pi^2 r k^{2t}\mathrm{V}\,.\,j^{2s+1}; \quad \mathrm{D}_t^2\rho = -\pi^2 r k^{2t}\mathrm{V}\,.\,i^{2s}.$$

(7.) The three *partial differentials* of the *first order*, of the same variable vector ρ, are the following:

$$\text{XXV.} \ldots \mathrm{d}_r\rho = r^{-1}\rho\mathrm{d}r; \quad \mathrm{d}_s\rho = \mathrm{D}_s\rho\,.\,\mathrm{d}s; \quad \mathrm{d}_t\rho = \mathrm{D}_t\rho\,.\,\mathrm{d}t;$$

with the products,

$$\text{XXVI.} \ldots \mathrm{d}_s\rho\,.\,\mathrm{d}_t\rho = -\pi r\rho\mathrm{dS}\,.\,\alpha^{2s}\,.\,\mathrm{d}t;$$

$$\text{XXVII.} \ldots \mathrm{d}_r\rho\,.\,\mathrm{d}_s\rho\,.\,\mathrm{d}_t\rho = \pi r^2\mathrm{d}r\,.\,\mathrm{dS}\,.\,\alpha^{2s}\,.\,\mathrm{d}t.$$

(8.) These *differential vectors*, $\mathrm{d}_r\rho$, $\mathrm{d}_s\rho$, $\mathrm{d}_t\rho$, are (in the present theory) *generally finite*; $\mathrm{d}_r\rho$, like $\mathrm{D}_r\rho$, being a line in the direction of ρ, or of the *radius* of this *sphere* round the origin, at least if $\mathrm{d}r$, like r, be positive; while $\mathrm{d}_s\rho$, like $\mathrm{D}_s\rho$, is (comp. 100, (9.)) a *tangent to the meridian* of that spheric surface, for which r and t are *constant*; but $\mathrm{d}_t\rho$, like $\mathrm{D}_t\rho$, is on the contrary a *tangent to the small circle* (or *parallel*), on the same sphere, for which r and s are constant.

(9.) Treating only the *radius* r as constant, and writing $\rho = \mathrm{OP}$, if we pass from the point P, or (s, t), to *another point* Q, or $(s + \Delta s, t)$, on the *same meridian*, the *chord* PQ is represented by the *finite difference*, $\Delta_s\rho$; and in like manner, if we pass from P to a point R, or $(s, t + \Delta t)$, on the *same parallel*, the new chord PR is represented by the *other* partial and finite difference, $\Delta_t \rho$; while the point $(s + \Delta s, t + \Delta t)$ may be denoted by S.

(10.) If now the *two points* Q and R be conceived to *approach to* P, and to come to be *very near it*, the *chords* PQ and PR will *very nearly coincide* with the two corresponding *arcs* of meridian and parallel; or with the *tangents* to the same two circles at P, so drawn as to have the *lengths* of those two arcs: or finally with the *differential* and *tangential vectors*, $d_s\rho$ and $d_t\rho$, if we suppose (as we may, comp. 322) that the two *arbitrary* and *scalar differentials*, ds and dt, are so *assumed* as to be constantly *equal* to the two *differences*, Δs and Δt, and consequently to *diminish with them*.

(11.) Whether the differentials ds and dt be *large* or *small*, the *product* $d_s\rho \,.\, d_t\rho$, like the product $D_s\rho \,.\, D_t\rho$, represents *rigorously* a *normal vector* (as in XXVI. and XX.); of which the *length* bears to the *unit of length* (comp. 281) the *same ratio*, as that which the *rectangle* under the *two perpendicular tangents*, $d_s\rho$ and $d_t\rho$, to the *sphere*, bears to the *unit of area*. Hence, with the recent suppositions (10.), we may regard this *product* $d_s\rho \,.\, d_t\rho$ as representing, with a continually and indefinitely increasing accuracy, even in the way of ratio, what we may call the *directed element of spheric surface*, PQRS, considered as thus *represented* (or constructed) by a *normal* at P; and the *tensor* of the same product, namely (by XXVI.),

$$\text{XXVIII.} \ldots \mathrm{T}(d_s\rho \,.\, d_t\rho) = -\pi r^2 d\mathrm{S} \,.\, a^{2s} \,.\, dt,$$

in which the *negative sign* is retained, because $\mathrm{S} \,.\, a^{2s}$ decreases from $+1$ to -1, while s increases from 0 to 1, is an expression on the same plan for what we may call by contrast the *undirected element of spheric area*, or that element considered with reference merely to *quantity*, and *not* with reference to *direction*.

(12.) *Integrating*, then, this last *differential expression* XXVIII., from $t = 0$ to $t = 2$, and from $s = s_0$ to $s = s_1$, that is, taking the *limit of the sum* of all the *elements* PQRS between these bounding values, we find the following equation:

$$\text{XXIX.} \ldots \textit{Area of Spheric Zone} = 2\pi r^2 \mathrm{S}(a^{2s_0} - a^{2s_1});$$

whence

$$\text{XXX.} \ldots \textit{Area of Spheric Cap } (s) = 2\pi r^2(1 - \mathrm{S} \,.\, a^{2s}) = 4\pi r^2(\mathrm{TV} \,.\, a^s)^2;$$

and finally,

$$\text{XXXI.} \ldots \textit{Area of Sphere} = 4\pi r^2, \text{ as usual.}$$

(13.) In like manner the expression XXVII., with its sign changed (on account of the decrease of $S \,.\, a^{2s}$, as in (11.)), represents the *element of volume*; and thus, by integrating from $r = r_0$ to $r = r_1$, from $s = 0$ to $s = 1$, and from $t = 0$ to $t = 2$, we obtain anew the known values:

$$\text{XXXII.}\ldots \textit{Volume of Spheric Shell} = \frac{4\pi}{3}(r_1^3 - r_0^3);$$

and

$$\text{XXXIII.}\ldots \textit{Volume of Sphere } (r) = \frac{4\pi r^3}{3}, \text{ as usual.}$$

(14.) *These* are however only specimens of what may be called *Scalar Integration*, although connected with *quaternion forms*; and it will be more *characteristic* of the present Calculus, if we apply it briefly to take the *Vector Integral*, or the *limit of the vector-sum* of the *directed elements* (11.) of a portion of a spheric surface: a problem which corresponds, in *hydrostatics*, to calculating the *resultant of the pressures* on that surface, each pressure having a *normal direction*, and a *quantity* proportional to the *element of area*.

(15.) For this purpose, we may employ the expression XXVI. with its sign changed, in order to denote an *inward normal*, or a *pressure* acting *from without*; and if we then substitute for ρ its value XV., and observe that

$$\text{XXXIV.}\ldots \int_0^2 k^{2t} \mathrm{d}t = 0, \quad \text{because} \quad k^2 = -1,*$$

and remember that $\mathrm{V}\,.\,k^{2s+1} = k\mathrm{S}\,.\,a^{2s}$, we easily deduce the expressions:

$$\text{XXXV.}\ldots \textit{Sum of Directed Elements of Elementary Zone} = \pi r^2 k \mathrm{d}\,.\,(\mathrm{S}\,.\,a^{2s})^2;$$

$$\text{XXXVI.}\ldots \textit{Sum of Directed Elements of Spheric Cap } (s) = -\pi r^2 k(1 - (\mathrm{S}\,.\,a^{2s})^2)$$
$$= \pi r^2 k(\mathrm{V}\,.\,a^{2s})^2 = \pi^{-1} k(\mathrm{D}_t\rho)^2 = \pi k(\mathrm{V}k\rho)^2.$$

(16.) But the *radius* of the *plane* and *circular base*, of the *spheric segment* corresponding, is $\mathrm{TV}k\rho$, so that its *area* is in *quantity* $= -\pi(\mathrm{V}k\rho)^2$; and the *common direction* of all its *inward normals* is that of $+k$; hence, if we still represent the directed *elements* by normals thus drawn *inwards*, we have this new expression:

$$\text{XXXVII.}\ldots \textit{Sum of Directed Elements of Circular Base} = -\pi k(\mathrm{V}k\rho)^2;$$

comparing which with XXXVI., we arrive at the formula,

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* [Since $\mathrm{d}k^{2t-1} = \frac{\pi}{2}k^{2t}\mathrm{d}(2t-1)$ by 333 (5.), the integral $\int_0^t k^{2t}\mathrm{d}t = \frac{1}{\pi}(k^{2t-1} - k^{-1})$.]

XXXVIII. . . *Sum of Directed Elements of Spheric Segment* = *Zero*;

a result which may be greatly extended, and which evidently answers to a known case of *equilibrium* in *hydrostatics*.

(17.) These few *examples* may serve to show already, that *Differentials of Quaternions* (or of *Vectors*) may be *applied* to various *geometrical* and *physical questions*; and that, when so applied, it is *permitted* to treat them as *small*, if any *convenience* be gained thereby, as in cases of *integration* there always is. But we must now pass to an important investigation of another kind, with which *differentials* will be found to have only a sort of *indirect* or *suggestive* connexion.

SECTION 6.

On the Differentiation of Implicit Functions of Quaternions; and on the General Inversion of a Linear Function, of a Vector or a Quaternion: with some connected Investigations.

346. We saw, when differentiating the square-root of a quaternion (332, (5.) and (6.)), that it was necessary for that purpose to *resolve* a *linear equation*,* or an equation of the *first degree*; namely the equation,

$$\text{I. . . } rr' + r'r = q',$$

in which r and q' represented two given quaternions, $q^{\frac{1}{2}}$ and $\mathrm{d}q$, while r' represented a sought quaternion, namely $\mathrm{d}r$ or $\mathrm{d}\,.\,q^{\frac{1}{2}}$. And generally, from the *linear* or *distributive form* (327), of the *quaternion differential*

$$\text{II. . . } \mathrm{d}Q = \mathrm{d}fq = f(q, \mathrm{d}q),$$

of any given and *explicit function* fq, when considered as depending on the differential $\mathrm{d}q$ of the quaternion variable q, we see that the *return* from the former differential to the latter, that is from $\mathrm{d}Q$ to $\mathrm{d}q$, or the *differentiation* of the *inverse* or *implicit* function $f^{-1}Q$, requires for its accomplishment the *Solution of an Equation of the First Degree*: or what may be called the *Inversion of a Linear Function of a Quaternion*. We are therefore led to consider here that general Problem; to which accordingly, and to investigations connected with which, we shall devote the present Section, dismissing however now the *special* consideration of the *Differentials* above mentioned, or treating

† Compare the Note to page 452.

them only as *Quaternions*, sought or given, of which the *relations* to each other are to be studied.

347. Whatever the *particular form* of the given *linear* or *distributive function*, fq, may be, we can always *decompose* it as follows:

$$\text{I.} \ldots fq = f(\mathrm{S}q + \mathrm{V}q) = f\mathrm{S}q + f\mathrm{V}q = \mathrm{S}q\,.f1 + f\mathrm{V}q\,;$$

taking then separately scalars and vectors, or operating with S and V on the proposed *linear equation*,

$$\text{II.} \ldots fq = r,$$

where r is a *given* quaternion, and q a *sought* one, we can in general *eliminate* $\mathrm{S}q$, and so reduce the problem to the solution of a *linear* and *vector equation*, of the form,

$$\text{III.} \ldots \phi\rho = \sigma\,;$$

where σ is a *given vector*, but $\rho\,(=\mathrm{V}q)$ is a *sought* one, and ϕ is used as the characteristic of a *given linear and vector function* of a vector, which function we shall throughout suppose to be a *real* one, or to involve *no imaginary constants* in its composition. But, to every *such* function $\phi\rho$, there always *corresponds* what may be called a *conjugate* linear and vector function $\phi'\rho$, connected with it by the following *Equation of Conjugation*,

$$\text{IV.} \ldots \mathrm{S}\lambda\phi\rho = \mathrm{S}\rho\phi'\lambda\,;$$

where λ and ρ are *any two vectors*. Assuming then, as we may, that μ and ν are *two auxiliary* vectors, so chosen as to satisfy the equation,

$$\text{V.} \ldots \mathrm{V}\mu\nu = \sigma,$$

and therefore also,

$$\text{VI.} \ldots \mathrm{S}\lambda\sigma = \mathrm{S}\lambda\mu\nu, \quad \mathrm{S}\mu\sigma = 0, \quad \mathrm{S}\nu\sigma = 0,$$

where λ is a *third* auxiliary and arbitrary vector, we may (comp. 312) replace the *one* vector equation III. by the *three* scalar equations,

$$\text{VII.} \ldots \mathrm{S}\rho\phi'\lambda = \mathrm{S}\lambda\mu\nu, \quad \mathrm{S}\rho\phi'\mu = 0, \quad \mathrm{S}\rho\phi'\nu = 0.$$

And these give, by principles with which the reader is supposed to be already familiar,* the expression,

$$\text{VIII.} \ldots m\rho = \psi\sigma, \quad \text{or} \quad \text{IX.} \ldots \rho = \phi^{-1}\sigma = m^{-1}\psi\sigma,$$

* A student might find it useful, at this stage, to read again the Sixth Section of the preceding Chapter; or at least the early sub-articles to Art. 294, a *familiar* acquaintance with which is presumed in the present Section.

if m be a *scalar-constant*, and ψ an *auxiliary linear and vector function*, of which the *value* and the *form* are determined by the two following equations:

$$\text{X.} \ldots m\mathrm{S}\lambda\mu\nu = \mathrm{S}(\phi'\lambda \,.\, \phi'\mu \,.\, \phi'\nu);$$

$$\text{XI.} \ldots \psi(\mathrm{V}\mu\nu) = \mathrm{V}(\phi'\mu \,.\, \phi'\nu);$$

or briefly,

$$\text{X}'. \ldots m\mathrm{S}\lambda\mu\nu = \mathrm{S} \,.\, \phi'\lambda\phi'\mu\phi'\nu,$$

and

$$\text{XI}'. \ldots \psi\mathrm{V}\mu\nu = \mathrm{V} \,.\, \phi'\mu\phi'\nu.$$

And thus the proposed *Problem of Inversion*, of the linear and vector function ϕ, may be considered to be, in all its generality, *resolved*; because it is always possible so to *prepare* the second members of the equations X. and XI., that they shall take the *forms* indicated in the first members of those equations.*

(1.) For example, if we *assume* any three diplanar vectors α, α', α'', and *deduce* from them three other vectors β_0, β'_0, β''_0, by the equations,

$$\text{XII.} \ldots \beta_0 \mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha'\alpha'', \quad \beta'_0\mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha''\alpha, \quad \beta''_0\mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha\alpha',$$

then *any* vector ρ may, by 294, XV., be expressed as follows:

$$\text{XIII.} \ldots \rho = \beta_0\mathrm{S}\alpha\rho + \beta'_0\mathrm{S}\alpha'\rho + \beta''_0\mathrm{S}\alpha''\rho;$$

if then we write,

$$\text{XIV.} \ldots \beta = \phi\beta_0, \quad \beta' = \phi\beta'_0, \quad \beta'' = \phi\beta''_0,\dagger$$

we shall have the following *General Expression*, or *Standard Trinomial Form*, for a *Linear and Vector Function of a Vector*,

$$\text{XV.} \ldots \phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho + \beta''\mathrm{S}\alpha''\rho;$$

containing, as we see, *three vector constants*, β, β', β'', or *nine scalar constants*, such as

$$\text{XVI.} \ldots \mathrm{S}\alpha\beta,\ \mathrm{S}\alpha'\beta,\ \mathrm{S}\alpha''\beta; \quad \mathrm{S}\alpha\beta',\ \mathrm{S}\alpha'\beta',\ \mathrm{S}\alpha''\beta'; \quad \mathrm{S}\alpha\beta'',\ \mathrm{S}\alpha'\beta'',\ \mathrm{S}\alpha''\beta'';$$

which may (and generally will) all *vary*, in passing from *one* linear and *vector function* $\phi\rho$ to *another* such function; but which are *all* supposed to be *real*, and *given*, for each *particular form* of that function.

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* [For a more elementary solution of the problem of Inversion, see sub-art. (4.).]

† [The equations XIV. lead to a useful expression for a linear vector function in terms of three diplanar vectors β_0, β'_0, and β''_0, and the derived vectors β, β', and β''.]

(2.) Passing to what we have called the *conjugate* linear function $\phi'\rho$, the form XV. gives, by IV., the expression,

$$\text{XVII.} \ldots \phi'\rho = a\text{S}\beta\rho + a'\text{S}\beta'\rho + a''\text{S}\beta''\rho;$$

but

$$\text{V}.(a\text{S}\beta\mu + a'\text{S}\beta'\mu)(a\text{S}\beta\nu + a'\text{S}\beta'\nu) = \text{V}aa'\text{S}.\beta'(\nu\text{S}\beta\mu - \mu\text{S}\beta\nu)$$
$$= \text{V}aa'\text{S}.\beta'\text{V}.\beta\text{V}\mu\nu = \text{V}aa'\text{S}.\beta'\beta\text{V}\mu\nu;$$

therefore the transformation XI. succeeds, and gives,

$$\text{XVIII.} \ldots \psi\rho = \text{V}a'a''\text{S}\beta''\beta'\rho + \text{V}a''a\text{S}\beta\beta''\rho + \text{V}aa'\text{S}\beta'\beta\rho,$$

as an expression for the *auxiliary* function ψ; of which the *conjugate* may be thus written,

$$\text{XIX.} \ldots \psi'\rho = \text{V}\beta'\beta''\text{S}a''a'\rho + \text{V}\beta''\beta\text{S}aa''\rho + \text{V}\beta\beta'\text{S}a'a\rho;$$

so that ψ is changed to ψ', when ϕ is changed to ϕ', by interchanging each of the three *alphas* with the corresponding *beta*.

(3.) If we write, as in this whole investigation we propose to do,

$$\text{XX.} \ldots \lambda' = \text{V}\mu\nu, \quad \mu' = \text{V}\nu\lambda, \quad \nu' = \text{V}\lambda\mu,$$

the formulæ XI. and X. become,

$$\text{XXI.} \ldots \psi\lambda' = \text{V}.\phi'\mu\phi'\nu, \quad \text{and} \quad \text{XXII.} \ldots m\text{S}\lambda\lambda' = \text{S}.\phi'\lambda\psi\lambda',$$

with the same sort of abridgment of notation as in XI'.; and because the coefficient of $\text{S}aa'a''$ in this last expression XXII. is by XVII. XVIII.,

$$\text{S}\beta\lambda\text{S}\beta''\beta'\lambda' + \text{S}\beta'\lambda\text{S}\beta\beta''\lambda' + \text{S}\beta''\lambda\text{S}\beta'\beta\lambda' = \text{S}\beta''\beta'\beta\text{S}\lambda\lambda',$$

the *division* by $\text{S}\lambda\lambda'$, or by $\text{S}\lambda\mu\nu$, succeeds, and we find the expression,

$$\text{XXIII.} \ldots m = \text{S}aa'a''\text{S}\beta''\beta'\beta;$$

which may also be thus written,

$$\text{XXIII'.} \ldots m = \text{S}\beta\beta'\beta''\text{S}a''a'a,$$

so that m does not change when we pass from ϕ to ϕ', on which account we may write also,

$$\text{XXIV.} \ldots m\text{S}\lambda\lambda' = \text{S}.\phi\lambda\psi'\lambda', \quad \text{or} \quad \text{XXIV'.} \ldots m\text{S}\lambda\mu\nu = \text{S}.\phi\lambda\phi\mu\phi\nu,$$

because, by (2.), we can deduce from XI. the conjugate expression,

$$\text{XXV.} \ldots \psi'\lambda' = \text{V}.\phi\mu\phi\nu.$$

(4.) We ought then to find that the *linear equation*,

$$\text{XXVI.} \ldots \sigma = \phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho + \beta'' S\alpha''\rho,$$

has its *solution* expressed (comp. VIII.) by the formula,

$$\text{XXVII.} \ldots \rho S\alpha\alpha'\alpha'' S\beta''\beta'\beta = V\alpha'\alpha'' S\beta''\beta'\sigma + V\alpha''\alpha S\beta\beta''\sigma + V\alpha\alpha' S\beta'\beta\sigma;$$

and accordingly, if we operate on the expression XXVI. for σ with the three symbols,

$$\text{XXVIII.} \ldots S.\beta''\beta', \quad S.\beta\beta'', \quad S.\beta'\beta,$$

we obtain the three scalar equations,

$$\text{XXIX.} \ldots S\beta''\beta'\sigma = S\beta''\beta'\beta S\alpha\rho, \ \&c.,$$

from which the equation XXVII. follows immediately, without any introduction of the auxiliary vectors λ, μ, ν, although these are useful in the theory generally.

(5.) Conversely, if the equation XXVII. were *given*, and the value of σ *sought*, we might operate with the three symbols,

$$\text{XXX.} \ldots S.\alpha, \quad S.\beta, \quad S.\gamma,$$

and so obtain the three scalar equations XXIX., from which the expression XXVI. for σ would follow.

(6.) It will be found a useful check on formulæ of this sort, to consider each *beta*, in what we have called the *Standard Form* (1.) of $\phi\rho$, as being of the *first dimension*; for then we may say that ϕ and ϕ' are *also* of the *first* dimension, but ψ and ψ' of the *second*, and m of the *third*; and every formula, into which these symbols enter, will thus be *homogeneous*: α, α', α'', and λ, μ, ν, ρ, being *not counted*, in this mode of estimating *dimensions*, but σ being treated as of the *first* dimension, when it is taken as representing $\phi\rho$.*

(7.) And although the *trinomial form* XV. has been seen to be *sufficiently general*, yet if we choose to take the more expanded form,

$$\text{XXXI.} \ldots \phi\rho = \Sigma\beta S\alpha\rho, \quad \text{which gives} \quad \text{XXXII.} \ldots \phi'\rho = \Sigma\alpha S\beta\rho,$$

any number of *terms* of $\phi\rho$, such as $\beta S\alpha\rho$, $\beta' S\alpha'\rho$, &c., being now included in the *sum* Σ, there is no difficulty in proving that the equations VIII. and IX. are satisfied, when we write,

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* [Compare the first Note to Art. 350.]

$$\text{XXXIII.} \ldots \psi\rho = \Sigma \mathrm{V}\alpha\alpha'\mathrm{S}\beta'\beta\rho, \quad \text{with} \quad \text{XXXIV.} \ldots \psi'\rho = \Sigma \mathrm{V}\beta\beta'\mathrm{S}\alpha'\alpha\rho,$$

and

$$\text{XXXV.} \ldots m = \Sigma \mathrm{S}\alpha\alpha'\alpha''\mathrm{S}\beta''\beta'\beta = \Sigma \mathrm{S}\beta\beta'\beta''\mathrm{S}\alpha''\alpha'\alpha.$$

(8.) The important property (2.), that the auxiliary function ψ is changed to its own *conjugate* ψ, when ϕ is changed to ϕ', may be proved without any reference to the form $\Sigma\beta\mathrm{S}\alpha\rho$ of $\phi\rho$, by means of the definitions IV. and XI., of ϕ' and ψ, as follows. Whatever four vectors μ, ν, μ_1, and ν_1 may be, if we write

$$\text{XXXVI.} \ldots \lambda'_1 = \mathrm{V}\mu_1\nu_1, \quad \text{and} \quad \text{XXXVII.} \ldots \psi'\mathrm{V}\mu\nu = \mathrm{V}.\phi\mu\phi\nu,$$

adopting here this last equation as a *definition* of the *function* ψ', we may proceed to *prove* that it is *conjugate* to ψ, by observing that we have the transformations,

$$\begin{aligned} \text{XXXVIII.} \ldots \mathrm{S}\lambda'_1\psi'\lambda' &= \mathrm{S}(\mathrm{V}\mu_1\nu_1 . \mathrm{V}.\phi\mu\phi\nu) = \mathrm{S}.\mu_1(\mathrm{V}.\nu_1\mathrm{V}.\phi\mu\phi\nu) \\ &= \mathrm{S}\mu_1\phi\nu . \mathrm{S}\nu_1\phi\mu - \mathrm{S}\mu_1\phi\mu . \mathrm{S}\nu_1\phi\nu \\ &= \mathrm{S}\mu\phi'\nu_1 . \mathrm{S}\nu\phi'\mu_1 - \mathrm{S}\mu\phi'\mu_1 . \mathrm{S}\nu\phi'\nu_1 \\ &= \mathrm{S}.\mu(\mathrm{V}.\nu\mathrm{V}.\phi'\mu_1\phi'\nu_1) = \mathrm{S}(\mathrm{V}\mu\nu.\mathrm{V}.\phi'\mu_1\phi'\nu_1) = \mathrm{S}\lambda'\psi\lambda'_1; \end{aligned}$$

which establish the relation in question, between ψ and ψ'.

(9.) And the not less important property (3.), that m remains *unchanged* when we pass from ϕ to ϕ', may in like manner be proved, without reference to the *form* XV. or XXXI. of $\phi\rho$, by observing that we have by XXXVII., &c., the transformations,

$$\text{XXXIX.} \ldots \mathrm{S}.\phi\lambda\phi\mu\phi\nu = \mathrm{S}.\phi\lambda\psi'\lambda' = \mathrm{S}\lambda'\psi\phi\lambda = m\mathrm{S}\lambda'\lambda = m\mathrm{S}\lambda\mu\nu,$$

because the equations III. and VIII. give,

$$\text{XL.} \ldots \psi\phi\rho = m\rho, \quad \text{whatever vector } \rho \text{ may be;}$$

so that the value of this scalar constant m may now be derived from the *original* linear function ϕ, exactly as it was in X. or X′. from the *conjugate* function ϕ'.

348. It is found, then, that the *linear and vector equation,*

$$\text{I.} \ldots \phi\rho = \sigma, \quad \text{gives} \quad \text{II.} \ldots m\rho = \psi\sigma,$$

as its *formula of solution*; with the *general method,* above explained, of deducing m and ψ from ϕ. We have therefore the two *identities,*

$$\text{III.} \ldots m\sigma = \phi\psi\sigma, \quad m\rho = \psi\phi\rho;$$

or briefly and symbolically,

$$\text{III}'. \ldots m = \phi\psi = \psi\phi;$$

with which, by what has been shown, we may connect these *conjugate equations*,

$$\text{III}''. \ldots m = \phi'\psi' = \psi'\phi'.$$

Changing then successively μ and ν to $\psi'\mu$ and $\psi'\nu$, in the equation of definition of the auxiliary function ψ, or in the formula,

$$\psi V\mu\nu = V.\phi'\mu\phi'\nu, \qquad 347,\ \text{XI}'.,$$

we get these two other equations,

$$\text{IV}. \ldots -\psi V.\nu\psi'\mu = mV.\mu\phi'\nu; \qquad \text{V}. \ldots \psi V.\psi'\mu\psi'\nu = m^2 V\mu\nu;$$

in the former of which the *points* may be omitted, while in each of them *accented* may be exchanged with unaccented symbols of operation: and we see that the *law of homogeneity* (347, (6.)) is preserved. And many other transformations of the same sort may be made, of which the following are a few examples.

(1.) Operating on V. by ψ^{-1}, or by $m^{-1}\phi$, we get this new formula,

$$\text{VI}. \ldots V.\psi'\mu\psi'\nu = m\phi V\mu\nu;$$

comparing which with the lately cited *definition* of ψ, we see that we *may change* ϕ to ψ, *if we at the same time change* ψ *to* $m\phi$, and therefore also m to m^2; ϕ' being then changed to ψ', and ψ' to $m\phi'$.

(2.) For example, we may thus pass from IV. and V. to the formulæ,

$$\text{VII}. \ldots -\phi V\nu\phi'\mu = V\mu\psi'\nu, \quad \text{and} \quad \text{VIII}. \ldots \phi V.\phi'\mu\phi'\nu = mV\mu\nu;$$

in which we see that the lately cited *law of homogeneity* is still observed.

(3.) The equation VII. might have been otherwise obtained, by interchanging μ and ν in IV., and operating with $-m^{-1}\phi$, or with $-\psi^{-1}$; and the formula VIII. may be at once deduced from the equation of definition of ψ, by operating on it with ϕ. In fact, our *rule of inversion*, of the *linear function* ϕ, may be said to be contained in the formula,

$$\text{IX}. \ldots \phi^{-1}V\mu\nu = m^{-1}V.\phi'\mu\phi'\nu;$$

where m is a scalar constant, as above.

(4.) By similar operations and substitutions,

$$\text{X}. \ldots \phi^2 V.\phi'\mu\phi'\nu = m\phi V\mu\nu = V.\psi'\mu\psi'\nu;$$

$$\text{XI}. \ldots m\phi V.\phi'\mu\phi'\nu = m^2 V\mu\nu = \psi V.\psi'\mu\psi'\nu;$$

$$\text{XII}. \ldots m^2 V.\phi'\mu\phi'\nu = m^2\psi V\mu\nu = \psi^2 V.\psi'\mu\psi'\nu;$$

$$\text{XIII}. \ldots V.\phi'^2\mu\phi'^2\nu = \psi V.\phi'\mu\phi'\nu = \psi^2 V\mu\nu; \ \&c.$$

(5.) But we have also,

$$\text{XIV.} \ldots \mathrm{S} \,.\, \lambda\phi^2\rho = \mathrm{S} \,.\, \phi\rho\phi'\lambda = \mathrm{S} \,.\, \rho\phi'^2\lambda,$$

so that the *second functions* ϕ^2 and ϕ'^2 are *conjugate* (compare 347, IV.); hence, by XIII., ψ^2 is formed from ϕ^2, as ψ from ϕ; and generally it will be found, that if n be *any whole number*, and if we change ϕ to ϕ^n, we change at the same time ϕ' to ϕ'^n, ψ to ψ^n, ψ' to ψ'^n, and m to m^n.

(6.) It may also be remarked that the changes (1.) conduct to the equation,

$$\text{XV.} \ldots (\mathrm{S} \,.\, \phi\lambda\phi\mu\phi\nu)^2 = \mathrm{S}\lambda\mu\nu\mathrm{S} \,.\, \psi\lambda\psi\mu\psi\nu;$$

and to many other analogous formulæ.

349. The expressions,

$$\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu, \quad \lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu$$

with the significations 347, XX. of λ', μ', ν', and others of the same type, are easily proved to *vanish* when λ, μ, ν are *complanar*,* and therefore to be *divisible* by $\mathrm{S}\lambda\mu\nu$, since each such expression involves each of the three auxiliary vectors λ, μ, ν in the *first degree* only; the *quotients* of such divisions being therefore certain *constant quaternions*, independent of λ, μ, ν, and depending only on the *particular form* of ϕ, or on the (scalar or vector, but real) *constants*, which enter into the composition of that given function. Writing, then,

$$\text{I.} \ldots q_1 = (\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) : \mathrm{S}\lambda\mu\nu,$$

and

$$\text{II.} \ldots q_2 = (\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) : \mathrm{S}\lambda\mu\nu,$$

we shall find it useful to consider separately the scalar and vector *parts* of these two *quaternion constants*, q_1 and q_2; which constants are, respectively, of the *first* and *second dimensions*, in a sense lately explained.†

(1.) Since $\mathrm{V}\lambda'\phi\lambda = \mu\mathrm{S}\nu\phi\lambda - \nu\mathrm{S}\lambda\phi'\mu$, &c., it follows that the vector parts of q_1 and q_2 change signs, when ϕ is changed to ϕ', and therefore ψ to ψ'. On the other hand, we may change the arbitrary vectors λ, μ, ν to λ', μ', ν', if we at the same time change λ' to $\mathrm{V}\mu'\nu'$, or to $-\lambda\mathrm{S}\lambda\mu\nu$, &c., and $\mathrm{S}\lambda\mu\nu$, or $\mathrm{S}\lambda\lambda'$, to $-(\mathrm{S}\lambda\mu\nu)^2$; dividing then by $-\mathrm{S}\lambda\mu\nu$, we find these new expressions,

$$\text{III.} \ldots q_1 = (\lambda\phi\lambda' + \mu\phi\mu' + \nu\phi\nu') : \mathrm{S}\lambda\mu\nu,$$

$$\text{IV.} \ldots q_2 = (\lambda\psi\lambda' + \mu\psi\mu' + \nu\psi\nu') : \mathrm{S}\lambda\mu\nu;$$

* [By putting $\nu = x\lambda + y\mu$.]

† [It may be instructive to the student to reduce these quaternion constants by replacing λ, μ, and ν by $xi + yj + zk$, $x'i + y'j + z'k$, and $x''i + y''j + z''k$.]

operating on which by S, we return to the scalars of the expressions I. and II., with ϕ and ψ changed to ϕ' and ψ'.

(2.) Hence the *conjugate quaternion constants*, $\mathrm{K}q_1$ and $\mathrm{K}q_2$, are obtained by passing to the *conjugate linear functions*; and thus we may write,

$$\text{V.} \ldots \mathrm{K}q_1 = (\lambda'\phi'\lambda + \mu'\phi'\mu + \nu'\phi'\nu) : \mathrm{S}\lambda\mu\nu;$$

$$\text{VI.} \ldots \mathrm{K}q_2 = (\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) : \mathrm{S}\lambda\mu\nu;$$

or, interchanging λ with λ', &c., in the dividends,

$$\text{VII.} \ldots \mathrm{K}q_1 = (\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') : \mathrm{S}\lambda\mu\nu;$$

$$\text{VIII.} \ldots \mathrm{K}q_2 = (\lambda\psi'\lambda' + \mu\psi'\mu' + \nu\psi'\nu') : \mathrm{S}\lambda\mu\nu;$$

where $\lambda' = \mathrm{V}\mu\nu$, &c., as before.

(3.) Operating with $\mathrm{V} \,.\, \rho$ on $\mathrm{V}q_1$, and observing that

$$\mathrm{V}.\rho\mathrm{V}\lambda'\phi\lambda = \phi(\lambda\mathrm{S}\lambda'\rho) - \lambda'\mathrm{S}\lambda\phi'\rho, \text{ \&c.},$$

while

$$\phi(\lambda\mathrm{S}\lambda'\rho + \mu\mathrm{S}\mu'\rho + \nu\mathrm{S}\nu'\rho) = \phi\rho\mathrm{S}\lambda\mu\nu,$$

and

$$\lambda'\mathrm{S}\lambda\phi'\rho + \mu'\mathrm{S}\mu\phi'\rho + \nu'\mathrm{S}\nu\phi'\rho = \phi'\rho\mathrm{S}\lambda\mu\nu,$$

with similar transformations for $\mathrm{V}.\rho\mathrm{V}q_2$, we find that

$$\text{IX.} \ldots \mathrm{V}.\rho\mathrm{V}q_1 = \phi\rho - \phi'\rho;$$

and

$$\text{X.} \ldots \mathrm{V}.\rho\mathrm{V}q_2 = \psi\rho - \psi'\rho.$$

(4.) Accordingly, since

$$\mathrm{S}\rho(\phi\rho - \phi'\rho) = - \mathrm{S}\rho(\phi\rho - \phi'\rho) = 0,$$

the vector $\phi\rho - \phi'\rho$, if it do not vanish, must be a line perpendicular to ρ, and therefore of the *form*,

$$\text{XI.} \ldots \phi\rho - \phi'\rho = 2\mathrm{V}\gamma\rho,$$

in which γ is some constant vector;* so that we may write,

$$\text{XII.} \ldots \phi\rho = \phi_0\rho + \mathrm{V}\gamma\rho, \quad \phi'\rho = \phi_0\rho - \mathrm{V}\gamma\rho,$$

where the function $\phi_0\rho$ is *its own conjugate*, or is the *common self-conjugate part* of $\phi\rho$ and $\phi'\rho$; namely the part,

$$\text{XIII.} \ldots \phi_0\rho = \tfrac{1}{2}(\phi\rho + \phi'\rho).$$

And we see that, with this signification of γ,

$$\text{XIV.} \ldots \mathrm{V}(\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) = - 2\gamma\mathrm{S}\lambda\mu\nu, \quad \text{or} \quad \text{XIV}'. \ldots \mathrm{V}q_1 = - 2\gamma;$$

* [This vector γ has been called the spin-vector of the function ϕ.]

while we have, in like manner,

$$\text{XV.} \ldots \mathrm{V}(\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) = -2\delta\mathrm{S}\lambda\mu\nu, \quad \text{or} \quad \text{XV}'. \ldots \mathrm{V}q_2 = -2\delta,$$

if

$$\text{XVI.} \ldots \psi\rho - \psi'\rho = 2\mathrm{V}\delta\rho.$$

As a confirmation, the part ϕ_0 of ϕ has by (1.) no effect in $\mathrm{V}q_1$; and if we change $\phi\lambda$ to $\mathrm{V}\gamma\lambda$, &c., in the first member of XIV., we have thus,

$$(\lambda\mathrm{S}\gamma\lambda' + \mu\mathrm{S}\gamma\mu' + \nu\mathrm{S}\gamma\nu') - \gamma\mathrm{S}(\lambda\lambda' + \mu\mu' + \nu\nu') = \gamma\mathrm{S}\lambda\mu\nu - 3\gamma\mathrm{S}\lambda\mu\nu.$$

(5.) Since $\mathrm{V}\lambda'\psi'\lambda = -\phi\mathrm{V}\lambda\phi'\lambda'$, &c., by 348, VII., while we may write, by (1.), (2.), and (4.),

$$\text{XVII.} \ldots \mathrm{V}(\lambda\phi\lambda' + \mu\phi\mu' + \nu\phi\nu') = -2\gamma\mathrm{S}\lambda\mu\nu,$$

$$\text{XVIII.} \ldots \mathrm{V}(\lambda\psi\lambda' + \mu\psi\mu' + \nu\psi\nu') = -2\delta\mathrm{S}\lambda\mu\nu,$$

or

$$\text{XIX.} \ldots \mathrm{V}(\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') = +2\gamma\mathrm{S}\lambda\mu\nu,$$

and

$$\text{XX.} \ldots \mathrm{V}(\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) = +2\delta\mathrm{S}\lambda\mu\nu,$$

we have this *relation* between the two new vector constants,

$$\text{XXI.} \ldots \delta = -\psi\gamma = -\phi'\gamma = -\phi_0\gamma;$$

for ϕ, ϕ', and ϕ_0 have all the *same effect*, on *this* particular vector, γ.

(6.) We may add that the vector constant γ is of the *first dimension*, and that δ is of the *second* dimension, with respect to the *betas* of the *standard form*; in fact, with that *form*, 347, XV., of $\phi\rho$, we have the expressions,

$$\text{XXII.} \ldots \gamma = \tfrac{1}{2}\mathrm{V}(\beta\alpha + \beta'\alpha' + \beta''\alpha''),$$

and

$$\text{XXIII.} \ldots \delta = \tfrac{1}{2}\mathrm{V}(\mathrm{V}\beta'\beta''\,.\,\mathrm{V}\alpha'\alpha'' + \mathrm{V}\beta''\beta\,.\,\mathrm{V}\alpha''\alpha + \mathrm{V}\beta\beta'\,.\,\mathrm{V}\alpha\alpha').$$

(7.) If we denote by ψ_0 and m_0, what ψ and m become when ϕ is changed to ϕ_0, we easily find that

$$\text{XXIV.} \ldots \psi\rho = \psi_0\rho - \gamma\mathrm{S}\gamma\rho + \mathrm{V}\delta\rho; \quad \text{XXV.} \ldots \psi'\rho = \psi_0\rho - \gamma\mathrm{S}\gamma\rho - \mathrm{V}\delta\rho;$$

so that the *self-conjugate part* of $\psi\rho$ contains a *term*, $-\gamma\mathrm{S}\gamma\rho$, which involves the vector γ, but only in the *second degree*; and in like manner,

$$\text{XXVI.} \ldots m = m_0 + \mathrm{S}\gamma\delta = m_0 - \mathrm{S}\gamma\phi\gamma;$$

γ again entering only in an *even degree*, because m remains unchanged, when we pass from ϕ to ϕ', or from γ to $-\gamma$.*

* [Expand $\psi\mathrm{V}\mu\nu = \mathrm{V}(\phi_0\mu - \mathrm{V}\gamma\mu)(\phi_0\nu - \mathrm{V}\gamma\nu)$.]

(8.) It is evident that we have the relations,

$$\text{XXVII.} \ldots m_0 = \phi_0\psi_0 = \psi_0\phi_0\,;$$

and that, in a sense already explained, ϕ_0, ψ_0, and m_0 are of the *first*, *second*, and *third* dimensions, respectively.

350. After thus considering the *vector parts* of the *two quaternion constants*, q_1 and q_2, we proceed to consider their *scalar parts*; which will introduce *two new scalar constants*, m'' and m', and will lead to the employment of *two new conjugate auxiliary functions*, $\chi\rho$ and $\chi'\rho$; whence also will result the establishment of a certain *Symbolic and Cubic Equation*,

$$\text{I.} \ldots 0 = m - m'\phi + m''\phi^2 - \phi^3,$$

which is *satisfied by the Linear Symbol of Operation*, ϕ, and is of great importance in this whole *Theory of Linear Functions.**

(1.) Writing, then,

$$\text{II.} \ldots m'' = \mathrm{S}q_1, \quad \text{and} \quad \text{III.} \ldots m' = \mathrm{S}q_2,$$

we see first that neither of these *two new constants* changes value, when we pass from ϕ to ϕ', or from γ to $-\gamma$; because, in such a passage, it has been seen that we only change q_1 and q_2 to $\mathrm{K}q_1$ and $\mathrm{K}q_2$. Accordingly, if we denote by m'_0 and m''_0 what m' and m'' become, when ϕ is changed to ϕ_0, we easily find the expressions,

$$\text{IV.} \ldots m'' = m''_0\,; \quad \text{and} \quad \text{V.} \ldots m' = m'_0 - \gamma^2.$$

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* [Or directly, without introducing χ or χ', for an arbitrary vector λ the relation

$$\phi^3\lambda \mathrm{S}\lambda\phi\lambda\phi^2\lambda = \lambda\mathrm{S}\phi\lambda\phi^2\lambda\phi^3\lambda + \phi\lambda\mathrm{S}\phi^2\lambda\lambda\phi^3\lambda + \phi^2\lambda\mathrm{S}\lambda\phi\lambda\phi^3\lambda$$

will generally exist. This may be briefly written in the form,

$$\phi^3\lambda - m_1\phi^2\lambda + m_2\phi\lambda - m_3\lambda = 0,$$

where the coefficients m can only depend on ϕ and λ. Operating on this by ϕ and ϕ^2,

$$\phi^4\lambda - m_1\phi^3\lambda + m_2\phi^2\lambda - m_3\phi\lambda = 0$$

and

$$\phi^5\lambda - m_1\phi^4\lambda + m_2\phi^3\lambda - m_3\phi^2\lambda = 0.$$

But an arbitrary vector ρ may be expressed in the form

$$x\lambda + y\phi\lambda + z\phi^2\lambda,$$

and hence from the three equations, on multiplying by x, y, and z, and adding, the equation

$$\phi^3\rho - m_1\phi^2\rho + m_2\phi\rho - m_3\rho = 0$$

results. This must be identical with the equation found by treating ρ directly, in the same manner as λ has been treated, and therefore the coefficients m must be independent of λ. The suffixes here printed serve to indicate the dimensions of the m. See 347 (6.).]

(2.) It may be noted that m'', or m''_0, is of the *first* dimension, but that m' and m'_0 are of the *second*, with respect to the standard form of ϕ; and accordingly, with that form we have,

$$\text{VI.} \ldots m'' = S\alpha\beta + S\alpha'\beta' + S\alpha''\beta'';$$

and

$$\text{VII.} \ldots m' = S(V\alpha'\alpha'' \,.\, V\beta''\beta' + V\alpha''\alpha \,.\, V\beta\beta'' + V\alpha\alpha' \,.\, V\beta'\beta).$$

(3.) If we introduce *two new linear functions*, $\chi\rho$ and $\chi'\rho$, such that

$$\text{VIII.} \ldots \chi V\mu\nu = V(\mu\phi'\nu - \nu\phi'\mu),$$

and

$$\text{IX.} \ldots \chi' V\mu\nu = V(\mu\phi\nu - \nu\phi\mu),$$

it is easily proved that these functions are *conjugate* to each other, and that each is of the *first* dimension; in fact, with the standard form of $\phi\rho$, we have the expressions,

$$\text{X.} \ldots \chi\rho = V(\alpha V\beta\rho + \alpha' V\beta'\rho + \alpha'' V\beta''\rho),$$

$$\text{XI.} \ldots \chi'\rho = V(\beta V\alpha\rho + \beta' V\alpha'\rho + \beta'' V\alpha''\rho);$$

and $S \,.\, \lambda\alpha V\beta\rho = S \,.\, \rho\beta V\alpha\lambda$, &c. Also, if χ_0 be formed from ϕ_0, as χ from ϕ, it will be found that

$$\text{XII.} \ldots \chi\rho = \chi_0\rho - V\gamma\rho, \quad \text{and} \quad \text{XIII.} \ldots \chi'\rho = \chi_0\rho + V\gamma\rho;$$

where χ_0 is of the first dimension.

(4.) Since

$$S\lambda\chi\lambda' = S \,.\, \lambda(\mu\phi'\nu - \nu\phi'\mu) = S(\mu'\phi'\mu + \nu'\phi'\nu),$$

the expression II. gives, by 349, V., the equation,

$$\text{XIV.} \ldots m'' S\lambda\lambda' = S \,.\, \lambda(\phi + \chi)\lambda',$$

λ and λ' being two *arbitrary* and *independent* vectors; which can only be, by our having the *functional relation*,

$$\text{XV.} \ldots \phi\rho + \chi\rho = m''\rho;$$

or briefly and symbolically,

$$\text{XVI.} \ldots \chi + \phi = m''.$$

Accordingly it is evident that the relation XV. is verified, by the form X. of $\chi\rho$, combined with the standard form of $\phi\rho$, and with the expression VI. for the constant m''.

(5.) The formula XVI. gives,

$$\text{XVII.} \ldots \chi\phi = m''\phi - \phi^2 = \phi\chi;$$

and accordingly the identity of $\chi\phi$ and $\phi\chi$ may easily be otherwise proved, by changing μ and ν to $\psi'\mu$ and $\psi'\nu$ in the definition VIII. of χ, and remembering that

$$\mathrm{V}\,.\,\psi'\mu\psi'\nu = m\phi\mathrm{V}\mu\nu, \quad \phi'\psi' = m, \quad \text{and} \quad \mathrm{V}\mu\psi'\nu = -\phi\mathrm{V}\nu\phi'\mu;$$

for thus we have,

$$\text{XVIII.} \ldots \chi\phi\mathrm{V}\mu\nu = \mathrm{V}(\mu\psi'\nu - \nu\psi'\mu) - \phi\mathrm{V}(\mu\phi'\nu - \nu\phi'\mu) = \phi\chi\mathrm{V}\mu\nu,$$

as required.

(6.) Since, then,

$$\mathrm{S}\,.\,\lambda\phi\chi\lambda' = \mathrm{S}\,.\,\lambda(\mu\psi'\nu - \nu\psi'\mu) = \mathrm{S}(\mu'\psi'\mu + \nu'\psi'\nu),$$

the value III. of m' gives, by 349, VI., the equation,

$$\text{XIX.} \ldots m'\mathrm{S}\lambda\lambda' = \mathrm{S}\,.\,\lambda(\psi + \phi\chi)\lambda',$$

λ and λ' being independent vectors; hence,

$$\text{XX.} \ldots \psi\rho + \phi\chi\rho = m'\rho,$$

or briefly,

$$\text{XXI.} \ldots \psi + \phi\chi = m'.$$

And in fact, with the standard form of $\phi\rho$, we have

$$\text{XXII.} \ldots \phi\chi\rho = \chi\phi\rho = \mathrm{V}(\mathrm{V}\beta'\beta''.\mathrm{V}\rho\mathrm{V}\alpha'\alpha'' + \mathrm{V}\beta''\beta\,.\mathrm{V}\rho\mathrm{V}\alpha''\alpha + \mathrm{V}\beta\beta'.\mathrm{V}\rho\mathrm{V}\alpha\alpha');$$

which verifies the equation XX., when it is combined with the value VII. of m', and with the expression 347, XVIII. for $\psi\rho$.

(7.) *Eliminating* the *symbol* χ, between the two equations XVI. and XXI., and remembering that $\phi\psi = \psi\phi = m$, we find the symbolic expression,

$$\text{XXIII.} \ldots m\phi^{-1} = \psi = m' - m''\phi + \phi^2;$$

and thus the *symbolic and cubic equation* I. is proved.

(8.) And because the *coefficients*, m, m', m'', of that equation, have been seen to remain unaltered, in the passage from ϕ to ϕ', we may write also this *conjugate equation*,

$$\text{XXIV.} \ldots 0 = m - m'\phi' + m''\phi'^2 - \phi'^3.*$$

* [This may also be proved thus:—If ρ and σ are arbitrary vectors, by 348 (5.),

$$\mathrm{S}\rho\,(\phi'^3 - m''\phi'^2 + m'\phi' - m)\,\sigma = \mathrm{S}\sigma\,(\phi^3 - m''\phi^2 + m'\phi - m)\,\rho,$$

and therefore vanishes. This requires $(\phi'^3 - m''\phi'^2 + m'\phi' - m)\,\sigma = 0,$ as ρ is arbitrary.]

(9.) Multiplying symbolically the equation I. by $-m^{-1}\psi^3$, and reducing by $\psi\phi = m$, we eliminate the symbol ϕ, and obtain this *cubic in* ψ,

$$\text{XXV.} \ldots 0 = m^2 - mm''\psi + m'\psi^2 - \psi^3;$$

in which ψ' may be substituted for ψ.

(10.) In general, it may be remarked, that when we change ϕ to ψ, and therefore ψ to $m\phi$, as before, we change not only m to m^2, but also m' to mm'', and m'' to m'; while χ is at the same time changed to $\phi\chi$, or to $\chi\phi$, and the quaternion q_1 is changed to q_2. Accordingly, we may thus pass from the relation XVI. to XXI.; and conversely, from the latter to the former.

(11.) And if the two new auxiliary functions, χ and χ', be considered as *defined* by the equations VIII. and IX., their *conjugate relation* (3.) to each other may be *proved*, without any reference to the *standard form* of $\phi\rho$, by reasonings similar to those which were employed in 347, (8.), to establish the corresponding conjugation of the functions ψ and ψ'.

(12.) It may be added that the relations between ϕ, ϕ', χ, χ', and m'' give the following additional transformations, which are occasionally useful:

$$\text{XXVI.} \ldots \phi'\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi\nu + \nu\phi\mu) = -\mathrm{V}(\nu\chi\mu + \mu\phi\nu);$$

$$\text{XXVII.} \ldots \phi\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi'\nu + \nu\phi'\mu) = -\mathrm{V}(\nu\chi'\mu + \mu\phi'\nu);$$

with others on which we cannot here delay.*

351. The cubic in ϕ may be thus written:

$$\text{I.} \ldots 0 = m\rho - m'\phi\rho + m''\phi^2\rho - \phi^3\rho;$$

where ρ is an arbitrary vector. If then it happen that for some *particular* but *actual* vector, ρ, the linear function $\phi\rho$ vanishes, so that $\phi\rho = 0$, $\phi^2\rho = 0$, $\phi^3\rho = 0$, &c., the *constant* m must be *zero*; or in symbols,

$$\text{II.} \ldots \text{if } \phi\rho = 0, \quad \text{and} \quad \mathrm{T}\rho > 0, \quad \text{then} \quad m = 0.$$

Hence, by the expression 347, XXIII. for m, when the standard form for $\phi\rho$ is adopted, we must have either

$$\text{III.} \ldots \mathrm{S}\alpha\alpha'\alpha'' = 0, \quad \text{or else} \quad \text{IV.} \ldots \mathrm{S}\beta''\beta'\beta = 0;$$

so that, in *each* case, that *generally trinomial form*, 347, XV., must admit of

* [Without introducing χ, since for any three vectors $m''\mathrm{S}\lambda\mu\nu = \mathrm{S}\lambda(\phi'\mathrm{V}\mu\nu + \mathrm{V}\mu\phi\nu + \mathrm{V}\phi\mu\nu)$, it follows, as λ is arbitrary, that $m''\mathrm{V}\mu\nu = \phi'\mathrm{V}\mu\nu + \mathrm{V}\mu\phi\nu + \mathrm{V}\phi\mu\nu$. This is equivalent to XXVI.]

being *reduced to a binomial.* Conversely, when we have thus a function of the *particular form*,

$$\text{V.} \ldots \phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho,$$

we have then,

$$\text{VI.} \ldots \phi V\alpha\alpha' = 0\,;$$

so that if α and α' be *actual* and *non-parallel* lines, the *real* and *actual vector* $V\alpha\alpha'$ will be a value of ρ, which will satisfy the equation $\phi\rho = 0$; but *no other real and actual value* of ρ, except $\rho = xV\alpha\alpha'$, will satisfy that equation, if β and β' be *actual*, and *non-parallel.* In this case V., the *operation* ϕ *reduces every other vector to the fixed plane* of β, β', which *plane* is therefore the *locus* of $\phi\rho$; and since we have also,

$$\text{VII.} \ldots \phi'\rho = \alpha S\beta\rho + \alpha' S\beta'\rho,$$

we see that the *locus of the functionally conjugate vector*, $\phi'\rho$, is *another fixed plane*, namely that of α, α'. Also, the *normal* to the *latter plane* is the *line* which is *destroyed* by the *former operation*, namely by ϕ; while the *normal* to the *former plane* is in like manner the *line*, which is *annihilated* by the *latter operation*, ϕ', since we have

$$\text{VIII.} \ldots \phi' V\beta\beta' = 0,$$

but *not* $\phi'\rho = 0$, for any *actual* ρ, in any *direction* except that of $V\beta\beta'$, or its *opposite*, which may however, for the *present purpose*, be regarded as the *same.** In this case we have also *monomial forms* for $\psi\rho$ and $\psi'\rho$, namely

$$\text{IX.} \ldots \psi\rho = V\alpha\alpha' S\beta'\beta\rho, \quad \text{and} \quad \text{X.} \ldots \psi'\rho = V\beta\beta' S\alpha'\alpha\rho\,;$$

so that the *operation* ψ *destroys every line in the first fixed plane* (of β, β'), and the *conjugate* operation ψ' annihilates every line in the *second* fixed plane (of α, α'). On the other hand, the operation ψ *reduces* every line, which is *out of the first plane*, to the *fixed direction* of the *normal* to the *second plane*; and the operation ψ' reduces every line which is *out of* the *second* plane, to that *other fixed direction*, which is *normal* to the *first* plane. And thus it comes to pass, that whether we operate first with ψ, and then with ϕ; or first with ϕ, and then with ψ; or first with ψ' and then with ϕ'; or first with ϕ', and then with ψ'; in *all* these cases, we *arrive at last at a null line*, in conformity with the symbolic equations,

$$\text{XI.} \ldots \phi\psi = \psi\phi = \phi'\psi' = \psi'\phi' = m = 0,$$

which belong to the case here considered.

* Accordingly, *in the present investigation*, whenever we shall speak of a "*fixed direction*," or the "*direction of a given line*," &c., we are always to be understood as meaning, "*or the opposite* of that direction."

(1.) Without recurring to the *standard form* of $\phi\rho$, the equation 348, VI., namely $\mathrm{V}.\psi'\mu\psi'\nu = m\phi\mathrm{V}\mu\nu$, and the analogous equation $\mathrm{V}.\psi\mu\psi\nu = m\phi'\mathrm{V}\mu\nu$, might have enabled us to foresee that $\psi'\rho$ and $\psi\rho$, if they do not both *constantly vanish*, must (if $m = 0$) have each a *fixed direction*; and therefore that each must be expressible by a *monome*, as above: the fixed *direction* of $\psi\rho$ being that of a line which is *annihilated* by the *operation* ϕ, and similarly for $\psi'\rho$ and ϕ'.

(2.) And because, by 347, XI. and XXV., we have

$$\psi\mathrm{V}\mu\nu = \mathrm{V}.\phi'\mu\phi'\nu, \quad \text{and} \quad \psi'\mathrm{V}\mu\nu = \mathrm{V}.\phi\mu\phi\nu,$$

so that the line $\phi'\mu$, if actual, is perpendicular to $\psi\mathrm{V}\mu\nu$, and the line $\phi\mu$ perpendicular to $\psi'\mathrm{V}\mu\nu$, we see that *each of the two lines*, $\phi'\rho$ and $\phi\rho$, must have (in the present case) a *plane locus*; whence the *binomial forms* of the two *conjugate vector functions*, $\phi\rho$ and $\phi'\rho$, might have been foreseen: $\psi\rho$ and $\psi'\rho$ being here supposed to be *actual* vectors.

(3.) The *relations of rectangularity*, of the *two fixed lines* (or *directions*), to the *two fixed planes*, might also have been thus deduced, through the two *conjugate binomial forms*, V. and VII., without the *previous* establishment of the more *general trinomial* (or *standard*) *form* of $\phi\rho$.

(4.) The existence of a plane locus for $\phi\rho$, and of another for $\phi'\rho$, for the case when $m = 0$, might also have been foreseen from the equations,

$$\mathrm{S}.\phi\lambda\phi\mu\phi\nu = \mathrm{S}.\phi'\lambda\phi'\mu\phi'\nu = m\mathrm{S}\lambda\mu\nu;$$

and the same equations might have enabled us to foresee, that the *scalar constant* m must be *zero*, if for any *one actual vector*, such as λ, either $\phi\lambda$ or $\phi'\lambda$ becomes *null.*

(5.) And the *reducibility* of the *trinomial* to the *binomial form*, when this last condition is satisfied, might have been anticipated, without any reference to the composition of the constant m, from the simple consideration (comp. 294, (10.)), that *no actual vector* ρ can be *perpendicular*, at once, to *three diplanar lines.*

352. It may happen, that besides the recent reduction (351) of the *linear function* $\phi\rho$ to a *binomial form*, when the *relation*

$$\mathrm{I.}\;.\;.\; m = 0$$

exists between the *constants* of that function, in which case the symbolic and *cubic equation* 350, I. reduces itself to the form,

$$\mathrm{II.}\;.\;.\; \phi^3 - m''\phi^2 + m'\phi = 0,$$

thus losing its absolute term, or having *one root* equal to *zero*, this equation may undergo a further *reduction*, by *two* of its roots becoming *equal* to *each other*; namely either by our having

$$\text{III.} \ldots m' = 0, \quad \text{and} \quad \text{IV.} \ldots \phi^2(\phi - m'') = 0;$$

or in another way, by the existence of these other equations,

$$\text{V.} \ldots m''^2 - 4m' = 0, \quad \text{and} \quad \text{VI.} \ldots \phi(\phi - \tfrac{1}{2}m'')^2 = 0.$$

In *each* of these two cases, we shall find that certain *new geometrical relations* arise, which it may be interesting briefly to investigate; and of which the principal is the mutual *rectangularity* of *two fixed planes*, which are the *loci* (comp. 351) of certain *derived*, and *functionally conjugate vectors*: namely, in the case III. IV., the loci of $\phi\rho$ and $\phi'\rho$; and in the case V. VI., the loci of $\Phi\rho$ and $\Phi'\rho$, if

$$\text{VII.} \ldots \Phi = \phi - \tfrac{1}{2}m'', \quad \text{and} \quad \text{VIII.} \ldots \Phi' = \phi' - \tfrac{1}{2}m'',$$

so that, in this last case, the symbol Φ satisfies this *new cubic*,

$$\text{IX.} \ldots 0 = \Phi^2(\Phi + \tfrac{1}{2}m'');$$

while Φ' satisfies at the same time a cubic equation with the *same coefficients* (comp. 350, (8.)), namely

$$\text{X.} \ldots 0 = \Phi'^2(\Phi' + \tfrac{1}{2}m'').$$

(1.) We saw in 351, (1.), (2.), that when $m = 0$ the line $\psi'\rho$ has *generally a fixed direction*, to which that of the line $\phi\rho$ is *perpendicular*; and that in like manner the line $\psi\rho$ has then *another* fixed direction, to which $\phi'\rho$ is perpendicular. If then the *plane loci* of $\phi\rho$ and $\phi'\rho$ be at *right angles* to each other, we must also have the *fixed lines* $\psi'\lambda$ and $\psi\mu$ *rectangular*, or

$$\text{XI.} \ldots 0 = \mathrm{S}.\psi'\lambda\psi\mu = \mathrm{S}\lambda\psi^2\mu,$$

independently of the directions of λ and μ; whence

$$\text{XII.} \ldots 0 = \psi^2\mu, \quad \text{or} \quad \text{XIII.} \ldots \psi^2 = 0,$$

since μ is an arbitrary vector.

(2.) Now *in general*, by the functional relation 350, XXI. combined with $\psi\phi = m$, we have the transformation,

$$\text{XIV.} \ldots \psi^2 = \psi(m' - \phi\chi) = m'\psi - m\chi;$$

if then $m = 0$, as in I., the symbol ψ must satisfy the *depressed* or *quadratic equation*,

$$\text{XV.} \ldots 0 = m'\psi - \psi^2;$$

which is accordingly a *factor* of the *cubic* equation,

$$\text{XVI.} \ldots 0 = m'\psi^2 - \psi^3,$$

whereto the general equation 350, XXV. is *reduced*, by this supposition of m vanishing.

(3.) If then we have *not only* $m = 0$, as in I., but *also* $m' = 0$, as in III., the condition XIII. is satisfied, by XV.; and the *two planes*, above referred to, are generally *rectangular*.

(4.) We might indeed propose to satisfy that condition XIII., by supposing that we had always,

$$\text{XVII.} \ldots \psi = 0, \quad \text{that is,} \quad \text{XVII}'. \ldots \psi\rho = 0,$$

for *every direction* of ρ; but in *this* case, the *quaternion constant* q_2 would *vanish* (by 349, II.); and therefore the constant m', as being its *scalar part* (by 350, III.), would *still* be equal to *zero*.

(5.) The particular supposition XVII. would however *alter* completely the *geometrical character* of the question; for it would imply (comp. 351, (2.)) that the *directions* of the *lines* $\phi\rho$ and $\phi'\rho$ (when not *evanescent*) are *fixed*, instead of those lines having only certain *planes* for their *loci*, as before.

(6.) On the side of *calculation*, we should thus have, for the two *conjugate functions*, $\phi\rho$ and $\phi'\rho$, *monomial expressions* of the forms,

$$\text{XVIII.} \ldots \phi\rho = \beta S\alpha\rho, \quad \phi'\rho = \alpha S\beta\rho;$$

whence, by 347, XVIII., and 350, VII., we should recover the equations, $\psi\rho = 0$ and $m' = 0$.

(7.) We should have also, in this particular case,

$$\text{XIX.} \ldots \phi\rho = 0, \quad \text{if} \quad \rho \perp \alpha, \quad \text{and} \quad \text{XX.} \ldots \phi'\rho = 0, \quad \text{if} \quad \rho \perp \beta;$$

so that $\phi\rho$ now *vanishes*, if ρ be *any line* in the *fixed plane* perpendiculur to α; and in like manner $\phi'\rho$ is a null line, if ρ be in that *other fixed plane*, which is at right angles to the *other given line*, β.

(8.) *These* two *planes*, or their *normals* α and β, or the fixed directions of the two *lines* $\phi'\rho$ and $\phi\rho$, will be *rectangular* (comp. (1.)), if we have this new equation,

$$\text{XXI.} \ldots \phi^2 = 0, \quad \text{or} \quad \text{XXI}'. \ldots \phi^2\rho = 0,$$

for every direction of ρ; and accordingly the expression XVIII. gives

$$\phi^2\rho = S\alpha\beta \, . \, \phi\rho = 0, \quad \text{if} \quad \beta \perp \alpha, \text{ and reciprocally.}$$

(9.) Without expressly introducing α and β, the equation 350, XXIII. shows that when $\psi = 0$, and therefore also $m' = 0$, as in (4.), the symbol ϕ satisfies (comp. (2.)) the *new quadratic* or *depressed equation*,

$$\text{XXII.} \ldots 0 = \phi^2 - m''\phi ;$$

which is accordingly a *factor* of the *cubic* IV., but to which that cubic is *not reducible*, *unless* we have thus $\psi = 0$, as well as $m' = 0$.

(10.) The *condition*, then, of the *existence* and *rectangularity* of the *two planes* (7.), for which we have respectively $\phi\rho = 0$ and $\phi'\rho = 0$, without $\phi\rho$ *generally* vanishing (a case which it would be useless to consider), is that the four following equations should subsist:

$$\text{XXIII.} \ldots m = 0, \quad m' = 0, \quad m'' = 0, \quad \text{and} \quad \text{XVII.} \ldots \psi = 0 ;$$

or that the *cubic* IV., and its *quadratic factor* XXII., should reduce themselves to the very simple forms,

$$\text{XXIV.} \ldots \phi^3 = 0, \quad \text{and} \quad \text{XXV.} \ldots \phi^2 = 0 ;$$

the cubic in ϕ having thus its *three roots equal*, and *null*, and $\psi\rho$ *vanishing*.

(11.) We may also observe that as, when even *one* root of the general cubic 350, I. is *zero*, that is when $m = 0$, the *vector equation*

$$\text{XXVI.} \ldots \phi\rho = 0$$

was seen (in 351) to be satisfied by *one real direction* of ρ, so when we have *also* $m' = 0$, or when the cubic in ϕ has *two null roots*, or takes the form IV., then the *two vector equations*,

$$\text{XXVII.} \ldots \phi\rho = 0, \quad \psi\rho = 0,$$

are satisfied by one *common direction* of the *real* and *actual line* ρ; because we have, by 350, XVII. and XX., the *general* relation,

$$\psi\rho = m'\rho - \chi\phi\rho.$$

(12.) And because, by 350, XV., we have also the relation $\chi\rho = m''\rho - \phi\rho$, it follows that when the *three roots* of the cubic *all vanish*, or when the *three* *scalar equations* XXIII. are satisfied, then the *three vector equations*,

$$\text{XXVIII.} \ldots \phi\rho = 0, \quad \psi\rho = 0, \quad \chi\rho = 0,$$

have a *common* (*real* and *actual*) *vector root*; or are all satisfied by one *common direction* of ρ.

(13.) Since $m'' - \phi = \chi$, the cubic IV. may be written under any one of the following forms,

$$\text{XXIX.} \ldots 0 = \phi^2\chi = \phi\chi\phi = \chi\phi^2 = \phi \cdot \phi\chi = \&\text{c.},$$

in which accented may be substituted for unaccented symbols: and its *geometrical signification* may be illustrated by a reference to certain *fixed lines*, and *fixed planes*, as follows.

(14.) Suppose first that m and m' both vanish, but that m'' is different from zero, so that the cubic in ϕ is reducible to the form IV., but *not* to the form XXIV.; and that the operation ψ, which is here equivalent to $-\phi\chi$, or to $-\chi\phi$, does not annihilate *every* vector ρ, so that (comp. (4.) (5.) (6.)) $\phi\rho$ and $\phi'\rho$ have *not* the directions of *two fixed lines*, but have only (comp. (1.) and (3.)) *two fixed* and *rectangular planes*, Π and Π', as their *loci*; and let the *normals* to these two planes be denoted by λ and λ', so that these two rectangular lines, λ and λ', are situated respectively in the planes Π' and Π.

(15.) Then it is easily shown (comp. 351) that the operation ϕ *destroys* the line λ' *itself*, while it *reduces** every *other* line (that is, every line which is not of the form $x\lambda'$, with $\mathrm{V}x = 0$) to the *plane* $\Pi \perp \lambda$; and that it reduces every line *in* that plane to a *fixed direction*, μ, in the same plane, which is thus the *common* direction of all the lines $\phi^2\rho$, whatever the direction of ρ may be. And the symbolical equation, $\chi \cdot \phi^2 = 0$, expresses that this fixed direction μ of $\phi^2\rho$ may also be denoted by $\chi^{-1}0$; or that we have the equation,

$$\text{XXX.} \ldots 0 = \chi\mu = m''\mu - \phi\mu, \quad \text{if} \quad \mu = \phi^2\rho,$$

which can accordingly be otherwise proved: with similar results for the conjugate symbols, ϕ' and χ'.

(16.) For example, we may represent the conditions of the present case by the following system of equations (comp. 351, V. VII. IX. X., and 350, VI. VII. X. XI.):

$$\text{XXXI.} \ldots \begin{cases} \phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho, \quad \phi'\rho = \alpha\mathrm{S}\beta\rho + \alpha'\mathrm{S}\beta'\rho, \\ 0 = m' = \mathrm{S}(\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta) = \mathrm{S}\alpha\beta\mathrm{S}\alpha'\beta' - \mathrm{S}\alpha\beta'\mathrm{S}\alpha'\beta, \\ m'' = \mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta'; \end{cases}$$

* We propose to include the case where an *operation* of this sort *destroys* a line, or reduces it to *zero*, under the case when the same operation *reduces* a line to a *fixed direction*, or to a *fixed plane*.

$$\text{XXXII.}\ldots\begin{cases}\chi\rho = \mathrm{V}(\alpha\mathrm{V}\beta\rho + \alpha'\mathrm{V}\beta'\rho) = m''\rho - \phi\rho,\\ \chi'\rho = \mathrm{V}(\beta\mathrm{V}\alpha\rho + \beta'\mathrm{V}\alpha'\rho) = m''\rho - \phi'\rho,\\ -\psi\rho = \phi\chi\rho = \chi\phi\rho = \mathrm{V}\alpha\alpha'\mathrm{S}\beta\beta'\rho,\\ -\psi'\rho = \phi'\chi'\rho = \chi'\phi'\rho = \mathrm{V}\beta\beta'\mathrm{S}\alpha\alpha'\rho;\end{cases}$$

and may then write (not *here* supposing $\lambda' = \mathrm{V}\mu\nu$, &c.),

$$\text{XXXIII.}\ldots\begin{cases}\lambda = \mathrm{V}\beta\beta', \quad \lambda' = \mathrm{V}\alpha\alpha', \quad \mathrm{S}\lambda\lambda' = 0,\\ \mu = \phi\beta \parallel \phi\beta', \quad \mu' = \phi'\alpha' \parallel \phi'\alpha, \quad \mathrm{S}\lambda\mu = \mathrm{S}\lambda'\mu' = 0;\end{cases}$$

after which we easily find that

$$\text{XXXIV.}\ldots\begin{cases}\phi\lambda' = 0, \quad \phi^2\rho \parallel \mu, \quad \phi\mu = m''\mu, \quad \chi\mu = 0;\\ \phi'\lambda = 0, \quad \phi'^2\rho \parallel \mu', \quad \phi'\mu' = m''\mu', \quad \chi'\mu' = 0.\end{cases}$$

(17.) Since we have thus $\chi'\mu' = 0$, where μ' is a line in the fixed direction of $\phi'^2\rho$, we have also the equation,

$$\text{XXXV.}\ldots 0 = \mathrm{S}\rho\chi'\mu' = \mathrm{S}\mu'\chi\rho, \quad \text{or} \quad \chi\rho \perp \mu';$$

the *locus* of $\chi\rho$ is therefore a *plane* perpendicular to the line μ'; and in like manner, μ is the *normal* to a plane, which is the *locus* of the line $\chi'\rho$. And the symbolical equations, $\phi\,.\,\phi\chi = 0$, $\phi^2\,.\,\chi = 0$, may be interpreted as expressing, that the operation ϕ *reduces* every line in this *new plane* of $\chi\rho$ to the *fixed direction* of $\phi^{-1}0$, or of λ'; and that the operation ϕ^2 *destroys* every line in this plane $\perp \mu'$; with analogous results, when accented are interchanged with unaccented symbols. Accordingly we see, by XXXII., that $\phi\chi\rho$ has the fixed direction of $\mathrm{V}\alpha\alpha'$, or of λ'; and that $\phi\,.\,\phi\chi\rho = 0$, because $\phi\lambda' = 0$.

(18.) We see also, that the operation $\phi\chi$, or $\chi\phi$, destroys every line in the plane Π, to which the operation ϕ reduces every line; and that thus the symbolical equations, $\phi\chi\,.\,\phi = 0$, $\chi\phi\,.\,\phi = 0$, may be interpreted.

(19.) As a verification, it may be remarked that the *fixed direction* λ', of $\phi\chi\rho$ or $\chi\phi\rho$, ought to be that of the *line of intersection* of the two *fixed planes* of $\phi\rho$ and $\chi\rho$; and accordingly it is perpendicular by XXXIII. to their two *normals*, λ and μ': with similar remarks respecting the fixed direction λ, of $\phi'\chi'\rho$ or $\chi'\phi'\rho$, which is perpendicular to λ' and to μ.

(20.) Let us next suppose, that besides $m = 0$, and $m' = 0$, we have $\psi = 0$, but that m'' is still different from zero. In this case, it has been seen (6.) that the expression for $\phi\rho$ reduces itself to the *monomial form*, $\beta\mathrm{S}\alpha\rho$; and therefore that the operation ϕ *destroys* every line in a *fixed plane* ($\perp \alpha$), while it *reduces* every *other* line to a *fixed direction* ($\parallel \beta$), which is *not contained* in that *plane*, because we have *not* now $\mathrm{S}\alpha\beta = 0$.

(21.) In *this* case we have by (16.), equating α' or β' to 0, the expressions,

$$\text{XXXVI.} \ldots \begin{cases} \phi\rho = \beta S\alpha\rho, \quad \phi'\rho = \alpha S\beta\rho, \quad m'' = S\alpha\beta \gtrless 0, \\ \chi\rho = V.\alpha V\beta\rho = (m'' - \phi)\rho, \quad \chi'\rho = V.\beta V\alpha\rho = (m'' - \phi')\rho, \end{cases}$$

so that the equations XVIII. are reproduced; and the *depressed cubic*, or the *quadratic* XXII. in ϕ, may be written under the very simple form,

$$\text{XXXVII.} \ldots 0 = \phi\chi = \chi\phi.$$

(22.) Accordingly (comp. (5.) and (7.)), the operation ϕ here reduces an arbitrary line to the fixed direction of β, while χ destroys every line in that direction; and conversely, the operation χ reduces an arbitrary line to the fixed plane perpendicular to α, and ϕ destroys every line in that fixed plane. But because we do not *here* suppose that $m'' = 0$, the *fixed direction* of $\phi\rho$ is *not contained* in the *fixed plane* of $\chi\rho$; and (comp. (8.) and (10.)) the directions of $\phi\rho$ and $\phi'\rho$ are *not rectangular* to each other.

(23.) On the other hand, if we suppose that the *three roots* of the cubic in ϕ *vanish*, or that we have $m = 0$, $m' = 0$, and $m'' = 0$, as in XXIII., but that the equation $\psi\rho = 0$ is *not* satisfied for *all* directions of ρ, then the *binomial forms* XXXI. of $\phi\rho$ and $\phi'\rho$ reappear, but with these *two* equations of condition between their *vector constants*, whereof only *one* had occurred before:

$$\text{XXXVIII.} \ldots 0 = S\alpha\beta S\alpha'\beta' - S\alpha\beta' S\alpha'\beta, \quad 0 = S\alpha\beta + S\alpha'\beta'.$$

(24.) We have also now the expressions,

$$\text{XXXIX.} \ldots \chi\rho = -\phi\rho, \quad \chi'\rho = -\phi'\rho;$$

and the cubic in ϕ becomes simply $\phi^3 = 0$, as in XXIV.; but it is important to observe that we have *not here* (comp. (9.)) the *depressed* or *quadratic* equation $\phi^2 = 0$, since we have *now* on the contrary the two conjugate expressions,

$$\text{XL.} \ldots \phi^2\rho = \psi\rho = V\alpha\alpha' S\beta'\beta\rho, \quad \phi'^2\rho = \psi'\rho = V\beta\beta' S\alpha'\alpha\rho,$$

which do not generally vanish. And the equation $\phi^3 = 0$ is now *interpreted*, by observing that ϕ^2 here reduces *every* line to the *fixed direction* of $\phi^{-1}0$; while ϕ reduces an arbitrary vector to that *fixed plane*, all lines in which are destroyed by ϕ^2.

(25.) In this last case (23.), in which *all* the roots of the cubic in ϕ are *equal*, and are *null*, the theorem (12.), of the existence of a *common vector root* of the three equations XXVIII., may be verified by observing that we have now,

$$\text{XLI.} \ldots \phi V\alpha\alpha' = 0, \quad \psi V\alpha\alpha' = 0, \quad \chi V\alpha\alpha' = 0;$$

the *third* of which would not have here held good, unless we had supposed $m'' = 0$.

(26.) This last condition allows us to write, by (16.),

$$\text{XLII.} \ldots \phi\mu = 0, \quad \phi'\mu' = 0, \quad V\mu\lambda' = 0, \quad V\mu'\lambda = 0, \quad S\mu\mu' = 0,$$

the lines μ' and μ thus coinciding in direction with the normals λ and λ', to the planes Π and Π'; if then we write,

$$\text{XLIII.} \ldots \nu = V\lambda\lambda' \parallel V\mu\mu', \quad \text{so that} \quad S\mu\nu = 0, \quad S\mu'\nu = 0,$$

this new vector ν will be a line in the *intersection* of those *two rectangular planes*, which were lately seen (14.) to be the *loci* of the lines $\phi\rho$ and $\phi'\rho$, and are *now* (comp. (17.)) the loci of $\chi\rho$ and $\chi'\rho$; and the *three lines* μ, μ', ν (or λ', λ, ν) will compose a *rectangular system.*

(27.) *In general*, it is easy to prove that the expressions,

$$\text{XLIV.} \ldots \begin{cases} \beta = a\beta_1 + b\beta'_1, & \beta' = a'\beta_1 + b'\beta'_1, \\ \alpha_1 = a\alpha + a'\alpha', & \alpha'_1 = b\alpha + b'\alpha', \end{cases}$$

in which α, β, α', β' may be *any four vectors*, and a, b, a', b' may be *any four scalars*, conduct to the following transformations (in which ρ may be *any vector*):

$$\text{XLV.} \ldots S\alpha_1\beta_1 + S\alpha'_1\beta'_1 = S\alpha\beta + S\alpha'\beta';$$

$$\text{XLVI.} \ldots \beta_1 S\alpha_1\rho + \beta'_1 S\alpha'_1\rho = \beta S\alpha\rho + \beta' S\alpha'\rho;$$

$$\text{XLVII.} \ldots V\alpha_1\alpha'_1 . V\beta'_1\beta_1 = V\alpha\alpha' . V\beta'\beta;$$

so that the *scalar*, $S\alpha\beta + S\alpha'\beta'$; the *vector*, $\beta S\alpha\rho + \beta' S\alpha'\rho$; and the *quaternion*,* $V\alpha\alpha' . V\beta'\beta$, remain *unaltered* in value, when we pass from a *given system* of *four vectors* $\alpha\beta\alpha'\beta'$, to *another system* of four vectors $\alpha_1\beta_1\alpha'_1\beta'_1$, by expressions of the forms XLIV.

(28.) With the help of this general principle (27.), and of the remarks in (26.), it may be shown, without difficulty, that in the case (23.) the vector

* We have, in these *transformations*, examples of what may be called *Quaternion Invariants.*

constants of the binomial expression $\beta S\alpha\rho + \beta' S\alpha'\rho$ for $\phi\rho$ may, without any real loss of generality, be supposed subject to the *four* following conditions,

$$\text{XLVIII.} \ldots 0 = S\alpha\beta = S\alpha'\beta = S\beta\beta' = S\alpha'\beta';$$

which evidently conduct to these other expressions,

$$\text{XLIX.} \ldots \phi^2\rho = \beta S\alpha\beta' S\alpha'\rho, \quad \phi^3\rho = 0;$$

and thus put in evidence, in a very simple manner, the *general non-depression* of the *cubic* $\phi^3 = 0$, to the *quadratic*, $\phi^2 = 0$.

(29.) The *case*, or *sub-case*, when we have not only $m = 0$, $m' = 0$, $m'' = 0$, but *also* $\psi = 0$, and therefore $\phi^2 = 0$, as a depressed form of $\phi^3 = 0$, by the linear function $\phi\rho$ reducing itself to the monomial $\beta S\alpha\rho$, with the relation $S\alpha\beta = 0$ between its constants, has been already considered (in (10.)); and thus the consequences of the supposition III., that there are (at least) *two equal* but *null roots* of the cubic in ϕ, have been perhaps sufficiently discussed.

(30.) As regards the *other principal case* of *equal roots*, of the *cubic* equation in ϕ, namely that in which the vector constants are connected by the relation V., or by the equation of condition,

$$\text{L.} \ldots 0 = m''^2 - 4m' = (S\alpha\beta + S\alpha'\beta')^2 - 4S(V\alpha\alpha'.V\beta'\beta)$$
$$= (S\alpha\beta - S\alpha'\beta')^2 + 4S\alpha\beta' S\alpha'\beta,$$

it may suffice to remark that it conducts, by VI., or by VII. and IX., to the symbolical equation,

$$\text{LI.} \ldots 0 = \phi\Phi^2, \quad \text{if} \quad \Phi = \phi - \tfrac{1}{2}m'';$$

and that thus its *interpretation* is precisely similar to that of the analogous equation,

$$\chi\phi^2 = 0, \quad \text{where} \quad \chi = m'' - \phi, \qquad \text{XXIX.},$$

as given in (14.), and in the following sub-articles.*

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* [The following resumé of the special cases discussed in recent articles may not be superfluous:—
Assuming arbitrarily any three constant and diplanar vectors β, β', and β'', any linear vector function $\phi\rho$ may be resolved along these three vectors; thus $\phi\rho = x\beta + x'\beta' + x''\beta''$. In this expression x, x', and x'' are linear and scalar functions of ρ, and may consequently be replaced by $S\alpha\rho$, $S\alpha'\rho$, and $S\alpha''\rho$. Hence the trinomial form $\phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho + \beta'' S\alpha''\rho$ is established, and the function ϕ is made to depend upon the three vectors α, α', and α''. When these are given, ϕ is determined; and conversely, when ϕ is given, the three vectors α, α', and α'' are determinate, retaining always the same set of vectors of reference β, β', and β''. Special cases will arise when special relations connect α, α', and α''.

If $\phi\rho = 0$ for a particular value of ρ, $S\alpha\rho = S\alpha'\rho = S\alpha''\rho = 0$ are necessary consequences, and

353. When we have $m = 0$, but *not* $m' = 0$, *nor* $m''^2 = 4m'$, the *three roots* of the cubic in ϕ are *all unequal*, while *one* of them is still *null*, as before; and the *two roots* of the *quadratic* and *scalar equation*, with *real coefficients* (347),

$$\text{I.} \ldots 0 = c^2 + m''c + m',$$

which is formed from the cubic by changing ϕ to $-c$, and then dividing by c, are also necessarily *unequal*, whether they be *real* or *imaginary*. We shall find that when these *two scalar roots*, c_1, c_2, are *real*, there are then *two real directions*, ρ_1 and ρ_2, in that *fixed plane* Π which is the *locus* (351, 352) of the line $\phi\rho$, possessing the property that for each of them the *homogeneous and vector equation of the second degree*,

$$\text{II.} \ldots \mathrm{V}\rho\phi\rho = 0, \quad \text{or} \quad \phi\rho \parallel \rho,$$

is satisfied, *without* ρ *vanishing*; namely by our having, for the *first* of these two directions, the equation

$$\text{III.} \ldots \phi\rho_1 = -c_1\rho_1, \quad \text{or} \quad \phi_1\rho_1 = 0, \quad \text{if} \quad \phi_1 = \phi + c_1;$$

and for the *second* of them the analogous equation,

$$\text{IV.} \ldots \phi\rho_2 = -c_2\rho_2, \quad \text{or} \quad \phi_2\rho_2 = 0, \quad \text{if} \quad \phi_2 = \phi + c_2:$$

but that *no other direction* of the *real* and *actual vector* ρ, satisfies the equation

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therefore α, α', and α'' (if actual) must be complanar. But if α, α', and α'' are complanar, the trinomial form reduces on rearrangement to the binomial form

$$\phi\rho = (\beta + a\beta'')\mathrm{S}\alpha\rho + (\beta' + a'\beta'')\mathrm{S}\alpha'\rho,$$

provided a and a' are the scalars determined by the relation of complanarity $\alpha'' = a\alpha + a'\alpha'$. Conversely, if the trinomial reduces to a binomial form, the three vectors α, α', and α'' (if actual) must be complanar.

Further reduction to the monomial form will not be possible unless these three vectors are parallel. In general, also, as $\psi\rho = \mathrm{V}\alpha'\alpha''\mathrm{S}\beta''\beta'\rho + \mathrm{V}\alpha''\alpha\mathrm{S}\beta\beta''\rho + \mathrm{V}\alpha\alpha'\mathrm{S}\beta'\beta\rho$, $\psi\rho$ will not vanish identically, or the equation $\psi = 0$ will not be true, unless the vectors are parallel. This easily follows on replacing ρ successively by β, β', and β''.

Remarking that, when ϕ is expressible in a binomial form, it reduces those vectors which it does not annul to a fixed plane, we may assume a plane containing a pair of arbitrarily chosen vectors β and β', and consider all those functions ϕ which reduce vectors to this particular plane. Just as in the case of the trinomial form, these functions ϕ may be expressed by the type $\phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho$, and they depend on and may be determined by the vectors α and α' if the vectors β and β' are preserved unchanged.

A second root of the cubic will vanish if $m' = \mathrm{SV}\alpha\alpha'\mathrm{V}\beta'\beta$ is equal to zero. This may happen in two ways—(1) when $\mathrm{V}\alpha\alpha' = 0$, in which case the binomial is reducible to the monomial form, and $\psi\rho$ will vanish for all values of ρ, or $\psi = 0$; (2) when $\mathrm{V}\alpha\alpha'$ is *actual* and perpendicular to $\mathrm{V}\beta\beta'$, that is, when the plane of α and α' is at right angles to that of β and β'. In this latter case, the assumptions

V., except that *third* which has already been considered (351), as satisfying the *linear* and *vector equation*,

$$\text{V.} \ldots \phi\rho = 0, \quad \text{with} \quad \text{T}\rho > 0.$$

It will also be shown that these *two* directions, ρ_1, ρ_2, are not only *real*, but *rectangular*, to each other and to the *third* direction ρ, when the *linear function* $\phi\rho$ is *self-conjugate* (349, (4.)), or when the condition

$$\text{VI.} \ldots \phi'\rho = \phi\rho, \quad \text{or} \quad \text{VI}' \ldots \text{S}\lambda\phi\rho = \text{S}\rho\phi\lambda,$$

is satisfied by the *given form* of ϕ, or by the *constants* which enter into the composition of that *linear symbol*; but that when this *condition of self-conjugation* is *not* satisfied, the roots of the quadratic I. *may* happen to be *imaginary*: and that in *this* case there exists *no real direction* of ρ, for which the vector equation II. of the *second degree* is satisfied, by *actual* values of ρ, except that *one* direction which has been seen before to satisfy the *linear* equation V.

(1.) The most obvious mode of seeking to satisfy II., otherwise than through V., is to assume an expression of the form, $\rho = x\beta + x'\beta'$, and to seek thereby to satisfy the equation, $(\phi + c)\rho = 0$, with $\phi\rho = \beta\text{S}\alpha\rho + \beta'\text{S}\alpha'\rho$, by satisfying separately the two scalar equations,

$$\text{VII.} \ldots 0 = x(c + \text{S}\alpha\beta) + x'\text{S}\alpha\beta', \quad 0 = x'(c + \text{S}\alpha'\beta') + x\text{S}\alpha'\beta,$$

$\alpha = a\beta'' + b\text{V}\beta\beta'$ and $\alpha' = a'\beta'' + b'\text{V}\beta\beta'$ are legitimate when a, a', b, and b' are scalars, while β'' is some vector in the plane of β and β', and not, as before, diplanar to them. Replacing α and α', the new binomial form, $\phi\rho = (a\beta + a'\beta')\text{S}\beta''\rho + (b\beta + b'\beta')\text{SV}\beta\beta'.\rho$ is obtained, and $\psi\rho = (ab' - a'b)$ $\text{V}.\beta''\text{V}\beta\beta'\text{S}\beta'\beta\rho$.

Again, a third root will vanish if $m'' = \text{S}\alpha\beta + \text{S}\alpha'\beta' = \text{S}(a\beta + a'\beta')\beta'' = 0$, or if $\beta'' \parallel \text{V}(a\beta + a'\beta')\text{V}\beta\beta'$.

Examining separately the case in which the symbolic equation of the binomial is depressed to a quadratic, it is seen at once that it must be of the form $\phi^2 + x\phi = 0$. It cannot be of the form $\phi^2 + x\phi + y = 0$, for $\phi\rho$, $\phi^2\rho$, &c., are in the plane of β and β', and ρ is not generally in that plane. On calculation of $\phi^2\rho$, it is found that

$$\phi^2\rho + x\phi\rho = \beta(\text{S}\alpha\beta\text{S}\alpha\rho + \text{S}\alpha\beta'\text{S}\alpha'\rho + x\text{S}\alpha\rho) + \beta'(\text{S}\alpha'\beta\text{S}\alpha\rho + \text{S}\alpha'\beta'\text{S}\alpha'\rho + x\text{S}\alpha'\rho);$$

and if this vanishes for all values of ρ,

$$x = -\text{S}\alpha\beta = -\text{S}\alpha'\beta', \quad \text{and} \quad \text{S}\alpha\beta' = \text{S}\alpha'\beta = 0.$$

The second pair of equations is satisfied by assuming $\alpha = \text{V}\tau'\beta'$ and $\alpha' = \text{V}\tau\beta$, and then from the first pair $x = -\text{S}\tau'\beta'\beta = -\text{S}\tau\beta\beta'$. Hence, it is easy to see that the general solutions are $\alpha = a\text{V}\beta\beta' - x\dfrac{\beta'}{\text{V}\beta\beta'}$, and $\alpha' = a'\text{V}\beta\beta' + x\dfrac{\beta}{\text{V}\beta\beta'}$, and that $\text{V}\alpha\alpha' = -x\left(a\beta + a'\beta' + \dfrac{x}{\text{V}\beta\beta'}\right)$.

From these $x = -\frac{1}{2}m''$, and $x^2 = +m' = \frac{1}{4}m''^2$. If x vanishes, the function becomes monomial.

Of course when m is zero, the usual solution $m\rho = \psi\sigma$ of the equation $\phi\rho = \sigma$ is nugatory. In this case, since $\phi^3\rho - m''\phi^2\rho + m'\phi\rho = 0$, or $\phi^2\sigma - m''\phi\sigma + m'\phi\rho = 0$, the solution is $m'\rho = m''\sigma - \phi\sigma + \phi^{-1}0$, and it is indeterminate; if in addition $m' = 0$, the solution is $m''\rho = \sigma + \phi^{-2}0$.]

which give, by elimination of $x' : x$, the following quadratic in c,

$$\text{VIII.} \ldots (c + S\alpha\beta)(c + S\alpha'\beta') = S\alpha\beta' S\alpha'\beta,$$

which is easily seen to be only another form of I. Denoting then, as above, by c_1 and c_2, the *roots* of that quadratic I., supposed for the present to be *real*, we have these two *real directions* for ρ, in the plane Π of β, β' :

$$\text{IX.} \ldots \rho_1 = \beta(c_1 + S\alpha'\beta') - \beta' S\alpha'\beta = c_1\beta + V\alpha' V\beta'\beta\,;$$

$$\text{X.} \ldots \rho_2 = \beta(c_2 + S\alpha'\beta') - \beta' S\alpha'\beta = c_2\beta + V\alpha' V\beta'\beta\,;$$

which satisfy the equations III. and IV. In fact, the expression IX. gives

$$\phi\rho_1 = c_1\phi\beta + m'\beta = -c_1\rho_1, \quad \text{or} \quad \phi_1\rho_1 = 0,$$

because we may write it thus,

$$\text{XI.} \ldots \rho_1 = (m'' + c_1)\beta - \phi\beta = -c_2\beta - \phi\beta = -\phi_2\beta = -\phi\beta - m'c_1^{-1}\beta\,;$$

and in like manner, the expression X. may be thus written,

$$\text{XII.} \ldots \rho_2 = (m'' + c_2)\beta - \phi\beta = -c_1\beta - \phi\beta = -\phi_1\beta = -\phi\beta - m'c_2^{-1}\beta,$$

and gives,

$$\phi\rho_2 = c_2\phi\beta + m'\beta = -c_2\rho, \quad \text{or} \quad \phi_2\rho_2 = 0.$$

(2.) We may also write,

$$\text{XIII.} \ldots \rho'_1 = \beta'(c_1 + S\alpha\beta) - \beta S\alpha\beta' = c_1\beta' + V\alpha V\beta\beta' = -\phi_2\beta' \parallel \rho_1\,;$$

$$\text{XIV.} \ldots \rho'_2 = \beta'(c_2 + S\alpha\beta) - \beta S\alpha\beta' = c_2\beta' + V\alpha V\beta\beta' = -\phi_1\beta' \parallel \rho_2\,;$$

and shall then have the equations,

$$\text{XV.} \ldots \phi_1\rho'_1 = 0, \quad \phi_2\rho'_2 = 0\,;$$

but the *directions* of ρ'_1 and ρ'_2 will be the *same* by VIII. as those of ρ_1 and ρ_2, and so will furnish *no new solution* of the problem just resolved.

(3.) Since we have thus,

$$\text{XVI.} \ldots \phi_2\beta' \parallel \phi_2\beta \parallel \rho_1 \parallel \phi_1^{-1}0, \quad \text{and} \quad \text{XVI.} \ldots \phi_1\beta' \parallel \phi_1\beta \parallel \rho_2 \parallel \phi_2^{-1}0,$$

it follows that the operation ϕ_2 *reduces* every line *in the fixed plane* of $\phi\rho$ to the *fixed direction* of $\phi_1^{-1}0$; and that, in like manner, the operation ϕ_1 reduces every line, in the *same* fixed plane of $\phi\rho$, to the *other* fixed direction of $\phi_2^{-1}0$.

(4.) Hence we may write the symbolic equations,

$$\text{XVII.} \ldots \phi_1 . \phi_2\phi = 0, \quad \phi_2 . \phi_1\phi = 0,$$

in which the points may be omitted; and in fact we have the transformations,

$$\text{XVIII.} \ldots \phi_1\phi_2 = \phi_2\phi_1 = (\phi + c_1)(\phi + c_2) = \phi^2 - m''\phi + m' = \psi,$$

so that

$$\phi_1\phi_2 . \phi = \phi_2\phi_1 . \phi = \psi\phi = m = 0.$$

(5.) If we propose to form ψ_1 from ϕ_1, by the same general rule (347, XI.) by which ψ is formed from ϕ, we have

$$\text{XIX.} \ldots \psi_1 \mathrm{V}\mu\nu = \mathrm{V}.\,\phi'_1\mu\phi'_1\nu = \mathrm{V}.\,(\phi'\mu + c_1\mu)\,(\phi'\nu + c_1\nu),$$

and therefore, by the definition 350, VIII. of χ,

$$\text{XX.} \ldots \psi_1\rho = \psi\rho + c_1\chi\rho + c_1^2\rho, \quad \text{or} \quad \text{XXI.} \ldots \psi_1 = \psi + c_1\chi + c_1^2\,;$$

and in like manner,

$$\text{XXII.} \ldots \psi_2 = \psi + c_2\chi + c_2^2,$$

even if m be different from *zero*, and if c_1, c_2 be *arbitrary scalars*.

(6.) Accordingly, *without* assuming that *m vanishes*, if we operate on $\psi_1\rho$ with ϕ_1, or symbolically multiply the expression XXI. for ψ_1 by ϕ_1, we get the symbolic product,

$$\begin{aligned}\text{XXIII.} \ldots \phi_1\psi_1 &= (\phi + c_1)\,(\psi + c_1\chi + c_1^2) \\ &= \phi\psi + c_1(\phi\chi + \psi) + c_1^2(\phi + \chi) + c_1^3 \\ &= m + c_1 m' + c_1^2 m'' + c_1^3 = m_1,\end{aligned}$$

where m_1 is what the scalar m becomes, when ϕ is changed to ϕ_1, or is such that

$$\text{XXIV.} \ldots m_1 \mathrm{S}\lambda\mu\nu = \mathrm{S}\,.\,\phi'_1\lambda\phi'_1\mu\phi'_1\nu = \mathrm{S}\,.\,(\phi'\lambda + c_1\lambda)\,(\phi'\mu + c_1\mu)(\phi'\nu + c_1\nu)\,;$$

as appears by the definitions of ϕ', ψ, χ, m, m', m'', and by the relations between those symbols which have been established in recent Articles, or in the sub-articles appended to them.

(7.) Supposing now again that $m = 0$, and that c_1, c_2 are the roots of the quadratic I. in c, we have by XXIII.,

$$\text{XXV.} \ldots \phi_1\psi_1 = m_1 = 0\,; \quad \text{and in like manner} \quad \text{XXVI.} \ldots \phi_2\psi_2 = m_2 = 0,$$

if m_2 be formed from m_1, by changing c_1 to c_2.

(8.) Comparing XXV. with XXVII., we may be led to suspect the existence of an intimate connexion existing between ψ_1 and $\phi_2\phi$, since each reduces an arbitrary vector to the fixed direction of $\phi_1^{-1}0$, or of ρ_1; and in fact these two operations are *identical*, because, by XXI., and by the known relations between the symbols, we have the transformations,

$$\begin{aligned}\text{XXVII.} \ldots \psi_1 = \psi + c_1\chi + c_1^2 &= (m' - m''\phi + \phi^2) + c_1(m'' - \phi) + c_1^2 \\ &= \phi^2 - (m'' + c_1)\phi = \phi^2 + c_2\phi = \phi\phi_2\,;\end{aligned}$$

and similarly

$$\text{XXVIII.} \ldots \psi_2 = \phi^2 + c_1\phi = \phi\phi_1\,;$$

while $\psi = \phi_1\phi_2$, as before.

(9.) We have thus the *new symbolic equation*,

$$\text{XXIX.}\ldots\ \phi\phi_1\phi_2 = 0,$$

in which the *three symbolic factors*, ϕ, ϕ_1, ϕ_2 may be in any manner *grouped* and *transposed*, so that it *includes* the two equations XVII.; and in which the subject of operation is an *arbitrary vector* ρ. Its interpretation has been already partly given; but we may add, that while ϕ *reduces* every vector *to* the *fixed plane* Π, ϕ_1 reduces every line to *another* fixed plane, Π_1, and ϕ_2 reduces to a *third* plane, Π_2; thus $\phi_1\phi_2$, or $\phi_2\phi_1$, while it *destroys two lines* ρ_1, ρ_2, and therefore *every* line *in* the plane Π, *reduces* an *arbitrary* line to the *fixed direction* of the *intersection* of the *two planes* $\Pi_1\Pi_2$, which intersection must thus have the direction of $\phi^{-1}0$; and in like manner, the fixed direction ρ_1 of $\phi_1^{-1}0$, as being that to which an arbitrary vector is reduced (3.) by the compound operation $\phi_2\phi$, or $\phi\phi_2$, must be that of the intersection of the planes $\Pi\Pi_2$; and ρ_2, or $\phi_2^{-1}0$, has the direction of the intersection of $\Pi\Pi_1$; while on the other hand $\phi\phi_2$ *destroys* every line in Π_1, and $\phi\phi_1$ every line in Π_2: so that *these three planes*, with their *three lines of intersection*, are the chief elements in the *geometrical interpretation* of the equation $\phi\phi_1\phi_2 = 0$.

(10.) The *conjugate equation*,

$$\text{XXX.}\ldots\ \phi'\phi'_1\phi'_2 = 0,$$

may be interpreted in a similar way, and so conducts to the consideration of a *conjugate system* of *planes and lines*; namely the planes Π', Π'_1, Π'_2, which are the *loci* of $\phi'\rho$, $\phi'_1\rho$, $\phi'_2\rho$, while the operations $\phi'_1\phi'_2$, $\phi'_2\phi'_1$, and $\phi'\phi'_1$ *destroy* all lines in these three planes respectively, and *reduce arbitrary* lines to the fixed directions of the *intersections*, $\Pi'_1\Pi'_2$, $\Pi'_2\Pi'$, $\Pi'\Pi'_1$, which are also those of $\phi'^{-1}0$, $\phi'^{-1}_1 0$, $\phi'^{-1}_2 0$.

(11.) It is important to observe that these *three last lines* are the *normals* to the *three first planes*, Π, Π', Π''; and that, in like manner, the three *former lines* are *perpendicular* to the three *latter planes*. To prove this, it is sufficient to observe that

$$\text{XXXI.}\ldots\ S\rho'\phi\rho = S\rho\phi'\rho' = 0, \quad \text{if} \quad \phi'\rho' = 0, \quad \text{or that} \quad \phi\rho \perp \phi'^{-1}0;$$

and similarly, $\phi'\rho \perp \phi^{-1}0$, &c.*

* [More symmetrically, without assuming one root to be zero, if ϕ satisfies the symbolical cubic $(\phi + c_1)(\phi + c_2)(\phi + c_3) = 0$, it is easy to show that ρ_1, the result of operating by $(\phi + c_2)(\phi + c_3)$ on any vector ρ, is parallel to a fixed direction. For a second arbitrary vector σ may be expressed in the form $x\phi^2\rho + y\phi\rho + z\rho$, and so $(\phi + c_2)(\phi + c_3)\sigma = x\phi^2\rho_1 + y\phi\rho_1 + z\rho_1 = (xc_1^2 - yc_1 + z)\rho_1$ (since by the symbolical cubic $(\phi + c_1)\rho_1 = 0$) is likewise $\parallel \rho_1$. Thus the operators $(\phi + c_2)(\phi + c_3)$, $(\phi + c_3)(\phi + c_1)$,

(12.) Instead of eliminating $x' : x$ between the two equations VII., we might have eliminated c; which would have given this *other quadratic*,

$$\text{XXXII.} \ldots 0 = x^2 S\alpha'\beta + xx'(S\alpha'\beta' - S\alpha\beta) - x'^2 S\alpha\beta';$$

also, if $x'_1 : x_1$ and $x'_2 : x_2$ be the two values of $x' : x$, then

$$\text{XXXIII.} \ldots \rho_1 \parallel x_1\beta + x'_1\beta', \quad \rho_2 \parallel x_2\beta + x'_2\beta',$$

and

$$\text{XXXIV.} \ldots x_1x_2 : (x_1x'_2 + x_2x'_1) : x'_1x'_2 = -S\alpha\beta' : (S\alpha\beta - S\alpha'\beta') : S\alpha'\beta;$$

hence the *condition of rectangularity* of the *two lines* ρ_1, ρ_2, or $\phi_1^{-1}0$, $\phi_2^{-1}0$, is expressed by the equation,

$$\text{XXXV.} \ldots 0 = -\beta^2 S\alpha\beta' + S\beta\beta'(S\alpha\beta - S\alpha'\beta') + \beta'^2 S\alpha'\beta = S.\beta\beta' V(\beta\alpha + \beta'\alpha');$$

and consequently it is *satisfied*, if the given function ϕ be *self-conjugate* (VI.), because we have then the relation,

$$\text{XXXVI.} \ldots V\beta\alpha + V\beta'\alpha' = 0;$$

in fact the binomial form of ϕ gives (comp. 349, XXII.),

$$\text{XXXVII.} \ldots \phi'\rho - \phi\rho = (\alpha S\beta\rho - \beta S\alpha\rho) + (\alpha' S\beta'\rho - \beta' S\alpha'\rho) = V.\rho V(\beta\alpha + \beta'\alpha'),$$

which cannot vanish independently of ρ, unless the constants satisfy the condition XXXVI.

(13.) With this *condition* then, of *self-conjugation* of ϕ, we have the relation of *rectangularity*,

$$\text{XXXVIII.} \ldots S\rho_1\rho_2 = 0,^* \quad \text{or} \quad \phi_1^{-1}0 \perp \phi_2^{-1}0;$$

at least if these *directions* ρ_1 and ρ_2 be *real*, which they can easily be proved to be, as follows. The condition XXXVI. gives,

$$\text{XXXIX.} \ldots 0 = S.\alpha\alpha' V(\beta\alpha + \beta'\alpha') = \alpha^2 S\alpha'\beta + S\alpha\alpha'(S\alpha'\beta' - S\alpha\beta) - \alpha'^2 S\alpha\beta';$$

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and $(\phi + c_1)(\phi + c_2)$ reduce any vector to lines parallel respectively to three fixed directions ρ_1, ρ_2, and ρ_3. Further, by the property of the conjugate function ϕ', $(\phi' + c_1)\rho$ is a general expression for a vector perpendicular to ρ_1. In the same way $(\phi' + c_2)(\phi' + c_3)\rho$ is perpendicular to ρ_2 and also to ρ_3 and parallel to a fixed direction ρ'_1 which satisfies $(\phi' + c_1)\rho'_1 = 0$; and ρ'_2 and ρ'_3 similarly found and satisfying $(\phi' + c_2)\rho'_2$ and $(\phi' + c_3)\rho'_3 = 0$ are at right angles respectively to the planes of ρ_3, ρ_1, and of ρ_1, ρ_2. Taking unit vectors through a common origin and parallel to these fixed vectors, $U\rho_1$, $U\rho_2$, and $U\rho_3$ determine a triangle on the unit sphere and $U\rho'_1$, $U\rho'_2$, and $U\rho'_3$ are the vectors to the vertices of the supplemental triangle. Again if γ is the spin-vector defined in 349 (4.), $U(\phi + c_1)\gamma$ or its equal $U(\phi' + c_1)\gamma$ terminates at the pole of the great circle through $U\rho_1$ and $U\rho'_1$, and the point determined by $UV\gamma\phi\gamma$ is the common orthocentre of the two triangles. When the function is self-conjugate, the two supplemental triangles coincide, and consequently the solutions of $V\rho\phi_0\rho = 0$ are mutually perpendicular (16.).]

* [In general by 349 (4.), $2S\gamma\rho_1\rho_2 = S(\phi - \phi')\rho_1\rho_2 = (c_2 - c_1)S\rho_1\rho_2$. So if ρ_1 is perpendicular to ρ_2, γ, if it does not vanish, lies in their plane. Conversely, if γ lies in the plane of ρ_1 and ρ_2, either $S\rho_1\rho_2 = 0$, or $c_1 = c_2$.]

hence $$(a^2 S\alpha'\beta - \alpha'^2 S\alpha\beta')^2 = (S\alpha\alpha')^2 (S\alpha\beta - S\alpha'\beta')^2,$$

$$\alpha^2\alpha'^2(m''^2 - 4m') = \alpha^2\alpha'^2\{(S\alpha\beta - S\alpha'\beta')^2 + 4S\alpha\beta' S\alpha'\beta\}$$

$$= (\alpha^2\alpha'^2 - (S\alpha\alpha')^2)(S\alpha\beta - S\alpha'\beta')^2 + (\alpha^2 S\alpha'\beta + \alpha'^2 S\alpha\beta')^2 > 0,$$

and $$\text{XL.} \ldots (S\alpha\beta - S\alpha'\beta')^2 + 4S\alpha\beta' S\alpha'\beta = m''^2 - 4m' > 0\,;$$

so that each of the two quadratics, I. (or VIII.), and XXXII., has *real and unequal roots*: a conclusion which may also be otherwise derived, from the expressions $\beta = a\alpha + b\alpha'$, $\beta' = b\alpha + a'\alpha'$, which the condition allows us to substitute for β and β'.

(14.) The same condition XXXVI. shows that the *four vectors* $\alpha\beta\alpha'\beta'$ are *complanar*, or that we have the relations,

$$\text{XLI.} \ldots S\alpha\beta\beta' = 0, \quad S\alpha'\beta\beta' = 0, \quad V(V\alpha\alpha'.V\beta'\beta) = 0\,;$$

hence $V\alpha\alpha'$, or $\phi^{-1}0$ is now *normal* to the *plane* Π; and therefore by (13.), *when the function* ϕ *is self-conjugate* (VI.), *the three directions*,

$$\text{XLII.} \ldots \rho, \rho_1, \rho_2, \quad \text{or} \quad \phi^{-1}0, \phi_1^{-1}0, \phi_2^{-1}0,$$

compose a real and rectangular system.

(15.) In the *present series* of sub-articles (to 353), we suppose that the *three roots* of the *cubic* in ϕ are *all unequal*, the cases of *equal* roots (with $m = 0$) having been discussed in a preceding series (352); but it may be remarked, in passing, that when a *self-conjugate function* $\phi\rho$ is reducible to the *monomial form* $\beta S\alpha\rho$, we must have the relation $V\beta\alpha = 0$; and that thus the *line* β, to the *fixed direction* of which (comp. 352, (5.) and (6.)) the operation ϕ then *reduces* an *arbitrary* vector, is *perpendicular* to the *fixed plane* (352, (7.)), every line in which is *destroyed* by that operation ϕ.

(16.) In general, if ϕ be thus self-conjugate, it is evident that the *three planes* Π', Π'_1, Π'_2, which are (comp. (10.)) the *loci* of $\phi'\rho$, $\phi'_1\rho$, $\phi'_2\rho$, *coincide* with the planes Π, Π_1, Π_2, which are the loci of $\phi\rho$, $\phi_1\rho$, $\phi_2\rho$.

(17.) When ϕ is *not* self-conjugate, so that $\phi\rho$ and $\phi'\rho$ are not generally equal, it has been remarked that the *scalar quadratic* I., and therefore also the symbolical cubic in ϕ, *may* have *imaginary roots*; and that, in *this* case, the *vector equation* II. of the *second degree* cannot be satisfied by *any real direction* of ρ, except that *one* which satisfies the *linear* equation V., or causes $\phi\rho$ itself to vanish, while ρ remains real and actual. As an *example* of such *imaginary scalars*, as *roots* of I., and of what may be called *imaginary directions*, or *imaginary vectors* (comp. 214, (4.)), which *correspond* to those scalars, and are themselves *imaginary roots* of II., we may take the very simple expressions (comp. 349, XII.),

$$\text{XLIII.} \ldots \phi\rho = \mathrm{V}\gamma\rho, \quad \phi'\rho = -\mathrm{V}\gamma\rho;$$

in which γ denotes some real and given vector, and which evidently do not satisfy the condition VI., the function ϕ being *here* the *negative* of its own conjugate, so that its *self-conjugate part* ϕ_0 is *zero* (comp. 349, XIII.). We have thus,

$$\text{XLIV.} \ldots m_0 = 0, \quad m'_0 = 0, \quad m''_0 = 0, \quad \phi_0 = 0, \quad \psi_0 = 0, \quad \chi_0 = 0,$$

and consequently, by the sub-articles to 349 and 350,

$$\text{XLV.} \ldots m = 0, \quad m' = -\gamma^2, \quad m'' = 0, \quad \psi\rho = -\gamma\mathrm{S}\gamma\rho, \quad \chi\rho = -\mathrm{V}\gamma\rho;$$

the quadratic I., and its roots c_1, c_2, become therefore,

$$\text{XLVI.} \ldots c^2 - \gamma^2 = 0, \quad c_1 = +\sqrt{-1}\,.\,\mathrm{T}\gamma, \quad c_2 = -\sqrt{-1}\,.\,\mathrm{T}\gamma,$$

where $\sqrt{-1}$ is the *imaginary of algebra* (comp. 214, (3.)); thus by XX. or XXI., and XXII. we have now

$$\text{XLVII.} \ldots \psi_1\sigma = -\gamma\mathrm{S}\gamma\sigma - c_1\mathrm{V}\gamma\sigma + c_1^2\sigma = (\gamma - c_1)\mathrm{V}\gamma\sigma, \quad \psi_2\sigma = (\gamma - c_2)\mathrm{V}\gamma\sigma;$$

hence

$$\mathrm{S}\gamma\psi_1\sigma = 0, \quad \mathrm{V}\gamma\psi_1\sigma = \gamma\psi_1\sigma, \text{ \&c.},$$

and

$$\text{XLVIII.} \ldots \phi_1\psi_1\sigma = (\phi + c_1)\psi_1\sigma = (\gamma + c_1)\,(\gamma - c_1)\mathrm{V}\gamma\sigma = (\gamma^2 - c_1^2)\mathrm{V}\gamma\sigma = 0,$$

and in like manner $\qquad \text{XLVIII}'. \ldots \phi_2\psi_2\sigma = 0;$

if then we take an *arbitrary vector* σ, and derive (or rather *conceive* as derived) from it *two* (*imaginary*) *vectors* ρ_1 and ρ_2 by the (*imaginary*) *operations* ψ_1 and ψ_2, we shall have (comp. III. and IV.) the equations,

$$\text{XLIX.} \ldots \rho_1 = \psi_1\sigma, \quad \phi_1\rho_1 = 0, \quad \phi\rho_1 = -c_1\rho_1, \quad \mathrm{V}\rho_1\phi\rho_1 = 0,$$

and

$$\text{L.} \ldots \rho_2 = \psi_2\sigma, \quad \phi_2\rho_2 = 0, \quad \phi\rho_2 = -c_2\rho_2, \quad \mathrm{V}\rho_2\phi\rho_2 = 0,$$

as ones which are at least *symbolically true*. We find then that the *two imaginary directions*, ρ_1 and ρ_2, *satisfy* (at least in a symbolical sense, or as far as *calculation* is concerned) the *vector equation* II., or that ρ_1 and ρ_2 are *two imaginary vector roots* of $\mathrm{V}\rho\phi\rho = 0$; but that, because the *scalar quadratic* I. has here *imaginary roots*, this vector equation II. has (as above stated) *no real vector root* ρ, except one in the *direction* of the *given* and *real vector* γ, which satisfies the *linear* equation V., or gives $\phi\rho = 0$.

(18.) This particular example might have been more simply treated, by a less general method, as follows. We wish to satisfy the equation,

$$\text{LI.} \ldots 0 = \text{V}.\rho\text{V}\gamma\rho = \rho\text{S}\gamma\rho - \rho^2\gamma;$$

which gives, when we operate on it by $\text{V}.\gamma$ and $\text{V}.\rho$, these others,

$$\text{LII.} \ldots 0 = \text{V}\gamma\rho.\text{S}\gamma\rho, \quad 0 = \rho^2\text{V}\gamma\rho;$$

if then we wish to *avoid* supposing $\phi\rho = \text{V}\gamma\rho = 0$, we must seek to satisfy the *two scalar equations*,

$$\text{LIII.} \ldots \text{S}\gamma\rho = 0, \quad \rho^2 = 0;$$

and conversely, if we can satisfy *these* by any (real or imaginary) ρ, we shall have satisfied (really or symbolically) the *vector* equation LI. Now the *first* equation LIII. is satisfied, when we assume the expression,

$$\text{LIV.} \ldots \rho = (c + \gamma)\text{V}\gamma\sigma = \text{V}\gamma\sigma.(c - \gamma),$$

where σ is an *arbitrary vector*, and c is *any scalar*, or *symbol* subject to the *laws of scalars*; and this expression LIV. for ρ, with its transformation just assigned, gives

$$\text{LV.} \ldots \rho^2 = (c^2 - \gamma^2)(\text{V}\gamma\sigma)^2 = 0, \quad \text{if} \quad c^2 - \gamma^2 = 0;$$

the *quadratic* XLVI. is therefore reproduced, and we have the *same imaginary roots*, and *imaginary directions*, as before.

(19.) *Geometrically*, the *imaginary character* of the recent problem, of satisfying the equation $\text{V}.\rho\text{V}\gamma\rho = 0$ by any direction of ρ except that of the given line γ, is apparent from the circumstance that $\phi\rho$, or $\text{V}\gamma\rho$, is here a vector *perpendicular* to ρ, if *both* be *actual* lines; and that therefore the one cannot be also *parallel* to the other, so long as both are *real*.*

354. In the three preceding Articles, and in the sub-articles annexed, we have supposed throughout that the *absolute term* of the cubic in ϕ is *wanting*, or that the condition $m = 0$ is satisfied; in which case we have seen (351)

* Accordingly *the two imaginary directions*, above found for ρ, are easily seen to be those which in modern geometry are called the directions of *lines drawn in a given plane* (perpendicular here to the given line γ), *to the circular points at infinity*: of which supposed *directions* the *imaginary character* may be said to be precisely this, that *each* is (in the given plane) *its own perpendicular*.

[As additional examples :—

If $\phi\rho = q\rho q^{-1}$, it is obvious that $\phi' = \phi^{-1}$. This shows that the cubic of ϕ is reciprocal, and it may easily be reduced to $(\phi - 1)(\phi^2 - 2\cos 2u\,\phi + 1) = 0$ if $u = \angle\, q$. The real direction is $\text{V}q$, and the imaginary directions are the lines to the circular points at infinity in the plane perpendicular to $\text{V}q$. Again, if ϕ changes α into β, β into γ, and γ into α, the cubic is $\phi^3 - 1 = 0$. The directions are $\alpha + \omega\beta + \omega^2\gamma$, where ω is an algebraic cube root of unity.]

that it is always possible to satisfy the *linear equation* $\phi\rho = 0$, by at least *one* real and actual value of ρ (with an arbitrary scalar coefficient); or by at least *one* real direction. It will be easy now to show, that although conversely (comp. 351, (4.)) the function $\phi\rho$ *cannot vanish* for *any actual* vector ρ, *unless* we have thus $m = 0$, yet there is *always at least one real direction* for which the *vector equation of the second degree*,

$$\text{I.} \ldots \mathrm{V}\rho\phi\rho = 0,$$

which has already been considered (353) in *combination* with the *condition* $m = 0$, is satisfied; and that if the function ϕ be a *self-conjugate* one, then this equation I. is *always* satisfied by *at least three real and rectangular directions*, but *not generally* by *more* directions than *three*; although, in this *case of self-conjugation*, namely when

$$\text{II.} \ldots \phi'\rho = \phi\rho, \quad \text{or} \quad \text{II}'. \ldots \mathrm{S}\lambda\phi\rho = \mathrm{S}\rho\phi\lambda,$$

for all values of the vectors ρ and λ, the equation I. may happen to become true, for *one* real direction of ρ, and for *every* direction *perpendicular* thereto: or even for *all* possible directions, according to the particular system of *constants*, which enter into the composition of the *function* $\phi\rho$. We shall show also that the *scalar* (or *algebraic*) and *cubic equation*,

$$\text{III.} \ldots 0 = m + m'c + m''c^2 + c^3,$$

which is formed from the *symbolic* and *cubic equation* 350, I., by changing ϕ to $-c$, enters importantly into this whole theory; and that if it have *one real* and *two imaginary roots*, the *quadratic* and *vector equation* I. is satisfied by *only one real direction* of ρ; but that it may *then* be said (comp. 353, (17.)) to be satisfied *also* by *two imaginary directions*, or to have *two imaginary and vector roots*: so that this *equation* I. may be said to represent *generally* a *system of three right lines*, whereof *one* at least must be *real*. For the *case* II., the *scalar roots* of III. will be proved to be *always real*; so that if m_0, m'_0, and m''_0 be formed (as in sub-articles to 349 and 350) from the *self-conjugate part* $\phi_0\rho$ of *any linear and vector function* $\phi\rho$, as m, m', and m'' are formed from that function $\phi\rho$ *itself*, then *the new cubic*,

$$\text{IV.} \ldots 0 = m_0 + m'_0c + m''_0c^2 + c^3,$$

which thus results, *can never have imaginary roots*.

(1.) If we write,

$$\text{V.} \ldots \Phi\rho = \phi\rho + c\rho, \quad \Phi'\rho = \phi'\rho + c\rho, \quad \text{or briefly,} \quad \text{V}'. \ldots \Phi = \phi + c, \quad \Phi' = \phi' + c,$$

where c is an arbitrary scalar, and if we denote by Ψ, Ψ', and M what ψ, ψ', and m, become, by this change of ϕ to $\phi + c$ or Φ, the calculations in 353, (5.), (6.), show that we have the expressions,

$$\text{VI.}\ldots \Psi = \psi + c\chi + c^2, \quad \Psi' = \psi' + c\chi' + c^2,$$

and

$$\text{VII.}\ldots M = m + m'c + m''c^2 + c^3,$$

with

$$\text{VIII.}\ldots M = \Phi\Psi = \Psi\Phi = \Phi'\Psi' = \Psi'\Phi'.$$

(2.) Hence it may be inferred that the functions χ, χ', and the constants m', m'' become,

$$\text{IX.}\ldots \mathrm{X} = \mathrm{D}_c\Psi = \chi + 2c, \quad \mathrm{X}' = \mathrm{D}_c\Psi' = \chi' + 2c,$$

$$\text{X.}\ldots \begin{cases} M' = \mathrm{D}_c M = m' + 2m''c + 3c^2, \\ M'' = \tfrac{1}{2}\mathrm{D}_c{}^2 M = m'' + 3c; \end{cases}$$

with the verifications,

$$\text{XI.}\ldots \Phi + \mathrm{X} = \Phi' + \mathrm{X}' = M'', \quad \Phi\mathrm{X} + \Psi = \Phi'\mathrm{X}' + \Psi' = M',$$

as we had, by the sub-articles to 350,

$$\phi + \chi = \phi' + \chi' = m'', \quad \phi\chi + \psi = \phi'\chi' + \psi' = m'.$$

(3.) The *new linear symbol* Φ must satisfy the *new cubic*,

$$\text{XII.}\ldots 0 = M - M'\Phi + M''\Phi^2 - \Phi^3;$$

which accordingly can be at once derived from the *old cubic* 350, I., under the form,

$$\text{XIII.}\ldots 0 = m + m'(c - \Phi) + m''(c - \Phi)^2 + (c - \Phi)^3.$$

(4.) Now it is always possible to satisfy the condition,

$$\text{XIV.}\ldots M = 0,$$

by substituting for c a *real root* of the *scalar cubic* III.; and thereby to *reduce* the *new symbolical cubic* XII. to the *form*,

$$\text{XV.}\ldots 0 = \Phi^3 - M''\Phi^2 + M'\Phi;$$

which is precisely similar to the form,

$$0 = \phi^3 - m''\phi^2 + m'\phi, \qquad 352, \text{ II.},$$

and conducts to analogous consequences, which need not here be developed in detail, since they can easily be supplied by anyone who will take the trouble to read again the few recent series of sub-articles.

(5.) For example, unless it happen that $\Psi\rho$ *constantly vanishes*, in which case $M' = 0$, and $\Phi\rho$ (if not *identically null*) takes a *monomial form*, which is reduced to *zero* (comp. 352, (7.)) for *every direction* of ρ in a *given plane*, the operation Ψ *reduces* (comp. 351) an *arbitrary vector* to a *given direction*; and the operation Φ *destroys* every line in *that* direction: so that, *in every case*, there is *at least one real way* of satisfying the *vector equation* $\Phi\rho = 0$, and therefore also (as above asserted) the equation I., without causing ρ *itself* to vanish.

(6.) And since that equation I. may be thus written,

$$\text{XVI.} \ldots \mathrm{V}\rho\Phi\rho = 0, \quad \text{or} \quad \Phi\rho \parallel \rho,$$

we see that it can be satisfied *without* $\Phi\rho$ *vanishing*, if this *new scalar and quadratic* equation,

$$\text{XVII.} \ldots 0 = C^2 + M''C + M', \qquad \text{comp. 353, I.,}$$

have *real and unequal roots*, C_1, C_2; for if we then write,

$$\text{XVIII.} \ldots \Phi_1 = \Phi + C_1, \quad \Phi_2 = \Phi + C_2,$$

the line $\Phi\rho$ will *generally* have for its *locus* a *given plane*, and there will be *two real and distinct directions* ρ_1 and ρ_2 in that plane, for one of which $\Phi_1\rho_1 = 0$, while $\Phi_2\rho_2 = 0$ for the other, so that *each* satisfies XVI., or I.; and these are precisely the *fixed directions* of $\Psi_1\rho$ and $\Psi_2\rho$, if Ψ_1 and Ψ_2 be formed from Ψ by changing Φ to Φ_1 and Φ_2 respectively.

(7.) Cases of *equal* and of *imaginary roots* need not be dwelt on here; but it may be remarked in passing, that if the function $\phi\rho$ have the *particular form* (g being any scalar constant),

$$\text{XIX.} \ldots \phi\rho = g\rho, \quad \text{then} \quad \text{XX.} \ldots (g - \phi)^3 = 0, \quad \text{and} \quad \text{XXI.} \ldots M = (g + c)^3;$$

the cubic XIV. or III. having thus *all* its roots *equal*, and the equation I. being satisfied by *every direction* of ρ, in this particular case.

(8.) The *general existence* of a *real* and *rectangular system* of *three directions* satisfying I., when the *condition* II. is satisfied, may be proved as in 353, (14.); and it is unnecessary to dwell on the case where, by *two roots* of the cubic becoming *equal*, *all lines* in a *given plane*, and *also* the *normal* to that plane, are *vector roots* of I., with the same condition II.

(9.) And because the *quadratic*, $0 = c^2 + m''c + m'$ (353, I.), has been proved to have always *real* roots (353, (13.)) when $\phi'\rho = \phi\rho$, the *analogous* quadratic XVII. must likewise then have real roots, C_1, C_2; whence it immediately follows (comp. XII. and XIII.), that (under the same *condition of self-conjugation*) the *cubic* III. has *three* real roots, c, $c + C_1$, $c + C_2$; and therefore that (as above stated) the *other cubic* IV., which is formed from the *self-conjugate part* ϕ_0 of the *general linear and vector function* ϕ, and which may on that account be thus denoted,

$$\text{XXII.} \ldots \mathrm{M}_0 = 0, \textit{ has its roots always real.}$$

(10.) If we denote in like manner by Φ_0 the symbol $\phi_0 + c$, the equation $m = m_0 - \mathrm{S}\gamma\Phi_0\gamma$ (349, XXVI., comp. 349, XXI.) becomes,

$$\text{XXIII.} \ldots M = M_0 - \mathrm{S}\gamma\Phi_0\gamma\,;$$

whence, by comparing powers of c, we recover the relations,

$$m' = m'_0 - \gamma^2, \quad \text{and} \quad m'' = m''_0, \text{ as in 350, (1.).*}$$

(11.) On a similar plan, the equation $m\phi'\mathrm{V}\mu\nu = \mathrm{V}.\psi\mu\psi\nu$ becomes,

$$\text{XXIV.} \ldots M\Phi'\mathrm{V}\mu\nu = \mathrm{V}.\Psi\mu\Psi\nu, \qquad \text{comp. 348, (1.),}$$

in which μ and ν are *arbitrary vectors*, and c is an *arbitrary scalar*; or more fully,

$$\text{XXV.} \ldots (m + m'c + m''c^2 + c^3)(\phi' + c)\mathrm{V}\mu\nu = \mathrm{V}.\,(\psi\mu + c\chi\mu + c^2\mu)(\psi\nu + c\chi\nu + c^2\nu)\,;$$

whence follow these new equations,

$$\text{XXVI.} \ldots (m + m'\phi')\mathrm{V}\mu\nu = \mathrm{V}(\psi\mu\,.\,\chi\nu - \psi\nu\,.\,\chi\mu),$$

$$\text{XXVII.} \ldots (m' + m''\phi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\psi\nu - \nu\psi\mu + \chi\mu\,.\,\chi\nu),$$

$$\text{XXVIII.} \ldots (m'' + \phi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi\nu - \nu\chi\mu),$$

which can all be otherwise proved, and from the last of which (by changing ϕ to ψ, &c.) we can infer this other of the same kind,

$$\text{XXIX.} \ldots (m' + \psi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\phi\chi\nu - \nu\phi\chi\mu).$$

(12.) As an *example* of the existence of a *real* and *rectangular system* of *three directions* (8.), represented jointly by an equation of the form I., and of a system of *three real roots* of the *scalar cubic* III., when the *condition* II. is satisfied, let us take the form

$$\text{XXX.} \ldots \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu = \phi'\rho,$$

g being here any *real* and *given scalar*, and λ, μ any real and non-parallel

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* [If

$$\phi\rho_1 = \phi_0\rho_1 + \mathrm{V}\gamma\rho_1 = -\,c_1\rho_1, \quad \text{then} \quad \rho_1 = -\,(\phi_0 + c_1)^{-1}\mathrm{V}\gamma\rho_1$$

$$(m_0 + m'_0c_1 + m''_0c_1^2 + c_1^3)\rho_1 = -\,\mathrm{V}(\phi_0 + c_1)\gamma(\phi_0 + c_1)\rho_1 = \mathrm{V}(\phi_0 + c_1)\gamma\mathrm{V}\gamma\rho_1$$
$$= \rho_1\mathrm{S}\gamma(\phi_0 + c_1)\gamma - \gamma\mathrm{S}\rho_1(\phi_0 + c_1)\gamma = \rho_1\mathrm{S}\gamma(\phi_0 + c_1)\gamma.$$

From this, c_1 is a root of

$$(m_0 - \mathrm{S}\gamma\phi\gamma) + (m'_0 - \gamma^2)c + m''_0c^2 + c^3 = 0,$$

and this cubic must be identical with $m + m'c + m''c^2 + c^3 = 0$, as they have three roots common.]

given vectors; to which *form*, indeed, we shall soon find that *every* self-conjugate function $\phi_0\rho$ can be brought. We have now (after some reductions),

$$\text{XXXI.}\ldots \psi\rho = \mathrm{V}\lambda\rho\mu\mathrm{S}\lambda\mu - \mathrm{V}\lambda\mu\mathrm{S}\lambda\rho\mu - g(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho) + g^2\rho,$$

$$\text{XXXII.}\ldots \chi\rho = -(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho) + 2g\rho,$$

and

$$\text{XXXIII.}\ldots m = (g - \mathrm{S}\lambda\mu)(g^2 - \lambda^2\mu^2), \quad m' = -\lambda^2\mu^2 - 2g\mathrm{S}\lambda\mu + 3g^2,$$
$$m'' = -\mathrm{S}\lambda\mu + 3g;$$

where the part of $\psi\rho$ which is independent of g may be put under several other forms, such as the following,

$$\text{XXXIV.}\ldots \mathrm{V}(\lambda\rho\mu\mathrm{S}\lambda\mu - \lambda\mu\mathrm{S}\lambda\rho\mu) = \lambda\rho\mu\mathrm{S}\lambda\mu - \lambda\mu\mathrm{S}\lambda\rho\mu$$
$$= \lambda(\rho\mathrm{S}\lambda\mu + \mathrm{S}\lambda\mu\rho)\mu = \tfrac{1}{2}\lambda(\lambda\mu\rho + \rho\lambda\mu)\mu = \lambda(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho - \lambda\rho\mu)\mu, \ \&\text{c.};$$

and Φ, Ψ, X, M, M', M'' may be formed from ϕ, ψ, χ, m, m', m'', by simply changing g to $c + g$. The equation $M = 0$ has therefore here three real and unequal roots, namely the three following

$$\text{XXXV.}\ldots c = -g + \mathrm{S}\lambda\mu, \quad c + C_1 = -g + \mathrm{T}\lambda\mu, \quad c + C_2 = -g - \mathrm{T}\lambda\mu;$$

and the corresponding forms of $\Psi\rho$ are found to be,

$$\text{XXXVI.}\ldots \Psi\rho = \mathrm{V}\lambda\mu\mathrm{S}\lambda\mu\rho, \quad \Psi_1\rho = -(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda)\mathrm{S}\,.\,\rho(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda),$$
$$\Psi_2\rho = -(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda)\mathrm{S}\,.\,\rho(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda).$$

Thus $\Psi\rho$, $\Psi_1\rho$, and $\Psi_2\rho$ have in fact the three fixed and rectangular directions of $\mathrm{V}\lambda\mu$, $\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda$, and $\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda$, namely of the normal to the given plane of λ, μ, and the bisectors of the angles made by those two given lines; and *these* are accordingly the *only* directions which satisfy the vector equation of the second degree,

$$\text{XXXVII.}\ldots (\mathrm{V}\rho\phi\rho = \mathrm{V}.\,\rho\mathrm{V}\lambda\rho\mu =)\mathrm{V}\rho\lambda\mathrm{S}\mu\rho + \mathrm{V}\rho\mu\mathrm{S}\lambda\rho = 0;$$

so that this last *equation* represents (as was expected) a *system of three right lines*, in these three respective *directions*.

(13.) *In general*, if c_1, c_2, c_3 denote the *three roots* (real or imaginary) of the *cubic* equation $M = 0$, and if we write,

$$\text{XXXVIII.}\ldots \Phi_1 = \phi + c_1, \quad \Phi_2 = \phi + c_2, \quad \Phi_3 = \phi + c_3,$$

the corresponding values of Ψ will be (comp. VI.),

$$\text{XXXIX.}\ldots \Psi_1 = \psi + c_1\chi + c_1^2, \quad \Psi_2 = \psi + c_2\chi + c_2^2, \quad \Psi_3 = \psi + c_3\chi + c_3^2;$$

also we have the relations,

$$\text{XL.}\ldots\begin{cases}c_1+c_2+c_3=-m''=-\phi-\chi,\\ c_2c_3+c_3c_1+c_1c_2=+m'=\phi\chi+\psi,\\ c_1c_2c_3=-m=-\phi\psi;\end{cases}$$

whence it is easy to infer the expressions,

$$\text{XLI.}\ldots\Phi_1=(c_2-c_3)^{-1}(\Psi_3-\Psi_2),\quad \Phi_2=(c_3-c_1)^{-1}(\Psi_1-\Psi_3),$$
$$\Phi_3=(c_1-c_2)^{-1}(\Psi_2-\Psi_1);$$

which enable us to express the *functions* $\Phi_1\rho$, $\Phi_2\rho$, $\Phi_3\rho$ as *binomials* (comp. 351, &c.), when $\Psi_1\rho$, $\Psi_2\rho$, $\Psi_3\rho$ have been expressed as *monomes*, and to assign the *planes* (real or imaginary), which are the *loci* of the *lines* $\Phi_1\rho$, $\Phi_2\rho$, $\Phi_3\rho$.

(14.) Accordingly, the *three operations*, Φ, Φ_1, Φ_2, by which lines in the three lately determined *directions* (12.) are *destroyed*, or reduced to *zero*, and which at first present themselves under the forms,

$$\text{XLII.}\ldots\Phi\rho=\lambda S\mu\rho+\mu S\lambda\rho,\quad \Phi_1\rho=V\lambda\rho\mu+\rho T\lambda\mu,\quad \Phi_2=V\lambda\rho\mu-\rho T\lambda\mu,$$

are found to admit of the transformations,

$$\text{XLIII.}\ldots\Phi\rho=\frac{\Psi_2\rho-\Psi_1\rho}{2T\lambda\mu};\quad \Phi_1\rho=\frac{\Psi_2\rho-\Psi\rho}{T\lambda\mu+S\lambda\mu};\quad \Phi_2\rho=\frac{\Psi\rho-\Psi_1\rho}{T\lambda\mu-S\lambda\mu};$$

where Ψ, Ψ_1, Ψ_2 have the recent forms XXXVI., and the *loci* of $\Phi\rho$, $\Phi_1\rho$, $\Phi_2\rho$ compose a system of *three rectangular planes*.

(15.) In general, the relations (13.) give also (comp. 353, (8.)),

$$\text{XLIV.}\ldots\Psi_1=\Phi_2\Phi_3,\quad \Psi_2=\Phi_3\Phi_1,\quad \Psi_3=\Phi_1\Phi_2,$$

and

$$\text{XLV.}\ldots\Phi_1\Psi_1=\Phi_2\Psi_2=\Phi_3\Psi_3=\Phi_1\Phi_2\Phi_3=0,$$

whence also,

$$\text{XLVI.}\ldots\Psi_1\Psi_2=\Psi_2\Psi_3=\Psi_3\Psi_1=0,$$

the *symbols* (in any *one* system of this sort) admitting of being *transposed* and *grouped* at pleasure; if then the roots of $M=0$ be *real* and *unequal*, there arises a system of *three real* and *distinct planes*, which are connected with the *interpretation* of the *symbolical equation*, $\Phi_1\Phi_2\Phi_3=0$, exactly as the three planes in 353, (9.) were connected with the analogous equation $\phi\phi_1\phi_2=0$.

(16.) And when the cubic has *two imaginary roots*, it may then be said that there is *one real plane* (such as the plane $\perp\gamma$ in 353, (18.), (19.)), containing the *two imaginary directions* which then satisfy the equation I.; and *two imaginary planes*, which respectively *contain* those two directions, and *intersect* each other in *one real line* (such as the line γ in the example cited), namely the *one real vector root* of the same equation I.

355. Some additional light may be thrown upon that *vector equation* of the *second degree*, by considering the system of the *two scalar equations*,

$$\text{I.} \ldots S\lambda\rho\phi\rho = 0, \quad \text{and} \quad \text{II.} \ldots S\lambda\rho = 0,$$

and investigating the condition of the *reality* of the *two** *directions*, ρ_1 and ρ_2, by which they are generally satisfied, and for each of which the *plane* of ρ and $\phi\rho$ contains generally the *given line* λ in I., or is *normal* to the plane *locus* II. of ρ. We shall find that these two directions are *always real* and *rectangular* (except that they *may* become *indeterminate*), when the linear function ϕ is *its own conjugate*; and that *then*, if λ be a *root* ρ_0 of the *vector equation*,

$$\text{III.} \ldots V\rho\phi\rho = 0,$$

which has been already otherwise discussed, the *lines* ρ_1 and ρ_2 are *also roots* of that equation; the *general existence* (354) of a *system* of *three real and rectangular directions*, which satisfy this equation III. when $\phi'\rho = \phi\rho$, being thus *proved anew*: whence also will follow a *new proof* of the *reality* of the *scalar roots* of the *cubic* $M = 0$, for this *case* of *self-conjugation* of ϕ; and therefore of the *necessary reality* of the roots of that *other cubic*, $M_0 = 0$, which is formed (354, IV. or XXII.) from the *self-conjugate part* ϕ_0 of the *general* linear and vector function ϕ, as $M = 0$ was formed from ϕ.

(1.) Let λ, μ, ν be a system of *three rectangular vector units*, following in all respects the laws (182, 183), of the symbols i, j, k. Writing then,

$$\text{IV.} \ldots \rho = y\mu + z\nu, \quad \text{and therefore,} \quad \lambda\rho = y\nu - z\mu, \quad \phi\rho = y\phi\mu + z\phi\nu,$$

the equation II. is satisfied, and I. becomes,

$$\text{V.} \ldots 0 = y^2 S\nu\phi\mu + yz(S\nu\phi\nu - S\mu\phi\mu) - z^2 S\mu\phi\nu;$$

the roots of which quadratic will be real and unequal, if

$$\text{VI.} \ldots (S\nu\phi\nu - S\mu\phi\mu)^2 + 4S\mu\phi\nu S\nu\phi\mu > 0;$$

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* Geometrically, the equation I. represents a *cone of the second order*, with λ for *one side*, and with the *three lines* ρ which satisfy III. for three *other* sides; and II. represents a *plane* through the vertex, *perpendicular* to the side λ. The *two directions* sought are thus the *two sides*, in which this plane *cuts* the cone. [The general equation of a quadric may be written in the form $S\rho\phi\rho = 1$ where the function ϕ is self-conjugate. The cone, through its intersection with a concentric sphere, is $S\rho(\phi + r^{-2})\rho = 0$ if r is the radius of the sphere. If this touches the plane $S\lambda\rho = 0$, it is geometrically evident that the edge of contact is a principal axis of the plane section of the quadric as it passes through the points of contact of the concentric sections of the quadric and the sphere. The condition for contact is $\lambda \parallel (\phi + r^{-2})\rho$, or $S\lambda\rho\phi\rho = 0$, coupled with $S\lambda\rho = 0$. The directions of the principal axes thus determined are always real whether the plane cuts the quadric in a real curve or not.]

and the corresponding directions of ρ will be rectangular, if

$$\text{VII.} \ldots 0 = \mathrm{S}(y_1\mu + z_1\nu)\ (y_2\mu + z_2\nu) = -(y_1y_2 + z_1z_2)\,;$$

that is, if

$$\text{VIII.} \ldots \mathrm{S}\nu\phi\mu = \mathrm{S}\mu\phi\nu,$$

at least for this particular pair of vectors, μ and ν.

(2.) Introducing now the expression, $\phi\rho = \phi_0\rho + \mathrm{V}\gamma\rho$ (349, XII.), the conditions VI. and VIII. take the forms,

$$\text{IX.} \ldots (\mathrm{S}\nu\phi_0\nu - \mathrm{S}\mu\phi_0\mu)^2 + 4\mathrm{S}(\mu\phi_0\nu)^2 > 4(\mathrm{S}\gamma\mu\nu)^2, \quad \text{and} \quad \text{X.} \ldots \mathrm{S}\gamma\mu\nu = 0\,;$$

which are *both* satisfied generally when $\gamma = 0$, or $\phi = \phi' = \phi_0$; the only exception being, that the quadratic V. *may* happen to become an *identity*, by all its coefficients vanishing: but the *opposite inequality* (to VI. and IX.) can *never* hold good, that is to say, the *roots* of that quadratic can never be *imaginary*, when ϕ is thus *self-conjugate*.

(3.) On the other hand, when γ is *actual*, or $\phi'\rho$ *not* generally $= \phi\rho$, the condition X. of *rectangularity* can only *accidentally* be satisfied, namely by the given or *fixed line* γ happening to be *in the assumed plane* of μ, ν; and when the *two directions* of ρ are thus *not rectangular*, or when the scalar $\mathrm{S}\gamma\mu\nu$ does not vanish, we have only to suppose that the square of this scalar becomes *large enough*, in order to render (by IX.) those directions *coincident*, or *imaginary*.

(4.) When $\phi' = \phi$, or $\gamma = 0$, we may take μ and ν for the two rectangular directions of ρ, or may reduce the quadratic to the very simple form $yz = 0$; but, for this purpose, we must establish the relations,

$$\text{XI.} \ldots \mathrm{S}\mu\phi\nu = \mathrm{S}\nu\phi\mu = 0.$$

(5.) And if, at the same time, λ satisfies the equation III., so that $\phi\lambda \parallel \lambda$, we shall have these other scalar equations,

$$\text{XII.} \ldots 0 = \mathrm{S}\mu\phi\lambda = \mathrm{S}\nu\phi\lambda = \mathrm{S}\lambda\phi\mu = \mathrm{S}\lambda\phi\nu\,;$$

whence

$$\phi\mu \parallel \mathrm{V}\nu\lambda \parallel \mu, \quad \text{and} \quad \phi\nu \parallel \mathrm{V}\lambda\mu \parallel \nu,$$

or,

$$\text{XIII.} \ldots 0 = \mathrm{V}\lambda\phi\lambda = \mathrm{V}\mu\phi\mu = \mathrm{V}\nu\phi\nu\,;$$

λ, μ, ν thus forming (as above stated) a *system*, of *three* real and rectangular *roots*, of that *vector* equation III.

(6.) But in general, if III. be satisfied by even *two real* and *distinct* directions of ρ, the *scalar* and *cubic* equation $M = 0$ can have *no imaginary* *root*; for if those two directions give *two unequal* but *real* and *scalar values*, c_1 and c_2, for the *quotient* $-\phi\rho : \rho$, then c_1 and c_2 are *two real roots* of the cubic, of which therefore the *third* root is *also* real; and if, on the other hand, the two directions ρ_1 and ρ_2 give one *common* real and scalar value, such as c_1, for that quotient, then $\phi\rho = -c_1\rho$, or $\Phi_1\rho = (\phi + c_1)\rho = 0$, for *every line in the plane* of ρ_1, ρ_2; so that $\phi\rho$ must be of the *form*, $-c_1\rho + \beta S\rho_1\rho_2\rho$, and the *cubic* will have at least *two equal roots*, since it will take the form,

$$\text{XIV.} \ldots 0 = (c - c_1)^2 (c - c_1 + S\rho_1\rho_2\beta),$$

as is easily shown from principles and formulæ already established.

(7.) It is then proved anew, that the equation $M = 0$ has *all* its roots *real*, if $\phi'\rho = \phi\rho$; and therefore that the equation $M_0 = 0$ (as above stated) can *never* have an *imaginary root*.

(8.) And we see, at the same time, how the *scalar cubic* $M = 0$ might have been deduced from the *symbolical cubic* 350, I., or from the equation 351, I., as the condition for the vector equation III. being satisfied by any *actual* ρ; namely by observing that if $\phi\rho = -c\rho$, then $\phi^2\rho = c^2\rho$, $\phi^3\rho = -c^3\rho$, &c., and therefore $M\rho = 0$, in which ρ, by supposition, is different from zero.

(9.) Finally, as regards the *case** *of indetermination*, above alluded to, when the quadratic V. *fails* to assign any *definite values* to $y : z$, or any *definite directions* in the given plane to ρ, this case is evidently distinguished by the condition,

$$\text{XV.} \ldots S\mu\phi\mu = S\nu\phi\nu,$$

in combination with the equations XI.

356. The existence of the *Symbolic and Cubic Equation* (350), which is satisfied by the *linear and vector symbol* ϕ, suggests a *Theorem*† *of Geometrical Deformation*, which may be thus enunciated :—

"*If, by any given Mode, or Law, of Linear Derivation*, of the kind above denoted by the symbol ϕ, *we pass from any assumed Vector* ρ *to a Series of Successively Derived Vectors*, $\rho_1, \rho_2, \rho_3, \ldots$ or $\phi^1\rho, \phi^2\rho, \phi^3\rho, \ldots$; *and if, by constructing a Parallelepiped, we decompose any Line of this Series*, such as ρ_3, *into three partial or component lines*, $m\rho$, $-m'\rho_1$, $m''\rho_2$, *in the Directions of the three*

* It will be found that this *case* corresponds to the *circular sections* of a *surface of the second order*; while the less particular case in which $\phi'\rho = \phi\rho$, but *not* $S\mu\phi\mu = S\nu\phi\nu$, so that the *two directions* of ρ are *determined, real, and rectangular*, corresponds to the *axes* of a *non-circular section* of such a surface.

† This theorem was stated, nearly in the same way, in page 568 of the *Lectures*; and the problem of *inversion* of a linear and vector function was treated, in the few preceding pages (559, &c.), though with somewhat less of completeness and perhaps of simplicity than in the present Section, and with a

which precede it, as here of ρ, ρ_1, ρ_2; *then the Three Scalar Coefficients*, m, $-m'$, m'', *or the Three Ratios which these three Components of the Fourth Line* ρ_3 *bear to the Three Preceding Lines of the Series, will depend only on the given Mode or Law of Derivation, and will be entirely independent of the assumed Length and Direction of the Initial Vector.*"

(1.) As an *Example* of such successive *Derivation*, let us take the *law*,

$$\text{I.} \ldots \rho_1 = \phi\rho = -\mathrm{V}\beta\rho\gamma, \quad \rho_2 = \phi^2\rho = -\mathrm{V}\beta\rho_1\gamma, \ \&\text{c.},$$

which answers to the *construction* in 305, (1.), &c., when we suppose that β and γ are *unit-lines*. Treating them at first as *any two given vectors*, our general method conducts to the equation,

$$\text{II.} \ldots \rho_3 = m\rho - m'\rho_1 + m''\rho_2,$$

with the following values of the coefficients,

$$\text{III.} \ldots m = -\beta^2\gamma^2\mathrm{S}\beta\gamma, \quad m' = -\beta^2\gamma^2, \quad m'' = \mathrm{S}\beta\gamma;$$

as may be seen, without any new calculation, by merely changing g, λ, and μ, in 354, XXXIII., to 0, β, and $-\gamma$.

(2.) Supposing next, for comparison with 305, that

$$\text{IV.} \ldots \beta^2 = \gamma^2 = -1, \quad \text{and} \quad \mathrm{S}\beta\gamma = -l,$$

so that β, γ are unit lines, and l is the cosine of their inclination to each other, the values III. become,

$$\text{V.} \ldots m = l, \quad m' = -1, \quad m'' = -l;$$

---

slightly different notation. The *general form* of such a function which was there adopted may now be thus expressed:

$$\phi\rho = \Sigma\beta\mathrm{S}\alpha\rho + \mathrm{V}r\rho, \quad r \text{ being a given quaternion};$$

the resulting value of m was found to be (page 561),

$$m = \Sigma\mathrm{S}\alpha\alpha'\alpha''\mathrm{S}\beta''\beta'\beta + \Sigma\mathrm{S}(r\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta) + \mathrm{S}r\Sigma\mathrm{S}\alpha\beta r - \Sigma\mathrm{S}\alpha r\mathrm{S}\beta r + \mathrm{S}r\mathrm{T}r^2;$$

and the auxiliary function which we now denote by ψ was,

$$m\phi^{-1}\sigma = \psi\sigma = \Sigma\mathrm{V}\alpha\alpha'\mathrm{S}\beta'\beta\sigma + \Sigma\mathrm{V}.\alpha\mathrm{V}(\mathrm{V}\beta\sigma.r) + (\mathrm{V}\sigma r\mathrm{S}r - \mathrm{V}r\mathrm{S}\sigma r);$$

where the sum of the two last terms of $\psi\sigma$ might have been written as $\sigma r\mathrm{S}r - r\mathrm{S}\sigma r$. A student might find it an useful exercise, to prove the correctness of these expressions by the principles of the present Section. One way of doing so would be, to treat $\Sigma\beta\mathrm{S}\alpha\rho$ and r as respectively equal to $\phi_0\rho + \mathrm{V}\gamma\rho$ and $c + \epsilon$; which would transform m and $\psi\sigma$, *as above written*, into the following,

$$M_0 - \mathrm{S}(\gamma + \epsilon)(\phi_0 + c)(\gamma + \epsilon), \quad \text{and} \quad \Psi_0\sigma - (\gamma + \epsilon)\,\mathrm{S}(\gamma + \epsilon)\sigma + \mathrm{V}\sigma(\phi_0 + c)(\gamma + \epsilon);$$

that is, into the new values which the M and $\Psi\sigma$ of the Section assume, when $\Phi\rho$ takes the new value, $\Phi\rho = (\phi_0 + c)\rho + \mathrm{V}(\gamma + \epsilon)\rho$.

and the equation II., connecting *four successive lines* of the series, takes the form,

$$\text{VI.}\ldots \rho_3 = l\rho + \rho_1 - l\rho_2 \quad \text{or} \quad \text{VII.}\ldots \rho_3 - \rho_1 = -\,l(\rho_2 - \rho)\,;$$

a result which agrees with 305, (2.), since we there found that if $\rho = \text{OP}$, &c., the *interval* P_1P_3 was $= -\,l \times \text{PP}_2$.

(3.) And as regards the *inversion* of a linear and vector function (347), or the *return* from any one line ρ_1 of such a *series* to the line ρ which *precedes* it, our general method gives, for the example I., by 354, (12.),

$$\text{VIII.}\ldots \psi\rho_1 = \tfrac{1}{2}\beta(\beta\gamma\rho_1 + \rho_1\beta\gamma)\gamma,$$

and

$$\text{IX.}\ldots \rho = \phi^{-1}\rho_1 = m^{-1}\psi\rho_1 = -\,\frac{\beta\rho_1\beta^{-1} + \gamma\rho_1\gamma^{-1}}{\beta\gamma + \gamma\beta}\,;$$

a result which it is easy to verify and to interpret, on principles already explained.

357. We are now prepared to assign some new and general *Forms*, to which the *Linear* and *Vector Function* (with real constants) of a variable vector can be brought, *without* assuming its *self-conjugation*; one of the simplest of which forms is the following,

$$\text{I.}\ldots \phi\rho = \text{V}q_0\rho + \text{V}\lambda\rho\mu, \quad \text{with} \quad \text{I}'.\ldots q_0 = g + \gamma\,;$$

q_0 being here a *real* and *constant quaternion*, and λ, μ *two real* and *constant vectors*, which can *all* be *definitely assigned*, when the *particular form* of ϕ is *given*: except that λ and μ may be *interchanged* (by 295, VII.), and that *either* may be *multiplied* by any scalar, if the *other* be *divided* by the same. It will follow that the *scalar*, *quadratic*, and *homogeneous function of a vector*, denoted by $\text{S}\rho\phi\rho$, can always be thus expressed:

$$\text{II.}\ldots \text{S}\rho\phi\rho = g\rho^2 + \text{S}\lambda\rho\mu\rho\,;$$

or thus,

$$\text{II}'.\ldots \text{S}\rho\phi\rho = g'\rho^2 + 2\text{S}\lambda\rho\text{S}\mu\rho, \quad \text{if} \quad g' = g - \text{S}\lambda\mu\,;$$

a *general* and (as above remarked) *definite transformation*, which is found to be one of great utility in the theory of *Surfaces** of the *Second Order*.

(1.) Attending first to the *case* of *self-conjugate* functions $\phi_0\rho$, from which we can pass to the *general* case by merely adding the *term* $\text{V}\gamma\rho$, and

* In the theory of such surfaces, the two constant and real vectors, λ and μ, have the directions of what are called the *cyclic normals*.

supposing (in virtue of what precedes) that $\alpha_1\alpha_2\alpha_3$ are three *real* and *rectangular vector-units*, and $c_1c_2c_3$ three *real scalars* (the roots of the cubic $M_0 = 0$), such that

$$\text{III.} \ldots \phi_1\alpha_1 = (\phi_0 + c_1)\alpha_1 = 0, \quad \phi_2\alpha_2 = (\phi_0 + c_2)\alpha_2 = 0, \quad \phi_3\alpha_3 = (\phi_0 + c_3)\alpha_3 = 0,$$

we may write

$$\text{IV.} \ldots \rho = -(\alpha_1 S\alpha_1\rho + \alpha_2 S\alpha_2\rho + \alpha_3 S\alpha_3\rho),$$

and therefore

$$\text{V.} \ldots \phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho\,;$$

so that

$$\text{VI.} \ldots \begin{cases} \phi_1\rho = (c_2 - c_1)\alpha_2 S\alpha_2\rho + (c_3 - c_1)\alpha_3 S\alpha_3\rho, \\ \phi_2\rho = (c_3 - c_2)\alpha_3 S\alpha_3\rho + (c_1 - c_2)\alpha_1 S\alpha_1\rho, \\ \phi_3\rho = (c_1 - c_3)\alpha_1 S\alpha_1\rho + (c_2 - c_3)\alpha_2 S\alpha_2\rho, \end{cases}$$

the *binomial forms* of ϕ_1, ϕ_2, ϕ_3 being thus put in evidence.

(2.) We have thus the general but *scalar* expressions:

$$\text{VII.} \ldots -\rho^2 = (S\alpha_1\rho)^2 + (S\alpha_2\rho)^2 + (S\alpha_3\rho)^2\,;$$

$$\begin{aligned} \text{VIII.} \ldots S\rho\phi\rho = S\rho\phi_0\rho &= c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2 \\ &= -c_1\rho^2 + (c_2 - c_1)\,(S\alpha_2\rho)^2 + (c_3 - c_1)\,(S\alpha_3\rho)^2 \\ &= -c_2\rho^2 - (c_2 - c_1)\,(S\alpha_1\rho)^2 + (c_3 - c_2)\,(S\alpha_3\rho)^2 \\ &= -c_3\rho^2 - (c_3 - c_1)\,(S\alpha_1\rho)^2 - (c_3 - c_2)\,(S\alpha_2\rho)^2 : \end{aligned}$$

in which it is in general permitted to assume that

$$\text{IX.} \ldots c_1 < c_2 < c_3, \quad \text{or that} \quad \text{X.} \ldots c_2 - c_1 = 2e^2, \quad c_3 - c_2 = 2e'^2,$$

e and e' being *real* scalars, and the numerical *coefficients* being introduced for a motive of convenience which will presently appear.

(3.) Comparing the last but one of the expressions VIII. with II'., we see that we may bring $S\rho\phi\rho$ to the proposed form II., by assuming,

$$\text{XI.} \ldots \lambda = e\alpha_1 + e'\alpha_3, \quad \mu = -e\alpha_1 + e'\alpha_3, \quad g = S\lambda\mu - c_2 = -\tfrac{1}{2}(c_1 + c_3),$$

because $S\lambda\mu = e^2 - e'^2 = c_2 - \frac{1}{2}(c_1 + c_3)$.

(4.) But in general (comp. 349, (4.)) we cannot have, for *all* values of ρ,

$$\text{XII.} \ldots S\rho\phi\rho = S\rho\phi'\rho, \quad \text{unless} \quad \text{XIII.} \ldots \phi_0\rho = \phi'_0\rho,$$

that is, unless the *self-conjugate parts* of ϕ and ϕ' be *equal*; we can therefore *infer* from II. that $\phi_0\rho = g\rho + V\lambda\rho\mu$, because $V\lambda\rho\mu = V\mu\rho\lambda$ = its own conjugate; and thus the transformation I. is proved to be *possible*, and *real*.

(5.) Accordingly, with the values XI. of λ, μ, g, the expression,

$$\text{XIV.} \ldots \phi_0\rho = g\rho + V\lambda\rho\mu = \rho(g - S\lambda\mu) + \lambda S\mu\rho + \mu S\lambda\rho,$$

becomes,

$$\text{XV.} \ldots \phi_0\rho = -c_2\rho + (e'a_3 + ea_1)\,S(e'a_3 - ea_1)\rho + (e'a_3 - ea_1)\,S(e'a_3 + ea_1)\rho$$
$$= -c_2\rho - 2e^2a_1Sa_1\rho + 2e'^2a_3Sa_3\rho\,;$$

which agrees, by X., with VI.

(6.) Conversely if g, λ, and μ be constants such that $\phi_0\rho = g\rho + V\lambda\rho\mu$, then $\phi_0V\lambda\mu = g'V\lambda\mu$, where $g' = g - S\lambda\mu$, as before; hence $-g'$ must be one of the three roots c_1, c_2, c_3 of the cubic $M_0' = 0$, and the normal to the plane of λ, μ must have one of the three directions of a_1, a_2, a_3; if then we assume, on trial, that this plane is that of a_1, a_3, and write accordingly,

$$\text{XVI.} \ldots \lambda = aa_1 + a'a_3, \quad \mu = ba_1 + b'a_3, \quad \phi_2\rho = \lambda S\mu\rho + \mu S\lambda\rho,$$

we are, by VI., to seek for scalars $aa'bb'$ which shall satisfy the three conditions,

$$\text{XVII.} \ldots 2ab = c_1 - c_2, \quad 2a'b' = c_3 - c_2, \quad ab' + ba' = 0\,;$$

but these give

$$\text{XVIII.} \ldots (2ab')^2 = (2ba')^2 = (c_3 - c_2)\,(c_2 - c_1),$$

so that if the transformation is to be a *real* one, we must suppose that $c_2 - c_1$ and $c_3 - c_2$ are either *both positive*, as in IX., or else *both negative*; or in other words, we must so *arrange* the three real roots of the cubic, that c_2 may be (algebraically) *intermediate* in value between the other two. Adopting then the *order* IX., with the values X., we satisfy the conditions XVII. by supposing that

$$\text{XIX.} \ldots a' = b' = e', \quad a = -b = e\,;$$

and are thus led back from XVI. to the expressions XI., as the *only real ones* for λ, μ, and g which render possible the transformations I. and II.; except that λ and μ may be *interchanged*, &c., as before.

(7.) We see, however, that *in an imaginary sense* there exist *two other solutions* of the problem, to *transform* $\phi\rho$ and $S\rho\phi\rho$ as above; for if we retain the order IX., and equate g' in II′. to either $-c_1$ or $-c_3$, we may in each case conceive the corresponding *sum of two squares* in VIII. as being the *product* of *two imaginary* but *linear factors*; the *planes* of the two *imaginary pairs* of vectors which result being *real*, and perpendicular respectively to a_1 and a_3.

(8.) And if the real expression XIV. for $\phi_0\rho$ be *given*, and it be required to pass from it to the expression V., with the order of inequality IX., the investigation in 354, (12.) enables us at once to establish the formulæ:

$$\text{XX.} \ldots c_1 = -g - \mathrm{T}\lambda\mu, \quad c_2 = -g + \mathrm{S}\lambda\mu, \quad c_3 = -g + \mathrm{T}\lambda\mu;$$

$$\text{XXI.} \ldots a_1 = \mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda), \quad a_2 = \mathrm{UV}\lambda\mu, \quad a_3 = \mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda);$$

in which however it is permitted to change the sign of any one of the three vector units. Accordingly the expressions XI. give,

$$\mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu = 2e^2 = c_2 - c_1, \quad \mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu = 2e'^2 = c_3 - c_2, \quad \mathrm{S}\lambda\mu = g + c_2;$$

$$\mathrm{T}\lambda = \mathrm{T}\mu, \quad \lambda - \mu = 2ea_1, \quad \mathrm{V}\lambda\mu = -2ee'a_3a_1 = \mp 2ee'a_2, \quad \lambda + \mu = 2e'a_3.$$

(9.) We have also the two *identical* transformations,

$$\text{XXII.} \ldots \mathrm{S}\lambda\rho\mu\rho = \rho^2\mathrm{T}\lambda\mu + \{(\mathrm{S}\lambda\mu\rho)^2 + (\mathrm{S}\lambda\rho\mathrm{T}\mu + \mathrm{S}\mu\rho\mathrm{T}\lambda)^2\}(\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)^{-1},$$

$$\text{XXIII.} \ldots \mathrm{S}\lambda\rho\mu\rho = -\rho^2\mathrm{T}\lambda\mu - \{(\mathrm{S}\lambda\mu\rho)^2 + (\mathrm{S}\lambda\rho\mathrm{T}\mu - \mathrm{S}\mu\rho\mathrm{T}\lambda)^2\}(\mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu)^{-1},$$

which hold good for *any three vectors*, λ, μ, ρ, and may (among other ways) be deduced, through the expressions XX. and XXI., from II. and VIII.

(10.) Finally, as regards the expressions VI. for $\phi_1\rho$, &c., if we denote the corresponding forms of $\psi\rho$ by $\psi_1\rho$, &c., we have (comp. 354, (15.)) these other expressions, which are as usual (comp. 351, &c.) of *monomial form*:

$$\text{XXIV.} \ldots \begin{cases} \psi_1\rho = \phi_2\phi_3\rho = (c_2 - c_1)(c_1 - c_3)a_1\mathrm{S}a_1\rho; \\ \psi_2\rho = \phi_3\phi_1\rho = (c_3 - c_2)(c_2 - c_1)a_2\mathrm{S}a_2\rho; \\ \psi_3\rho = \phi_1\phi_2\rho = (c_1 - c_3)(c_3 - c_2)a_3\mathrm{S}a_3\rho; \end{cases}$$

and which verify the relations 354, XLI., and several other parts of the whole foregoing theory.

358. The *general linear* and *vector function* $\phi\rho$ of a *vector* has been seen (347, (1.)) to contain, at least implicitly, *nine scalar constants*; and accordingly the expression 357, I. involves that number, namely *four* in the term $\mathrm{V}q_0\rho$, on account of the constant *quaternion* q_0, and *five* in the other term $\mathrm{V}\lambda\rho\mu$, *each* of the *two unit-vectors*, $\mathrm{U}\lambda$ and $\mathrm{U}\mu$, counting as *two scalars*, and the *tensor* $\mathrm{T}\lambda\mu$ as *one more*. But a *self-conjugate* linear and vector function, or the *self-conjugate part* $\phi_0\rho$ of the *general* function $\phi\rho$, involves only *six* scalar constants; either because *three* disappear with the *term* $\mathrm{V}\gamma\rho$ of $\phi\rho$; or because the

condition of self-conjugation, $\Sigma V\beta a = 2\gamma = 0$ (comp. 349, XXII. and 353, XXXVI.), which arises when we take for $\phi\rho$ the form $\Sigma\beta Sa\rho$ (347, XXXI.), is equivalent to a system of *three scalar equations*, connecting the *nine constants*. And for the same reason the *general quadratic* but *scalar* function, $S\rho\phi\rho$, involves in like manner only *six* scalar constants. Accordingly there enter only six such constants into the expressions 357, II., II′., V., VIII., XIV.; c_1, c_2, c_3, for instance, being *three* such, and the rectangular unit system a_1, a_2, a_3 answering to *three others*. The following *other general transformations* of $S\rho\phi\rho$ and $\phi_0\rho$, although not quite so simple as 357, II. and XIV., involve the *same number* (*six*) of scalar constants, and deserve to be briefly considered: namely the forms,

$$\text{I.} \ldots S\rho\phi\rho = a(Va\rho)^2 + b(S\beta\rho)^2;$$

$$\text{II.} \ldots \phi_0\rho = -aaVa\rho + b\beta S\beta o;$$

in which a, b are two real scalars, and a, β are two real unit-vectors. We shall merely set down the leading formulæ, leaving the reader to supply the analysis, which at this stage he cannot find difficult.

(1.) In accomplishing the reduction of the expressions,

$$S\rho\phi\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2, \qquad 357, \text{VIII.}$$

and

$$\phi_0\rho = c_1a_1Sa_1\rho + c_2a_2Sa_2\rho + c_3a_3Sa_3\rho, \qquad 357, \text{V.},$$

to these new forms I. and II., it is found that, if the result is to be a *real* one, $-a$ must be *that* root of the scalar cubic $M_0 = 0$, the *reciprocal* of which is algebraically *intermediate*, between the reciprocals of the other two. It is therefore convenient *here* to assume this *new condition*, respecting the *order* of the *inequalities*,

$$\text{III.} \ldots c_1^{-1} > c_2^{-1} > c_3^{-1};$$

which will indeed *coincide* with the arrangement 357, IX., if the three roots c_1, c_2, c_3, be *all positive*, but will be *incompatible* with it in every *other* case.

(2.) This being laid down (or even, if we choose, the *opposite* order being taken), the (real) values of a, b, a, β may be thus expressed:

$$\text{IV.} \ldots a = -c_2, \quad b = c_1 - c_2 + c_3;$$

$$\text{V.} \ldots a = xa_1 + za_3, \quad \beta = x'a_1 + z'a_3;$$

in which

$$\text{VI.} \ldots x^2 = \frac{c_1^{-1} - c_2^{-1}}{c_1^{-1} - c_3^{-1}} \quad z^2 = \frac{c_2^{-1} - c_3^{-1}}{c_1^{-1} - c_3^{-1}};$$

$$\text{VII.} \ldots \frac{c_1 x}{x'} = \frac{c_3 z}{z'} = b(xx' + zz') = -bS\alpha\beta = (\text{say})\, b';$$

$$\text{VIII.} \ldots b'^2 = c_1 c_2^{-1} c_3 b = c_1^2 x^2 + c_3^2 z^2; \quad \text{IX.} \ldots x^2 + y^2 = x'^2 + y'^2 = 1;$$

$$\text{X.} \ldots bx'z' = c_2 xz;$$

$$\text{XI.} \ldots c_1 x^2 + c_3 z^2 = c_1 c_2^{-1} c_3 = b^{-1} b'^2 = b(S\alpha\beta)^2, \quad c_1 c_3 = -ab(S\alpha\beta)^2;$$

$$\text{XII.} \ldots b'\beta = -b\beta S\alpha\beta = c_1 x\alpha_1 + c_3 z\alpha_3; \ \&c.$$

(3.) And there result the transformations:

$$\text{XIII.} \ldots \phi_2\rho = (c_1 - c_2)\alpha_1 S\alpha_1\rho + (c_3 - c_2)\alpha_3 S\alpha_3\rho$$
$$= -c_2(x\alpha_1 + z\alpha_3)\ S(x\alpha_1 + z\alpha_3)\rho + \frac{c_2}{c_1 c_3}(xc_1\alpha_1 + zc_3\alpha_3)\ S(xc_1\alpha_1 + zc_3\alpha_3)\rho;$$

$$\text{XIV.} \ldots \phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho$$
$$= c_2(x\alpha_1 + z\alpha_3)\ V(x\alpha_1 + z\alpha_3)\rho + \frac{c_2}{c_1 c_3}(xc_1\alpha_1 + zc_3\alpha_3)\ S(xc_1\alpha_1 + zc_3\alpha_3)\rho;$$

$$\text{XV.} \ldots S\rho\phi\rho = -c_2(V(x\alpha_1 + z\alpha_3)\rho)^2 + \frac{c_2}{c_1 c_3}(S(xc_1\alpha_1 + zc_3\alpha_3)\rho)^2;$$

which last, *if* $c_1 c_3$ be *positive*, gives this *other real form*,

$$\text{XVI.} \ldots S\rho\phi\rho = \frac{c_2}{c_1 c_3} N\{S(xc_1\alpha_1 + zc_3\alpha_3)\rho + (c_1 c_3)^{\frac{1}{2}}\, V(x\alpha_1 + z\alpha_3)\rho\};$$

x^2 and z^2 being determined by the expressions VI.

(4.) Those expressions allow us to change the *sign* of $z : x$, and thereby to determine a *second pair* of *real unit lines*, α' and β', which may be substituted for α and β in the forms I. and II.; the order of inequalities III. (or the opposite order), and the values IV. of a and b, remaining unchanged. We have therefore the *double transformations*:

$$\text{XVII.} \ldots S\rho\phi\rho = -c_2(V\alpha\rho)^2 + (c_1 - c_2 + c_3)\,(S\beta\rho)^2 = -c_2(V\alpha'\rho)^2$$
$$+ (c_1 - c_2 + c_3)\,(S\beta'\rho)^2;$$

$$\text{XVIII.} \ldots \phi_0\rho = c_2\alpha V\alpha\rho + (c_1 - c_2 + c_3)\beta S\beta\rho = c_2\alpha' V\alpha'\rho + (c_1 - c_2 + c_3)\beta' S\beta'\rho.$$

(5.) If either of the two connected *forms* I. and II. had been *given*, we might have proposed to *deduce* from it the values of $c_1 c_2 c_3$, and of $\alpha_1\alpha_2\alpha_3$, by the *general method* of this Section. We should thus have had the cubic,

$$\text{XIX.} \ldots 0 = M_0 = (c + a)\{c^2 + (a - b)c - ab(S\alpha\beta)^2\};$$

and because the quadratic $(c + a)^{-1}M_0 = 0$ may be thus written,

$$\text{XX.} \ldots (c^{-1} + a^{-1})^2 (\mathrm{S}\alpha\beta)^2 - (c^{-1} + a^{-1}) (a^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-1}) + a^{-2}(\mathrm{V}\alpha\beta)^2 = 0,$$

it gives two real values of $c^{-1} + a^{-1}$, one positive and the other negative; if then we *arrange* the reciprocals of the three roots of $M_0 = 0$ in the order III., we have the expressions,

$$\text{XXI.} \ldots \begin{cases} c_1 = \tfrac{1}{2}(b - a) + \tfrac{1}{2}ab \sqrt{(a^{-2} + 2a^{-1}b^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-2})}; \quad c_2 = -a; \\ c_3 = \tfrac{1}{2}(b - a) - \tfrac{1}{2}ab \sqrt{(a^{-2} + 2a^{-1}b^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-2})}; \end{cases}$$

the signs of the radical being determined by the condition that $(c_1 - c_3) : ab(\mathrm{S}\alpha\beta)^2 = c_1^{-1} - c_3^{-1} > 0$. Accordingly these expressions for the roots agree evidently with the former results, IV. and XI., because $\mathrm{S}.(\alpha\beta)^2 = 2(\mathrm{S}\alpha\beta)^2 - 1$.

(6.) The *roots* c_1, c_2, c_3 being thus known, the same general method gives for the *directions* of $\alpha_1, \alpha_2, \alpha_3$ the *versors* of the following expressions (or of their negatives):

$$\text{XXII.} \ldots \begin{cases} \psi_1\rho = ac_3^{-1}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)\ \mathrm{S}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)\rho; \\ \psi_2\rho = ab\mathrm{V}\alpha\beta\mathrm{S}\beta\alpha\rho; \\ \psi_3\rho = ac_1^{-1}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)\ \mathrm{S}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)\rho; \end{cases}$$

of which the *monomial forms* may again be noted, and which give,

$$\text{XXII}'. \ldots \alpha_1 = \pm\mathrm{U}(c_3\alpha + b\beta\mathrm{S}\alpha\beta), \quad \alpha_2 = \pm\mathrm{UV}\alpha\beta, \quad \alpha_3 = \pm\mathrm{U}(c_1\alpha + b\beta\mathrm{S}\alpha\beta).$$

(7.) Accordingly the expressions in (2.) give (if we suppose $\alpha_3\alpha_1 = +\alpha_2$),

$$\text{XXIII.} \ldots c_3\alpha + b\beta\mathrm{S}\alpha\beta = (c_3 - c_1)x\alpha_1, \quad \mathrm{V}\alpha\beta = (x'z - xz')\alpha_2, \quad c_1\alpha + b\beta\mathrm{S}\alpha\beta = (c_1 - c_3)z\alpha_3;$$

and as an additional verification of the *consistency* of the various parts of this whole theory, it may be observed (comp. 357, XXIV.), that

$$\text{XXIV.} \ldots -ac_3^{-1}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)^2 = (c_2 - c_1)(c_1 - c_3), \quad ab(\mathrm{V}\alpha\beta)^2 = (c_3 - c_2)(c_2 - c_1), \quad -ac_1^{-1}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)^2 = (c_1 - c_3)(c_3 - c_2).$$

(8.) As regards the *second transformations*, XVII. and XVIII., it is easy to prove that we may write,

$$\text{XXV.} \ldots (c_3 - c_1)\alpha' = b\beta\alpha\beta - a\alpha, \quad (c_3 - c_1)\beta' = a\alpha\beta\alpha - b\beta,$$

$$\text{XXVI.} \ldots - (c_3 - c_1)^2 = (b\beta\alpha\beta - a\alpha)^2 = (a\alpha\beta\alpha - b\beta)^2;$$

so that we have the following equation,

$$\text{XXVII.} \ldots (a(\mathrm{V}\alpha\rho)^2 + b(\mathrm{S}\beta\rho)^2)\,(a^2 + 2ab\mathrm{S}.(\alpha\beta)^2 + b^2)$$
$$= a(\mathrm{V}(b\beta\alpha\beta - a\alpha)\rho)^2 + b(\mathrm{S}(a\alpha\beta\alpha - b\beta)\rho)^2,$$

which is true for any vector ρ, any two unit lines α, β, and any two scalars a, b.

(9.) Accordingly it is evident from (4.), that α_1, α_3 must be the bisectors of the angles made by α, α', and also of those made by β, β'; and the expressions XXV. may be thus written (because $b - a = c_1 + c_3$),

$$\text{XXVIII.} \ldots (c_3 - c_1)\alpha' = (c_3 + c_1)\alpha + 2b\beta\mathrm{S}\alpha\beta, \quad (c_1 - c_3)\beta' = (c_1 + c_3)\beta - 2a\alpha\mathrm{S}\alpha\beta;$$

whence, by XXIII., we may write,

$$\text{XXIX.} \ldots \alpha + \alpha' = 2x\alpha_1, \quad \alpha - \alpha' = 2z\alpha_3;$$

so that α_1 bisects the internal angle, and α_3 the external angle, of the lines α, α'.

(10.) At the same time we have these other expressions,

$$\text{XXX.} \ldots (c_1 - c_3)\,(\beta + \beta') = 2(c_1\beta - a\alpha\mathrm{S}\alpha\beta), \quad (c_3 - c_1)\,(\beta - \beta') = 2(c_3\beta - a\alpha\mathrm{S}\alpha\beta);$$

which can easily be reduced to the simple forms,

$$\text{XXXI.} \ldots \beta + \beta' = 2x'\alpha_1, \quad \beta - \beta' = 2z'\alpha_3,$$

with the recent meanings of the coefficients x' and z'.

(11.) And although, for the sake of obtaining *real transformations*, we have supposed (comp. III.) that

$$\text{XXXII.} \ldots (c_1^{-1} - c_2^{-1})\,(c_2^{-1} - c_3^{-1}) > 0,$$

because the assumed relation $\alpha = x\alpha_1 + z\alpha_3$ between the three unit vectors $\alpha\alpha_1\alpha_3$, whereof the two latter are rectangular, gives $x^2 + z^2 = 1$, as in IX., so that each of the two expressions VI. involves the other, and their comparison gives the ratio,

$$\text{XXXIII.} \ldots x^2 : z^2 = (c_1^{-1} - c_2^{-1}) : (c_2^{-1} - c_3^{-1}),$$

yet we see that, *without* this inequality XXXII. existing, the foregoing transformations hold good in an *imaginary* (or merely *symbolical*) *sense*: so that we may say, in general, that the functions $\mathrm{S}\rho\phi\rho$ and $\phi_0\rho$ can be brought to the *forms* I. and II. in *six distinct ways*, whereof *two* are *real*, and the *four others* are *imaginary*.

(12.) It may be added that the first equation XXII. admits of being replaced by the following,

$$\text{XXXIV.} \ldots \psi_1\rho = -bc_1^{-1}(c_1\beta - a\alpha S\alpha\beta)\, S(c_1\beta - a\alpha S\alpha\beta)\rho,$$

with a corresponding form for $\psi_3\rho$; and that thus, instead of XXII'., we are at liberty to write the expressions,

$$\text{XXXV.} \ldots \alpha_1 = U(c_1\beta - a\alpha S\alpha\beta), \quad \alpha_2 = UV\alpha\beta, \quad \alpha_3 = U(c_3\beta - a\alpha S\alpha\beta),$$

for the rectangular unit system, deduced from I. or II.

359. If we call, as we naturally may, the expressions

$$\text{I.} \ldots \phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho, \qquad 357, \text{V.},$$

and

$$\text{II.} \ldots S\rho\phi\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2, \qquad 357, \text{VIII.},$$

the *Rectangular Transformations* of the *Functions* $\phi_0\rho$ and $S\rho\phi\rho$, then by another *geometrical analogy*, which will be seen when we come to speak briefly of the theory of *Surfaces of the Second Order*, we may call the expressions,

$$\text{III.} \ldots \phi_0\rho = g\rho + V\lambda\rho\mu, \qquad 357, \text{XIV.},$$

and

$$\text{IV.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho, \qquad 357, \text{II.},$$

the *Cyclic** *Transformations* of the same two functions; and may say that the two other and more recent expressions,

$$\text{V.} \ldots \phi_0\rho = -a\alpha V\alpha\rho + b\beta S\beta\rho, \qquad 358, \text{II.},$$

and

$$\text{VI.} \ldots S\rho\phi\rho = a(V\alpha\rho)^2 + b(S\beta\rho)^2, \qquad 358, \text{I.},$$

are *Focal*† *Transformations* of the same. We have already shown (357) how to exchange *rectangular forms* with *cyclic* ones; and also (358) how to pass from *rectangular* expressions to *focal* ones, and reciprocally: but it may be worth while to consider briefly the mutual relations which exist, between *cyclic* and *focal* expressions, and the modes of passing from either to the other.

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* Compare the Note to Art. 357.

† It will be found that the *two real vectors* α, α', of 358, are the *two real focal lines* of the *real or imaginary cone*, which is *asymptotic* to the *surface of the second order*, $S\rho\phi\rho = \text{const.}$

(1.) To pass from IV. to VI., or from the *cyclic* to the *focal* form, we may first accomplish the *rectangular* transformation II., with the values 357, XX., and XXI., of c_1, c_2, c_3, and of a_1, a_2, a_3, the order of inequality being assumed to be

$$\text{VII.} \ldots c_3 > c_2 > c_1, \qquad \text{as in 357, IX.;}$$

and then shall have (comp. 358, XV.) the following expressions:

$$\text{VIII.} \ldots 4S\rho\phi\rho = \{S.\rho(c_1^{\frac{1}{2}}(U\lambda - U\mu) + c_3^{\frac{1}{2}}(U\lambda + U\mu))\}^2 - \{V.\rho(c_1^{\frac{1}{2}}(U\lambda + U\mu) + c_3^{\frac{1}{2}}(U\lambda - U\mu))\}^2;$$

$$\text{VIII}'. \ldots 4S\rho\phi\rho = -\{S.\rho((-c_1)^{\frac{1}{2}}(U\lambda - U\mu) + (-c_3)^{\frac{1}{2}}(U\lambda + U\mu))\}^2 + \{V.\rho((-c_1)^{\frac{1}{2}}(U\lambda + U\mu) + (-c_3)^{\frac{1}{2}}(U\lambda - U\mu))\}^2$$

$$\text{IX.} \ldots (c_3 - c_2)^2 S\rho\phi\rho = \{V.\rho(c_3^{\frac{1}{2}} V\lambda\mu + (-c_2)^{\frac{1}{2}}(\lambda T\mu + \mu T\lambda))\}^2 + \{S.\rho((-c_2)^{\frac{1}{2}} V\lambda\mu - c_3^{\frac{1}{2}}(\lambda T\mu + \mu T\lambda))\}^2;$$

$$\text{X.} \ldots (c_2 - c_1)^2 S\rho\phi\rho = -\{V.\rho((-c_1)^{\frac{1}{2}} V\lambda\mu + c_2^{\frac{1}{2}}(\lambda T\mu - \mu T\lambda))\}^2 - \{S.\rho(-c_2^{\frac{1}{2}} V\lambda\mu + (-c_1)^{\frac{1}{2}}(\lambda T\mu - \mu T\lambda))\}^2;$$

in which it is to be remembered that (by 357, XX.),

$$\text{XI.} \ldots c_1 = -g - T\lambda\mu, \quad c_2 = -g + S\lambda\mu, \quad c_3 = -g + T\lambda\mu;$$

and of which *all* are *symbolically true*, or give (as in IV.) the *real value* $g\rho^2 + S\lambda\rho\mu\rho$ for $S\rho\phi\rho$, if g, λ, μ, ρ be *real*. And in *this* symbolical *sense*, although they have been written down as *four*, they only *count* as *three* distinct *focal transformations*, of a *given* and *real cyclic form*; because the expression VIII′. is an *immediate* consequence of VIII.; and other formulæ IX′. and X′. might in like manner be at once derived from IX. and X.

(2.) But if we wish to confine ourselves to *real focal forms*, there are then *four cases* to be considered, in *each* of which *some one* of the *four equations* VIII. VIII′. IX. X. is to be adopted, to the *exclusion* of the *other three*. Thus, if

$$\text{XII.} \ldots c_3 > c_2 > c_1 > 0, \quad \text{and therefore} \quad c_1^{-1} > c_2^{-1} > c_3^{-1} > 0,$$

the *form* VIII. is the *only real* one. If

$$\text{XIII.} \ldots c_3 > c_2 > 0 > c_1, \quad c_2^{-1} > c_3^{-1} > 0 > c_1^{-1}, \text{ then X. is the real form.}$$

If $\quad \text{XIV.} \ldots c_3 > 0 > c_2 > c_1, \quad c_3^{-1} > 0 > c_1^{-1} > c_2^{-1},$ the only real form is IX.

Finally if $$\text{XV.} \ldots 0 > c_3 > c_2 > c_1, \quad 0 > c_1^{-1} > c_2^{-1} > c_3^{-1},$$

that is, if *all* the roots of the cubic $M_0 = 0$ be *negative*, then VIII′. is the form to be adopted, under the same condition of *reality*.

(3.) When *all* the roots c are *positive*, or in the case when VIII. is the *real focal form*, the unit lines α, β in VI. may be thus expressed:

$$\text{XVI.} \ldots \begin{cases} \alpha = \frac{1}{2}\left(\frac{c_3}{c_2}\right)^{\frac{1}{2}}(\mathrm{U}\lambda - \mathrm{U}\mu) + \frac{1}{2}\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}}(\mathrm{U}\lambda + \mathrm{U}\mu); \\ \beta = \frac{1}{2}\left(\frac{c_1}{b}\right)^{\frac{1}{2}}(\mathrm{U}\lambda - \mathrm{U}\mu) + \frac{1}{2}\left(\frac{c_3}{b}\right)^{\frac{1}{2}}(\mathrm{U}\lambda + \mathrm{U}\mu); \end{cases}$$

with $$b = c_1 - c_2 + c_3 \text{ as before (358, IV.).}$$

(4.) In the same case VIII., the expressions for $4\mathrm{S}\rho\phi\rho$ may be written (comp. 358, XVI.) under either of these two *other real forms*:

$$\text{XVII.} \ldots 4\mathrm{S}\rho\phi\rho = \mathrm{N}\{(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})\, \rho \,.\, \mathrm{U}\lambda + (c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})\, \mathrm{U}\mu \,.\, \rho\};$$

$$\text{XVII}'. \ldots 4\mathrm{S}\rho\phi\rho = \mathrm{N}\{(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})\, \mathrm{U}\lambda \,.\, \rho + (c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})\, \rho \,.\, \mathrm{U}\mu\};$$

so that if we write, for abridgment,

$$\text{XVIII.} \ldots \iota_0 = \tfrac{1}{2}(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})\mathrm{U}\lambda, \quad \kappa_0 = \tfrac{1}{2}(c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})\mathrm{U}\mu,$$

we shall have, briefly,

$$\text{XIX.} \ldots \mathrm{S}\rho\phi\rho = \mathrm{N}(\iota_0\rho + \rho\kappa_0) = \mathrm{N}(\rho\iota_0 + \kappa_0\rho).$$

(5.) Or we may make

$$\text{XX.} \ldots \iota = \tfrac{1}{2}(c_1^{-\frac{1}{2}} + c_3^{-\frac{1}{2}})\mathrm{U}\lambda, \quad \kappa = \tfrac{1}{2}(c_1^{-\frac{1}{2}} - c_3^{-\frac{1}{2}})\mathrm{U}\mu, \quad \text{whence} \quad \kappa^2 - \iota^2 = c_1^{-\frac{1}{2}}c_3^{-\frac{1}{2}};$$

and shall then have the transformation,

$$\text{XXI.} \ldots \mathrm{S}\rho\phi\rho = \mathrm{N}\frac{\iota\rho + \rho\kappa}{\kappa^2 - \iota^2},$$

which may be compared with the equation 281, XXIX. of the *ellipsoid*, and for the *reality* of which form, or of its two *vector constants*, ι, κ, it is necessary that the roots c of the cubic should all be *positive* as above.

(6.) It was lately shown (in 358, (8.), &c.) how to pass from a *given* and *real focal form* to a *second* of the same kind, with its *new real unit lines* α', β' in the *same plane* as the two *old* or *given lines*, α, β; but we have not yet shown how to pass from a *focal* form to a *cyclic* one, although the *converse* passage has been recently discussed. Let us then now suppose that the *form* VI. is *real* and *given*, or that the two scalar constants a, b, and the two unit vectors α, β, have real and given values; and let us seek to reduce this expression VI. to the earlier form IV.

(7.) We might, for this purpose, begin by assuming that

$$\text{XXII.} \ldots c_1^{-1} > c_2^{-1} > c_3^{-1}, \quad \text{as in 358, III.};$$

which would give the expressions 358, XXI. and XXII., for $c_1c_2c_3$ and $a_1a_2a_3$, and so would supply the *rectangular transformation*, from which we could pass, as before, to the *cyclic* one.

(8.) But to vary a little the analysis, let us now suppose that the *given focal form* is some one of the four following (comp. (1.)):

$$\text{XXIII.} \ldots S\rho\phi\rho = (S\beta_0\rho)^2 - (V\alpha_0\rho)^2; \quad \text{XXIII}'. \ldots S\rho\phi\rho = (V\alpha_0\rho)^2 - (S\beta_0\rho)^2;$$

$$\text{XXIV.} \ldots S\rho\phi\rho = (S\beta_0\rho)^2 + (V\alpha_0\rho)^2; \quad \text{XXIV}'. \ldots S\rho\phi\rho = -(V\alpha_0\rho)^2 - (S\beta_0\rho)^2;$$

in each of which α_0 and β_0 are conceived to be *given* and *real vectors*, but *not* generally *unit lines*; and which are in fact the *four cases* included under the *general form*, $a(V\alpha\rho)^2 + b(S\beta\rho)^2$, according as the scalars a and b are positive or negative. It will be sufficient to consider the two cases, XXIII. and XXIV., from which the two others will follow at once.

(9.) For the case XXIII. we easily derive the *real cyclic transformation*,

$$\begin{aligned}\text{XXV.} \ldots S\rho\phi\rho &= (S\beta_0\rho)^2 - (S\alpha_0\rho)^2 + \alpha_0^2\rho^2 \\ &= S(\beta_0 + \alpha_0)\rho \,.\, S(\beta_0 - \alpha_0)\rho + \alpha_0^2\rho^2 \\ &= g\rho^2 + S\lambda\rho\mu\rho = (g - S\lambda\mu)\rho^2 + 2S\lambda\mu S\mu\rho,\end{aligned}$$

where

$$\text{XXVI.} \ldots \lambda = \beta_0 + \alpha_0, \quad \mu = \tfrac{1}{2}(\beta_0 - \alpha_0), \quad g = \tfrac{1}{2}(\alpha_0^2 + \beta_0^2);$$

and the equations 357, (9.) enable us to pass thence to the two *imaginary cyclic forms*.

(10.) For example, if the proposed function be (comp. XIX.),

$$\text{XXVII.} \ldots S\rho\phi\rho = N(\iota_0\rho + \rho\kappa_0) = (S(\iota_0 + \kappa_0)\rho)^2 - (V(\iota_0 - \kappa_0)\rho)^2,$$

we may write

$$\alpha_0 = \iota_0 - \kappa_0, \quad \beta_0 = \iota_0 + \kappa_0, \quad \lambda = 2\iota_0, \quad \mu = \kappa_0, \quad g = \iota_0^2 + \kappa_0^2;$$

and the required transformation is (comp. 336, XI.),

$$\text{XXVIII.} \ldots \mathrm{N}(\iota_0\rho + \rho\kappa_0) = (\iota_0^2 + \kappa_0^2)\rho^2 + 2\mathrm{S}\iota_0\rho\kappa_0\rho.$$

(11.) To treat the case XXIV. by our general method, we may omit for simplicity the subindices $_0$, and write simply (comp. V. and VI.) the expressions,

$$\text{XXIX.} \ldots \phi\rho = -a\mathrm{V}a\rho + \beta\mathrm{S}\beta\rho, \quad \text{and} \quad \text{XXX.} \ldots \mathrm{S}\rho\phi\rho = (\mathrm{V}a\rho)^2 + (\mathrm{S}\beta\rho)^2;$$

in which, however, it is to be observed that a and β, though *real vectors*, are *not now unit lines* (8.). Hence, because $-a\mathrm{V}a\rho = a\mathrm{S}a\rho - a^2\rho$, we easily form the expressions:

$$\text{XXXI.} \ldots m = a^2(\mathrm{S}a\beta)^2, \quad m' = a^2(a^2 - \beta^2) - (\mathrm{S}a\beta)^2, \quad m'' = \beta^2 - 2a^2;$$

$$\text{XXXII.} \ldots \begin{cases} \psi\rho = \mathrm{V}a\beta\mathrm{S}\beta a\rho - a^2(a\mathrm{V}a\rho + \beta\mathrm{V}\beta\rho) + a^4\rho \\ \quad\;\; = \mathrm{V}a\rho\beta\mathrm{S}a\beta + a(a^2 - \beta^2)\mathrm{S}a\rho, \\ \chi\rho = -(a\mathrm{S}a\rho + \beta\mathrm{S}\beta\rho) + (\beta^2 - a^2)\rho; \end{cases}$$

and therefore

$$\text{XXXIII.} \ldots M = (c - a^2)\,(c^2 + (\beta^2 - a^2)c - (\mathrm{S}a\beta)^2),$$

and

$$\text{XXXIV.} \ldots \Psi\rho = \mathrm{V}a\rho\beta\mathrm{S}a\beta + (\beta^2 - a^2)\,(c\rho - a\mathrm{S}a\rho) - c(a\mathrm{S}a\rho + \beta\mathrm{S}\beta\rho) + c^2\rho$$
$$= (a(a^2 - \beta^2 - c) + \beta\mathrm{S}a\beta)\mathrm{S}a\rho + (a\mathrm{S}a\beta - c\beta)\mathrm{S}\beta\rho + (c^2 + (\beta^2 - a^2)c - (\mathrm{S}a\beta)^2)\rho.$$

(12.) Introducing then a real and positive scalar constant, r, such that

$$\text{XXXV.} \ldots r^4 = (a^2 - \beta^2)^2 + 4(\mathrm{S}a\beta)^2 = (a^2 + \beta^2)^2 + 4(\mathrm{V}a\beta)^2$$
$$= a^4 + (a\beta)^2 + (\beta a)^2 + \beta^4 = a^4 + 2\mathrm{S}\,.\,(a\beta)^2 + \beta^4$$
$$= a^{-2}(a^3 + \beta a\beta)^2 = \beta^{-2}(\beta^3 + a\beta a)^2 = \&\text{c.},$$

in which (by 199, &c.),

$$\mathrm{S}\,.\,(a\beta)^2 = (\mathrm{S}a\beta)^2 + (\mathrm{V}a\beta)^2 = 2(\mathrm{S}a\beta)^2 - a^2\beta^2 = 2(\mathrm{V}a\beta)^2 + a^2\beta^2,$$

the roots of $M = 0$ admit of being expressed as follows:

$$\text{XXXVI.} \ldots c_1 = \tfrac{1}{2}(a^2 - \beta^2 + r^2), \quad c_2 = a^2, \quad c_3 = \tfrac{1}{2}(a^2 - \beta^2 - r^2);$$

and when they are thus arranged, we have the inequalities,

$$\text{XXXVII.} \ldots c_1 > 0 > c_3 > c_2, \quad c_1^{-1} > 0 > c_2^{-1} > c_3^{-1}.$$

(13.) The corresponding forms of $\Psi\rho$ are the three monomial expressions,

$$\text{XXXVIII.} \ldots \begin{cases} \psi_1\rho = c_3^{-1}(\alpha c_3 + \beta S\alpha\beta)\, S(\alpha c_3 + \beta S\alpha\beta)\rho, \quad \psi_2\rho = V\alpha\beta S\beta\alpha\rho, \\ \psi_3\rho = c_1^{-1}(\alpha c_1 + \beta S\alpha\beta)\, S(\alpha c_1 + \beta S\alpha\rho)\rho\,; \end{cases}$$

which may be variously transformed and verified, and give the three following rectangular vector units,

$$\text{XXXIX.} \ldots \alpha_1 = U(\alpha c_3 + \beta S\alpha\beta), \quad \alpha_2 = UV\alpha\beta, \quad \alpha_3 = U(\alpha c_1 + \beta S\alpha\beta)\,;$$

in connexion with which it is easy to prove that

$$\text{XL.} \ldots \begin{cases} T(\alpha c_3 + \beta S\alpha\beta) = (-c_3)^{\frac{1}{2}} (c_1 - c_2)^{\frac{1}{2}} (c_1 - c_3)^{\frac{1}{2}} = r\,(c_1 - c_2)^{\frac{1}{2}} (-c_3)^{\frac{1}{2}}, \\ TV\alpha\beta = \qquad (c_1 - c_2)^{\frac{1}{2}} (c_3 - c_2)^{\frac{1}{2}}\,; \\ T(\alpha c_1 + \beta S\alpha\beta) = c_1^{\frac{1}{2}}(c_3 - c_2)^{\frac{1}{2}} (c_1 - c_3)^{\frac{1}{2}} = r\,(c_3 - c_2)^{\frac{1}{2}} c_1^{\frac{1}{2}}\,; \end{cases}$$

the radicals being all real, by XXXVII.

(14.) We have thus, for the *given focal form* XXX., the *rectangular transformation*,

$$\text{XLI.} \ldots S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2$$
$$= \frac{c_1(S(\alpha c_3 + \beta S\alpha\beta)\rho)^2}{-c_3(c_1 - c_2)r^2} + \frac{c_2(S\alpha\beta\rho)^2}{(c_1 - c_2)\,(c_3 - c_2)} + \frac{c_3(S(\alpha c_1 + \beta S\alpha\beta)\rho)^2}{c_1(c_3 - c_2)r^2},$$

or briefly,

$$\text{XLII.} \ldots S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2 = c_1(S\,.\,\rho U(\alpha c_3 + \beta S\alpha\beta)\rho)^2$$
$$+ a^2(S\,.\,\rho UV\alpha\beta)^2 + c_3(S\,.\,\rho U(\alpha c_1 + \beta S\alpha\beta))^2\,;$$

in which the first term is positive, but the two others are negative, and c_1, c_3 are the roots of the quadratic,

$$\text{XLIII.} \ldots 0 = c^2 + (\beta^2 - \alpha^2)c - (S\alpha\beta)^2.$$

(15.) We have also the parallelisms,

$$\text{XLIV.} \ldots \alpha c_3 + \beta S\alpha\beta \parallel \beta c_1 - \alpha S\alpha\beta, \quad \alpha c_1 + \beta S\alpha\beta \parallel \beta c_3 - \alpha S\alpha\beta,$$

because

$$c_1 c_3 = -\,(S\alpha\beta)^2\,;$$

and may therefore write,

$$\text{XLV.} \ldots S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2 = c_1(S\,.\,\rho U(\beta c_1 - \alpha S\alpha\beta))^2$$
$$+ a^2(S\,.\,\rho UV\alpha\beta)^2 + c_3(S\,.\,\rho U(\beta c_3 - \alpha S\alpha\beta))^2\,;$$

while

$$\text{XLVI.} \ldots \mathrm{T}(\beta c_1 - a\mathrm{S}a\beta) = rc_1^{\frac{1}{2}}(c_1 - c_2)^{\frac{1}{2}}, \quad \mathrm{T}(\beta c_3 - a\mathrm{S}a\beta) = r(-c_3)^{\frac{1}{2}}(c_3 - c_2)^{\frac{1}{2}},$$

and $r = (c_1 - c_3)^{\frac{1}{2}}$, with real radicals as before.

(16.) Multiplying then by $r^2(\mathrm{TV}a\beta)^2$, or by $(c_1 - c_2)(c_1 - c_3)(c_3 - c_2)$, we obtain this new equation,

$$\begin{aligned} \text{XLVII.} \ldots (c_1 - c_3)\{(\mathrm{TV}a\beta)^2((\mathrm{V}a\rho)^2 + (\mathrm{S}\beta\rho)^2) - a^2(\mathrm{S}a\beta\rho)^2\} \\ = (c_3 - a^2)(c_1\mathrm{S}\beta\rho - \mathrm{S}a\beta\mathrm{S}a\rho)^2 - (c_1 - a^2)(c_3\mathrm{S}\beta\rho - a\mathrm{S}a\beta)^2; \end{aligned}$$

which is only another way of expressing the same rectangular transformation as before, but has the advantage of being freed from *divisors*.

(17.) Developing the second member of XLVII., and dividing by $c_1 - c_3$, we obtain this new transformation:

$$\begin{aligned} \text{XLVIII.} \ldots (\mathrm{TV}a\beta)^2\mathrm{S}\rho\phi\rho = -(\mathrm{V}a\beta)^2((\mathrm{V}a\rho)^2 + (\mathrm{S}\beta\rho)^2) \\ = a^2(\mathrm{S}a\beta\rho)^2 - (\mathrm{S}a\beta)^2(\mathrm{S}a\rho)^2 + 2a^2\mathrm{S}a\beta\mathrm{S}a\rho\mathrm{S}\beta\rho + C(\mathrm{S}\beta\rho)^2; \end{aligned}$$

in which we have written for abridgment,

$$\text{XLIX.} \ldots C = c_1c_3 - a^2(c_1 + c_3).$$

(18.) The expressions XXXVI. for c_1, c_3 give thus,

$$\text{L.} \ldots C = -a^4 - (\mathrm{V}a\beta)^2;$$

and accordingly, when this value is substituted for C in XLVIII., that *equation* becomes an *identity*, or holds good for *all values* of the *three vectors*, a, β, ρ; as may be proved* in various ways.

(19.) Admitting this result, we see that for the mere establishment of the equation XLVII., it is *not necessary* that c_1 and c_2 should be roots of the *particular quadratic* XLIII. It is sufficient, for *this* purpose, that they should be roots of *any* quadratic,

$$\text{LI.} \ldots c^2 + Ac + B = 0, \quad \text{with the } \textit{relation} \quad \text{LII.} \ldots Aa^2 + B + a^4 + (\mathrm{V}a\beta)^2 = 0,$$

between its coefficients. But when we *combine* with this the *condition of rectangularity*, $a_3 \perp a_1$, or

$$\text{LIII.} \ldots 0 = \mathrm{S}.(c_1\beta - a\mathrm{S}a\beta)(c_3\beta - a\mathrm{S}a\beta) = A(\mathrm{S}a\beta)^2 + B\beta^2 + a^2(\mathrm{S}a\beta)^2,$$

* Many such proofs, or verifications, as the one here alluded to, are purposely left, at this stage, as exercises, to the student.

we obtain thus a *second* relation, which gives *definitely*, for the two coefficients, the values,

$$\text{LIV.} \ldots A = \beta^2 - \alpha^2, \quad B = -(S\alpha\beta)^2;$$

and so conducts, in a new way, to the equation XLIII.

(20.) In this manner, then, we *might* have been led to perceive the truth of the rectangular transformation XLVII., with the *quadratic* equation XLIII. of which c_1 and c_3 are roots, without having previously found the *cubic* XXXIII., of which the quadratic is a *factor*, and of which the *other* root is $c_2 = \alpha^2$. But if we had not employed the *general method* of the present Section, which conducted us to form *first* that *cubic* equation, there would have been nothing to *suggest* the *particular form* XLVII., which could thus have only been by some sort of *chance* arrived at.

(21.) The values of $\alpha_1\alpha_2\alpha_3$ give also (comp. 357, VII.),

$$\text{LV.} \ldots -\rho^2 = (S\,.\,\rho U(\beta c_1 - \alpha S\alpha\beta))^2 + (S\,.\,\rho UV\alpha\beta)^2 + (S\,.\,\rho U(\beta c_3 - \alpha S\alpha\beta))^2;$$

that is, by XL. and XLVI.,

$$\begin{aligned}\text{LVI.} \ldots c_1c_3(c_1 - c_3)(\rho^2(V\alpha\beta)^2 - (S\alpha\beta\rho)^2) &= c_3(c_3 - \alpha^2)(c_1S\beta\rho - S\alpha\beta S\alpha\rho)^2 \\ &\quad - c_1(c_1 - \alpha^2)(c_3S\beta\rho - S\alpha\beta S\alpha\rho)^2;\end{aligned}$$

and accordingly the values XXXVI. of c_1, c_3 enable us to express each member of this last equation under the common form, $-c_1c_3(c_1 - c_3)$ $(\alpha S\beta\rho - \beta S\alpha\rho)^2$.

(22.) Comparing the recent inequalities $c_1 > c_3 > c_2$ (XXXVII.) with the arrangement 357, IX., we see, by 357, (6.), that for the *real cyclic transformation* (6.) at present sought, the plane of λ, μ is to be perpendicular to α_3 (and not to α_2, as in 357, (3.), &c.). We are therefore to eliminate $(c_3S\beta\rho - S\alpha\beta S\alpha\rho)^2$ between the equations XLVII. and LVI., which gives (after a few reductions) the real transformation:

$$\begin{aligned}\text{LVII.} \ldots ((S\alpha\beta)^2 - c_1\beta^2)((V\alpha\rho)^2 + (S\beta\rho)^2) - (c_1 - \alpha^2)(S\alpha\beta)^2\rho^2 \\ = (c_1S\beta\rho - S\alpha\beta S\alpha\rho)^2 - c_1(S\alpha\beta\rho)^2 \\ = S\,.\,\rho(c_1\beta - \alpha S\alpha\beta + c_1^{\frac{1}{2}}V\alpha\beta)\,S\,.\,\rho(c_1\beta - \alpha S\alpha\beta - c_1^{\frac{1}{2}}V\alpha\beta);\end{aligned}$$

which is of the kind required.

(23.) Accordingly it will be found that the following equation,

$$\begin{aligned}\text{LVIII.} \ldots ((S\alpha\beta)^2 - c\beta^2)(V\alpha\rho)^2 + (c - \alpha^2)(c(S\beta\rho)^2 - \rho^2S(\alpha\beta)^2) \\ = (cS\beta\rho - S\alpha\beta S\alpha\rho)^2 - c(S\alpha\beta\rho)^2,\end{aligned}$$

is an *identity*, or that it holds good for *all values* of the scalar c, and of the vectors α, β, ρ; since, by addition of $c(\mathrm{V}\alpha\beta)^2\rho^2$ on both sides, it takes this *obviously* identical form,

$$\text{LIX.}\ \ldots\ ((\mathrm{S}\alpha\beta)^2 - c\beta^2)(\mathrm{S}\alpha\rho)^2 + c(c-\alpha^2)(\mathrm{S}\beta\rho)^2 = (c\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 \\ - c(\alpha\mathrm{S}\beta\rho - \beta\mathrm{S}\alpha\rho)^2;$$

so that if c_1 be *either root* of the quadratic XLIII., or if $c_1(c_1 - \alpha^2) = (\mathrm{S}\alpha\beta)^2 - c_1\beta^2$, the *transformation* LVII. is at least *symbolically valid*: but we must take, as above, the *positive root* of that quadratic for c_1, if we wish that transformation to be a *real* one, as regards the *constants* which it employs. And if we had *happened* (comp. (20.)) to perceive this *identity* LIX., and to see its transformation LVIII., we might have been in that way led to form the *quadratic* XLIII., without having previously formed the *cubic* XXXIII.

(24.) Already, then, we see how to obtain *one* of the two *imaginary cyclic transformations* of the *given focal form* XXX., namely by changing c_1 to c_3 in LVII.; and the *other* imaginary transformation is had, on principles before explained, by eliminating $(\mathrm{S}\alpha\beta\rho)^2$ between XLVII. and LVI.; a process which easily conducts to the equation,

$$\text{LX.}\ \ldots\ (\mathrm{V}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2 + \alpha^2\rho^2 = (c_1 - c_3)^{-1}\{c_1^{-1}(c\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 \\ - c_3^{-1}(c_3\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2\},$$

where the second member is the *sum of two squares* (c_1 being > 0, but $c_3 < 0$), as the second expression LVII. would also become, if c_1 were replaced by c_3. Accordingly, each member of LX. is equal to $(\mathrm{S}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2$, if c_1, c_3 be the roots of *any* quadratic LI., with only the *one* condition,

$$\text{LXI.}\ \ldots\ c_1c_3 = B = -\ (\mathrm{S}\alpha\beta)^2;$$

which however, when *combined* with the *condition of rectangularity* LIII., suffices to give also $A = \beta^2 - \alpha^2$, as in LIV., and so to lead us back to the quadratic XLIII., which had been deduced by the general method, as a *factor* of the *cubic* equation XXXIII.

(25.) And since the values XXXVI. of c_1, c_3 reduce, as above, the second member of LX. to the simple form $(\mathrm{S}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2$, we may thus, or even without employing the *roots* c_1, c_3 at all, deduce the following expression for the last imaginary cyclic transformation:

$$\text{LXII.}\ \ldots\ \mathrm{S}\rho\phi\rho = (\mathrm{V}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2 = -\alpha^2\rho^2 + \mathrm{S}(\alpha + \sqrt{-1}\beta)\rho\,.\,\mathrm{S}(\alpha - \sqrt{-1}\beta)\rho,$$

where $\sqrt{-1}$ is the imaginary of algebra (comp. 214, (6.)); while the *real scalar* r^4 of XXXV. may at the same time receive the connected *imaginary form*,

$$\text{LXIII.} \ldots r^4 = (\alpha^2 - \beta^2)^2 + 4(S\alpha\beta)^2 = (\alpha + \sqrt{-1}\beta)^2 (\alpha - \sqrt{-1}\beta)^2.$$

(26.) Finally, as regards the passage from the *given form* XXX., to a *second real focal form* (comp. 358, (4.)), or the transformation,

$$\text{LXIV.} \ldots (V\alpha\rho)^2 + (S\beta\rho)^2 = (V\alpha'\rho)^2 + (S\beta'\rho)^2,$$

in which α' and β' are real vectors, distinct from $\pm\alpha$ and $\pm\beta$, but in the same plane with them, it may be sufficient (comp. 358, (8.)), to write down the formulæ:

$$\text{LXV.} \ldots r^2\alpha' = -(\alpha^3 + \beta\alpha\beta), \quad r^2\beta' = -(\beta^3 + \alpha\beta\alpha),$$

with the same real value of r^2 as before; so that (by XXXV., &c.) we have the relations,

$$\text{LXVI.} \ldots T\alpha' = T\alpha, \quad T\beta' = T\beta, \quad S\alpha'\beta' = S\alpha\beta;$$

$$\text{LXVII.} \ldots \begin{cases} r^2(\alpha + \alpha') = \alpha(r^2 - \alpha^2 + \beta^2) - 2\beta S\alpha\beta = -2(\alpha c_3 + \beta S\alpha\beta) \parallel \alpha_1, \\ r^2(\alpha - \alpha') = \alpha(r^2 + \alpha^2 - \beta^2) + 2\beta S\alpha\beta = 2(\alpha c_1 + \beta S\alpha\beta) \parallel \alpha_3; \end{cases}$$

$$\text{LXVIII.} \ldots \begin{cases} r^2(\beta + \beta') = \beta(r^2 + \alpha^2 - \beta^2) - 2\alpha S\alpha\beta = 2(\beta c_1 - \alpha S\alpha\beta) \parallel \alpha_1, \\ r^2(\beta - \beta') = \beta(r^2 - \alpha^2 + \beta^2) + 2\alpha S\alpha\beta = -2(\beta c_3 - \alpha S\alpha\beta) \parallel \alpha_3. \end{cases}$$

(27.) We have then the identity,

$$\text{LXIX.} \ldots (V(\alpha^3 + \beta\alpha\beta)\rho)^2 + (S(\beta^3 + \alpha\beta\alpha)\rho)^2 \\ = (\alpha^4 + 2S.(\alpha\beta)^2 + \beta^4)((V\alpha\rho)^2 + (S\beta\rho)^2);$$

with which may be combined this other of the same kind,

$$\text{LXX.} \ldots -(V(\alpha^3 - \beta\alpha\beta)\rho)^2 + (S(\beta^3 - \alpha\beta\alpha)\rho)^2 \\ = (\alpha^4 - 2S.(\alpha\beta)^2 + \beta^4)(-(V\alpha\rho)^2 + (S\beta\rho)^2),$$

which enables us to pass from the focal form XXIII., to a second real focal form, with its two new lines in the same plane as the two old ones: and it may be noted that we can pass from LXIX. to LXX., by changing α to $\alpha\sqrt{-1}$.

360. Besides the rectangular, cyclic, and focal transformations of $S\rho\phi\rho$, which have been already considered, there are others, although perhaps of less importance: but we shall here mention only two of them, as specimens, whereof one may be called the *Bifocal*, and the other the *Mixed Transformation*.

(1.) The two lines α, α', of 359, LXV., being called *focal lines*,* an expression which shall introduce them *both* may be called on that account a *bifocal transformation*.

(2.) Retaining then the value 359, XXXV. of r^4, and introducing a new auxiliary constant e, which shall satisfy the equation,

$$\text{I.} \ldots \beta^2 - \alpha^2 = r^2 e, \quad \text{and therefore} \quad \text{II.} \ldots 4(S\alpha\beta)^2 = r^4(1 - e^2),$$

so that

$$\text{III.} \ldots 4e^2(S\alpha\beta)^2 = (1 - e^2)(\beta^2 - \alpha^2)^2,$$

the first equation 359, LXV. gives,

$$\text{IV.} \ldots r^2(e\alpha - \alpha') = 2\beta S\alpha\beta, \qquad \text{V.} \ldots r^2(eS\alpha\rho - S\alpha'\rho) = 2S\alpha\beta S\beta\rho;$$

and therefore, with the form 359, XXX. of $S\rho\phi\rho$,

$$\begin{aligned} \text{VI.} \ldots (1 - e^2)S\rho\phi\rho &= (1 - e^2)\left((V\alpha\rho)^2 + (S\beta\rho)^2\right) \\ &= (1 - e^2)(V\alpha\rho)^2 + (eS\alpha\rho - S\alpha'\rho)^2 \\ &= (e^2 - 1)\alpha^2\rho^2 + (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2; \end{aligned}$$

in which $\alpha^2 = \alpha'^2$, by 359, LXVI., so that α and α' may be considered to enter *symmetrically* into this last transformation, which is of the *bifocal* kind above mentioned.

(3.) For the same reason, the expression last found for $S\rho\phi\rho$ involves again (comp. 358) *six* scalar constants; namely, e, $T\alpha(= T\alpha')$, and the four involved in the two unit lines, $U\alpha$, $U\alpha'$.

(4.) In all the foregoing transformations, the scalar and quadratic function $S\rho\phi\rho$ has been *evidently homogeneous*, or has been seen to involve no terms below the *second degree* in ρ. We may however also employ this *apparently heterogeneous* or *mixed* form,

$$\text{VII.} \ldots S\rho\phi\rho = g'(\rho - \epsilon)^2 + 2S\lambda(\rho - \zeta)S\mu(\rho - \zeta) + e;$$

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* Compare the Note to Art. 359 [p. 535].

in which g', λ, μ have the same significations as in 357, but e, ε, ζ are *three new constants*, subject to the two *conditions of homogeneity*,

$$\text{VIII.} \ldots g'\varepsilon + \lambda S\mu\zeta + \mu S\lambda\zeta = 0,$$

and

$$\text{IX.} \ldots g'\varepsilon^2 + 2S\lambda\zeta S\mu\zeta + e = 0,$$

in order that the expression VII. may admit of reduction to the form,

$$\text{X.} \ldots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho, \quad \text{as in 357, II}'.$$

(5.) Other general *homogeneous* transformations of $S\rho\phi\rho$, which are themselves *real*, although *connected* with *imaginary* cyclic forms* (comp. 357, (7.)), because a *sum of two squares* of linear and scalar functions is, in an imaginary sense, a *product* of two such functions, are the two following (comp. 357, (9.)):

$$\text{XI.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho = g_1\rho^2 + (S\lambda_1\rho)^2 + (S\mu_1\rho)^2;$$

$$\text{XII.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho = g_3\rho^2 - (S\lambda_3\rho)^2 - (S\mu_3\rho)^2;$$

in which (comp. 357, (2.) and (8.)),

$$\text{XIII.} \ldots g_1 = g + T\lambda\mu = -c_1, \quad g_3 = g - T\lambda\mu = -c_3,$$

$$\text{XIV.} \ldots \lambda_1 = V\lambda\mu(T\lambda\mu - S\lambda\mu)^{-\frac{1}{2}}, \quad \mu_1 = (\lambda T\mu + \mu T\lambda)(T\lambda\mu - S\lambda\mu)^{-\frac{1}{2}},$$

and

$$\text{XV.} \ldots \lambda_3 = V\lambda\mu(T\lambda\mu + S\lambda\mu)^{-\frac{1}{2}}, \quad \mu_3 = (\lambda T\mu - \mu T\lambda)(T\lambda\mu + S\lambda\mu)^{-\frac{1}{2}};$$

so that g_1, λ_1, μ_1, and g_3, λ_3, μ_3 are *real*, if g, λ, μ be such.

(6.) We have therefore the *two new mixed transformations* following:

$$\text{XVI.} \ldots S\rho\phi\rho = g_1(\rho - \varepsilon_1)^2 + (S\lambda_1(\rho - \zeta_1))^2 + (S\mu_1(\rho - \zeta_1))^2 + e_1;$$

$$\text{XVII.} \ldots S\rho\phi\rho = g_3(\rho - \varepsilon_3)^2 - (S\lambda_3(\rho - \zeta_3))^2 - (S\mu_3(\rho - \zeta_3))^2 + e_3;$$

with these two new pairs of equations, as *conditions* of *homogeneity*,

$$\text{XVIII.} \ldots g_1\varepsilon_1 + \lambda_1 S\zeta_1\lambda_1 + \mu_1 S\zeta_1\mu_1 = 0,$$

$$\text{XIX.} \ldots g_1\varepsilon_1^2 + (S\zeta_1\lambda_1)^2 + (S\zeta_1\mu_1)^2 + e_1 = 0,$$

and

$$\text{XX.} \ldots g_3\varepsilon_3 - \lambda_3 S\zeta_3\lambda_3 - \mu_3 S\zeta_3\mu_3 = 0,$$

$$\text{XXI.} \ldots g_3\varepsilon_3^2 - (S\zeta_3\lambda_3)^2 - (S\mu_3\zeta_3)^2 + e_3 = 0.$$

* $\lambda_1 \pm \sqrt{-1}\,\mu_1$, and $\lambda_3 \pm \sqrt{-1}\,\mu_3$, may here be said to be two pairs of *imaginary cyclic normals*, of that *real surface* of the second order, of which the equation is, as before, $S\rho\phi\rho = \text{const.}$ Compare the Notes to pages 527, 534.

361. We saw, in the sub-articles to 336, that the *differential*, $\mathrm{d}f\rho$, of a *scalar function of a vector*, may in general be expressed under the form,

$$\text{I.} \ldots \mathrm{d}f\rho = n\mathrm{S}\nu\mathrm{d}\rho,$$

where ν is a *derived vector function*, of the same variable vector ρ, and n is a *scalar coefficient*. And we now propose to show, that if

$$\text{II.} \ldots f\rho = \mathrm{S}\rho\phi\rho,$$

$\phi\rho$ still denoting the linear and vector function which has been considered in the present Section, and of which $\phi_0\rho$ is still the self-conjugate part, we shall have the equation I. with the values,

$$\text{III.} \ldots n = 2, \quad \nu = \phi_0\rho\,;$$

so that the *part* $\phi_0\rho$ may thus be *deduced* from $\phi\rho$ by *operating* with $\frac{1}{2}\mathrm{dS}\,.\,\rho$, and seeking the coefficient of $\mathrm{d}\rho$ under the sign S. in the result: while there exist certain general *relations of reciprocity* (comp. 336, (6.)), between the *two vectors* ρ and ν, which are in this way *connected*, as *linear functions of each other*.

(1.) We have here, by the supposed *linear form* of $\phi\rho$, the differential equation (comp. 334, VI.),

$$\text{IV.} \ldots \mathrm{d}\phi\rho = \phi\mathrm{d}\rho\,;$$

also

$$\mathrm{S}(\mathrm{d}\rho\,.\,\phi\rho) = \mathrm{S}(\phi\rho\,.\,\mathrm{d}\rho), \quad \text{and} \quad \mathrm{S}(\rho\,.\,\phi\mathrm{d}\rho) = \mathrm{S}(\phi'\rho\,.\,\mathrm{d}\rho)\,;$$

hence, by 349, XIII., we have, as asserted,

$$\text{V.} \ldots \mathrm{dS}\rho\phi\rho = \mathrm{S}(\phi\rho + \phi'\rho)\mathrm{d}\rho = 2\mathrm{S}\,.\,\phi_0\rho\mathrm{d}\rho.$$

(2.) As an example of the employment of this formula, in the deduction of $\phi_0\rho$ from $\phi\rho$, let us take the expression,

$$\text{VI.} \ldots \phi\rho = \Sigma\beta\mathrm{S}\alpha\rho, \qquad 347, \text{XXXI.},$$

which gives,

$$\text{VII.} \ldots f\rho = \mathrm{S}\rho\phi\rho = \Sigma\mathrm{S}\alpha\rho\mathrm{S}\beta\rho,$$

and therefore

$$\text{VIII.} \ldots \mathrm{d}f\rho = \Sigma\mathrm{S}(\beta\mathrm{S}\alpha\rho + \alpha\mathrm{S}\beta\rho)\mathrm{d}\rho.$$

Comparing this with the general formula,

$$\text{IX.} \ldots \tfrac{1}{2}\mathrm{d}f\rho = \mathrm{S}\nu\mathrm{d}\rho = \mathrm{S}\,.\,\phi_0\rho\mathrm{d}\rho,$$

we find that the form VI. of $\phi\rho$ has for its self-conjugate *part*,

$$\text{X.} \ldots \nu = \phi_0\rho = \tfrac{1}{2}\Sigma(\beta S\alpha\rho + \alpha S\beta\rho);$$

and in fact we saw (347, XXXII.) that this form gives, as its *conjugate*, the expression,

$$\text{XI.} \ldots \phi'\rho = \Sigma\alpha S\beta\rho.$$

(3.) Supposing now, for simplicity, that the function ϕ is *given*, or *made*, *self-conjugate*, by taking (if necessary) the semisum of itself and its own conjugate function, we may write ϕ instead of ϕ_0, and shall thus have, simply,

$$\text{XII.} \ldots \nu = \phi\rho, \quad \text{XIII.} \ldots f\rho = S\nu\rho, \quad \text{XIV.} \ldots df\rho = 2S\nu d\rho;$$

whence also (comp. 348, I. II.),

$$\text{XV.} \ldots \rho = \phi^{-1}\nu = m^{-1}\psi\nu, \quad \text{and} \quad \text{XVI.} \ldots S\nu d\rho = S\rho d\nu.$$

(4.) Writing, then,

$$\text{XVII.} \ldots F\nu = S\nu\phi^{-1}\nu = m^{-1}S\nu\psi\nu,$$

we shall have the equations,

$$\text{XVIII.} \ldots F\nu = f\rho, \quad \text{XIX.} \ldots dF\nu = 2S\rho d\nu = 2S \,.\, \phi^{-1}\nu d\nu;$$

so that ρ *may be deduced from* $F\nu$, *as* ν *was deduced from* $f\rho$; and generally, as above stated, there exists a perfect *reciprocity* of relations, between the *vectors* ρ and ν, and also between their *scalar functions*, $f\rho$ and $F\nu$.

(5.) As regards the *deduction*, or *derivation*, of ν from $f\rho$, and of ρ from $F\nu$, it may occasionally be convenient to *denote* it thus:*

$$\text{XX.} \ldots \nu = \tfrac{1}{2}(S \,.\, d\rho)^{-1}df\rho; \qquad \text{XXI.} \ldots \rho = \tfrac{1}{2}(S \,.\, d\nu)^{-1}dF\nu;$$

* [Hamilton suggested the notation $\mathfrak{q} = (S \,.\, d\rho)^{-1}d$ in page 291 of a paper published in the "Proceedings of the Royal Irish Academy," vol. iii. On the same page he introduced the "more general characteristic of operation,

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz} = \triangleleft,$$

in which x, y, and z are ordinary rectangular coordinates," while i, j, and k are unit vectors parallel to the coordinate axes. More recently $\triangleleft$ has been printed ∇, and in accordance with the notation for partial differentiation used in the "Elements" $\nabla = iD_x + jD_y + kD_z$. Now if $\rho = ix + jy + kz$, for *any* system of rectangular axes,

$$df\rho = (dxD_x + dyD_y + dzD_z)f\rho = -\,Sd\rho\nabla \,.\, f\rho.$$

Comparing this with $df\rho = nS\nu d\rho$, it is evident, as $d\rho$ may have any direction whatever, that the equation $n\nu = -\,\nabla \,.\, f(\rho)$ must be true. Hence it may be inferred that ∇ is independent of any particular set of coordinate axes.]

in fact, these last may be considered as only *symbolical transformations* of the expressions,

$$\text{XXII.} \ldots \mathrm{d}f\rho = 2\mathrm{S}(\mathrm{d}\rho \,.\, \nu), \quad \mathrm{d}F\nu = 2\mathrm{S}(\mathrm{d}\nu \,.\, \rho),$$

which follow immediately from XIV. and XIX.

(6.) As an *example* of the passage from an expression such as $f\rho$, to an *equal expression* of the *reciprocal form* $F\nu$, let us resume the *cyclic form* 357, II., writing thus,

$$\text{XXIII.} \ldots f\rho = \mathrm{S}\rho\phi\rho = g\rho^2 + \mathrm{S}\lambda\rho\mu\rho,$$

and supposing that g, λ, and μ are real. Here, by what has been already shown (in sub-articles to 354 and 357), if $\phi\rho$ be supposed self-conjugate, as in (3.), we have,

$$\text{XXIV.} \ldots \nu = \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu\,;$$

$$\text{XXV.} \ldots m = (g - \mathrm{S}\lambda\mu)\,(g^2 - \lambda^2\mu^2) = -c_1c_2c_3\,;$$

$$\text{XXVI.} \ldots \psi\nu = \mathrm{V}\lambda\nu\mu\mathrm{S}\lambda\mu - \mathrm{V}\lambda\mu\mathrm{S}\lambda\nu\mu - g(\lambda\mathrm{S}\mu\nu + \mu\mathrm{S}\lambda\nu) + g^2\nu\,;^*$$

and therefore

$$\text{XXVII.} \ldots mF\nu = \mathrm{S}\nu\psi\nu$$
$$= \mathrm{S}\lambda\nu\mu\nu\mathrm{S}\lambda\mu + (\mathrm{S}\lambda\nu\mu)^2 - 2g\mathrm{S}\lambda\nu\mathrm{S}\mu\nu + g^2\nu^2$$
$$= (g^2 - \lambda^2\mu^2)\nu^2 + \lambda^2(\mathrm{S}\mu\nu)^2 + \mu^2(\mathrm{S}\lambda\nu)^2 - 2g\mathrm{S}\lambda\nu\mathrm{S}\mu\nu\,;$$

which last, when compared with 360, VI., is seen to be what we have called a *bifocal form*: its *focal lines* α, α' (360, (1.)) having here the directions of λ, μ, that is of what may be called the *cyclic lines*† of the *form* XXIII. The *cyclic* and *bifocal transformations* are therefore *reciprocals* of each other.

(7.) As another example of this reciprocal relation between cyclic and focal lines, in the passage from $f\rho$ to $F\nu$, or conversely from the latter to the former, let us now *begin* with the *focal form*,

$$\text{XXVIII.} \ldots f\rho = \mathrm{S}\rho\phi\rho = (\mathrm{V}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2, \qquad 359, \text{XXX.},$$

---

* [Since $\nu = g\rho + \lambda\rho\mu - \mathrm{S}\lambda\rho\mu,$

it follows that $\lambda\nu\mu = g\lambda\rho\mu + \lambda^2\mu^2\rho - \lambda\mu\mathrm{S}\lambda\rho\mu = (\lambda^2\mu^2 - g^2)\rho + (g - \lambda\mu)\mathrm{S}\lambda\rho\mu + g\nu.$

From this $\mathrm{S}\lambda\nu\mu = (g - \mathrm{S}\lambda\mu)\mathrm{S}\lambda\rho\mu,$

and, on substitution, equation XXVI. may at once be found, remembering that $\psi = m\phi^{-1}$.]

† They are in fact (compare the Note to page 527) the *cyclic normals*, or the normals to the *cyclic planes*, of that *surface of the second order*, which has for its equation $f\rho = $ const.; while they are, as above, the *focal lines* of that *other* or *reciprocal surface*, of which ν is the variable vector, and the equation is $F\nu = $ const.

in which α and β are supposed to be given and real vectors. We have now, by 359, (11.),

$$\text{XXIX.}\ldots\begin{cases}\nu = \phi\rho = -\alpha V\alpha\rho + \beta S\beta\rho, \quad m = \alpha^2(S\alpha\beta)^2,\\ \psi\nu = V\alpha\nu\beta S\alpha\beta + \alpha(\alpha^2 - \beta^2)S\alpha\nu,\end{cases}$$

and therefore,

$$\begin{aligned}\text{XXX.}\ldots mF\nu &= \alpha^2(S\alpha\beta)^2F\nu = S\nu\psi\nu\\ &= S\alpha\nu\beta\nu S\alpha\beta + (\alpha^2 - \beta^2)(S\alpha\nu)^2\\ &= -\nu^2(S\alpha\beta)^2 + S\alpha\nu((\alpha^2 - \beta^2)S\alpha\nu + 2S\alpha\beta S\beta\nu)\\ &= -\nu^2(S\alpha\beta)^2 + S\alpha\nu S(\alpha^3 + \beta\alpha\beta)\nu,\end{aligned}$$

an expression which is of *cyclic form*; *one* cyclic line of $F\nu$ being the *given* focal line α of $f\rho$; and the *other* cyclic line of $F\nu$ having the direction of $\pm(\alpha^3 + \beta\alpha\beta)$, and consequently (by 359, LXV.) of $\mp\alpha'$, where α' is the *second* real and focal line of $f\rho$.

(8.) And to verify the equation XVIII., or to show by an example that the *two functions* $f\rho$ and $F\nu$ are *equal in value*, although they are (generally) *different in form*, it is sufficient to substitute in XXX. the value XXIX. of ν; which, after a few reductions, will exhibit the asserted equality.

362. It is often convenient to introduce a certain *scalar and symmetric function* of *two independent vectors*, ρ and ρ', which is *linear* with respect to *each* of them, and is deduced from the linear and *self-conjugate vector function* $\phi\rho$, of a *single* vector ρ, as follows:

$$\text{I.}\ldots f(\rho, \rho') = f(\rho', \rho) = S\rho'\phi\rho = S\rho\phi\rho'.$$

With this notation, we have

$$\text{II.}\ldots f(\rho + \rho') = f\rho + 2f(\rho, \rho') + f\rho';$$

$$\text{III.}\ldots f(\rho, \rho' + \rho'') = f(\rho, \rho') + f(\rho, \rho'');$$

$$\text{IV.}\ldots f(\rho, \rho) = f\rho; \qquad \text{V.}\ldots df\rho = 2f(\rho, d\rho);$$

$$\text{VI.}\ldots f(x\rho, y\rho') = xyf(\rho, \rho'), \quad \text{if} \quad Vx = Vy = 0;$$

and as a verification,

$$\text{VII.}\ldots f(x\rho) = x^2f\rho,$$

a result which might have been obtained, without introducing this new function I.

(1.) It appears to be unnecessary, at this stage, to write down proofs of the foregoing consequences, II. to VI., of the definition I.; but it may be worth remarking, that we *here* depart a little, in the formula V., from a *notation* (325) which was used in some early Articles of the present Chapter, although avowedly only as a *temporary* one, and adopted merely for convenience of exposition of the *principles* of Quaternion Differentials.

(2.) In *that* provisional notation (comp. 325, IX.) we should have had, for the differentiation of the recent function $f\rho$ (361, II.), the formulæ,

$$\mathrm{d}f\rho = f(\rho, \mathrm{d}\rho), \quad f(\rho, \rho'), = 2\mathrm{S}\rho'\phi\rho\,;$$

the numerical coefficient being thus *transferred* from one of them to the other, as compared with the recent equations, I. and V. But there is a convenience *now* in adopting these last equations V. and I., namely,

$$\mathrm{d}f\rho = 2f(\rho, \mathrm{d}\rho), \quad f(\rho, \rho') = \mathrm{S}\rho'\phi\rho\,;$$

because this *function* $\mathrm{S}\rho'\phi\rho$, or $\mathrm{S}\rho\phi\rho'$, occurs frequently in the applications of quaternions to surfaces of the second order, and not always with the *coefficient* 2.

(3.) Retaining then the recent notations, and treating $\mathrm{d}\rho$ as constant, or $\mathrm{d}^2\rho$ as null, successive differentiation of $f\rho$ gives, by IV. and V., the formulæ,

$$\text{VIII.} \ldots \mathrm{d}^2 f\rho = 2f(\mathrm{d}\rho)\,; \quad \mathrm{d}^3 f\rho = 0\,; \ \&\text{c.}\,;$$

so that the theorem 342, I. is here verified, under the form,

$$\text{IX.} \ldots \varepsilon^{\mathrm{d}} f\rho = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2) f\rho = f\rho + 2f(\rho, \mathrm{d}\rho) + f\mathrm{d}\rho\,;$$

or briefly,

$$\text{X.} \ldots \varepsilon^{\mathrm{d}} f\rho = f(\rho + \mathrm{d}\rho),$$

an equation which by II. is *rigorously exact* (comp. 339, (4.)), without any supposition whatever being made, respecting any *smallness* of the *tensor*, $\mathrm{Td}\rho$.

363. *Linear* and *vector functions* of vectors, such as those considered in the present Section, although *not generally* satisfying the condition of *self-conjugation*, present themselves generally in the *differentiation* of *non-linear* but *vector functions* of vectors. In fact, if we denote for the moment *such* a non-linear function by $\omega(\rho)$, or simply by $\omega\rho$, the general *distributive property* (326) of differential expressions allows us to write,

$$\text{I.} \ldots \mathrm{d}\omega(\rho) = \phi(\mathrm{d}\rho), \quad \text{or briefly,} \quad \text{I}'. \ldots \mathrm{d}\omega\rho = \phi\mathrm{d}\rho\,;$$

where ϕ has all the properties hitherto employed, including that of *not* being *generally* self-conjugate, as has been just observed. There is, however, as we shall soon see, an extensive and important *case*, in which the property of self-conjugation *exists*, for such a function ϕ; namely when the *differentiated function*, $\omega\rho$, is *itself* the result ν of the *differentiation* of a *scalar function* $f\rho$ of the variable vector ρ, although *not necessarily* a function of the *second dimension*, such as has been recently considered (361); or more fully, when it is the coefficient of $\mathrm{d}\rho$, under the sign S., in the differential (361, I.) of that scalar function $f\rho$, whether it be multiplied or not by any *scalar constant* (such as n, in the formula last referred to). And generally (comp. 346), the *inversion* of the linear and vector function ϕ in I. corresponds to the *differentiation* of the *inverse* (or *implicit*) function ω^{-1}; in such a manner that the equation I. or I′. may be written under this other form,

$$\text{II.} \ldots \mathrm{d}\omega^{-1}\sigma = \phi^{-1}\mathrm{d}\sigma = m^{-1}\psi\mathrm{d}\sigma, \quad \text{if} \quad \sigma = \omega\rho.$$

(1.) As a very simple *example* of a non-linear but vector function, let us take the form,

$$\text{III.} \ldots \sigma = \omega(\rho) = \rho a \rho, \quad \text{where } a \text{ is a constant vector.}$$

This gives, if $\mathrm{d}\rho = \rho'$,

$$\text{IV.} \ldots \phi\rho' = \phi\mathrm{d}\rho = \mathrm{d}\omega\rho = \rho' a \rho + \rho a \rho' = 2\mathrm{V}\rho a \rho';$$

$$\text{V.} \ldots \mathrm{S}\lambda\phi\rho' = 2\mathrm{S}\lambda\rho a \rho' = \mathrm{S}\rho'\phi'\lambda;$$

$$\text{VI.} \ldots \phi'\lambda = 2\mathrm{V}\lambda\rho a = 2\mathrm{V}a\rho\lambda, \quad \phi'\rho' = 2\mathrm{V}a\rho\rho';$$

so that $\phi\rho'$ and $\phi'\rho'$ are unequal, and the linear function $\phi\rho'$ is *not* self-conjugate.

(2.) To find its self-conjugate *part* $\phi_0\rho'$, by the method of Art. 361, we are to form the scalar expression,

$$\text{VII.} \ldots \tfrac{1}{2}f\rho' = \tfrac{1}{2}\mathrm{S}\rho'\phi\rho' = \rho'^2\mathrm{S}a\rho;$$

of which the differential, taken with respect to ρ', is

$$\text{VIII.} \ldots \tfrac{1}{4}\mathrm{d}f\rho' = \mathrm{S}\,.\,\phi_0\rho'\mathrm{d}\rho' = 2\mathrm{S}a\rho\mathrm{S}\rho'\mathrm{d}\rho', \quad \text{giving} \quad \text{IX.} \ldots \phi_0\rho' = 2\rho'\mathrm{S}a\rho;$$

and accordingly this is equal to the semisum of the two expressions, IV. and VI., for $\phi\rho'$ and its conjugate.

(3.) On the other hand, as an *example of* the *self-conjugation* of the linear and vector function,

$$\text{X.} \ldots \mathrm{d}\nu = \mathrm{d}\omega\rho = \phi\mathrm{d}\rho, \quad \text{when} \quad \text{X}'. \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\,.\,\omega\rho\mathrm{d}\rho,$$

even if the *scalar* function $f\rho$ be of a higher dimension than the second, let this last function have the form,

$$\text{XI.} \ldots f\rho = \mathrm{S}q\rho q'\rho q''\rho, \quad q,\ q',\ q'' \text{ being three constant quaternions.}$$

Here

$$\text{XII.} \ldots \nu = \omega\rho = \tfrac{1}{2}\mathrm{V}(q\rho q'\rho q'' + q'\rho q''\rho q + q''\rho q\rho q');$$

$$\text{XIII.} \ldots \mathrm{d}\nu = \phi\mathrm{d}\rho = \phi\rho' = \tfrac{1}{2}\mathrm{V}(q\rho' q'\rho q'' + q'\rho q''\rho' q) + \tfrac{1}{2}\mathrm{V}(q'\rho' q''\rho q + q''\rho q\rho' q') + \tfrac{1}{2}\mathrm{V}(q''\rho' q\rho q' + q\rho q'\rho' q'');$$

and

$$\text{XIV.} \ldots \mathrm{S}\lambda\phi\rho' = \tfrac{1}{2}\mathrm{S}\,.\,q'\rho q''(\lambda q\rho' + \rho' q\lambda) + \&\text{c.} = \mathrm{S}\rho'\phi\lambda;$$

so that $\phi' = \phi$, as asserted.

(4.) In general, if δ be used as a *second* and *independent symbol* of differentiation, we may write (comp. 345, IV.),

$$\text{XV.} \ldots \delta\mathrm{d}fq = \mathrm{d}\delta fq,$$

where fq may denote any function of a quaternion; in fact, each member is, by the principles of the present Chapter (comp. 344, I., and 345, IX.), an expression for the *limit*,*

$$\text{XVI.} \ldots \lim_{\substack{n=\infty \\ n'=\infty}} nn'\{f(q + n^{-1}\mathrm{d}q + n'^{-1}\delta q) - f(q + n^{-1}\mathrm{d}q) - f(q + n'^{-1}\delta q) + fq\}.$$

(5.) As another statement of the same theorem, we may remark that a first differentiation of fq, with each symbol separately taken, gives results of the forms,

$$\text{XVII.} \ldots \mathrm{d}fq = f(q,\ \mathrm{d}q), \quad \delta fq = f(q,\ \delta q);$$

and then the assertion is, that if we differentiate the first of these with δ, and the second with d, operating only on q with each, and not on dq nor on δq, we obtain *equal results*, of these other forms,

$$\text{XVIII.} \ldots \delta\mathrm{d}fq = f(q,\ \mathrm{d}q,\ \delta q) = f(q,\ \delta q,\ \mathrm{d}q) = \mathrm{d}\delta fq.$$

For example, if

$$\text{XIX.} \ldots fq = qcq, \quad \text{where } c \text{ is a constant quaternion,}$$

* We may also say that each of the two symbols XV. represents the coefficient of x^1y^1, in the development of $f(q + x\mathrm{d}q + y\delta q)$ according to ascending powers of x and y, when such development is possible.

the common value of these last expressions is,

$$\text{XX.} \ldots \delta \mathrm{d} fq = \mathrm{d}\delta fq = \delta q \,.\, c \,.\, \mathrm{d}q + \mathrm{d}q \,.\, c \,.\, \delta q.$$

(6.) Writing then, by X.,

$$\text{XXI.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\omega\rho\mathrm{d}\rho, \quad \delta f\rho = 2\mathrm{S}\omega\rho\delta\rho,$$

and

$$\text{XXII.} \ldots \delta\omega\rho = \phi\delta\rho, \quad \text{with} \quad \mathrm{d}\omega\rho = \phi\mathrm{d}\rho, \text{ as before,}$$

we have the general equation,

$$\text{XXIII.} \ldots \mathrm{S}(\mathrm{d}\rho \,.\, \phi\delta\rho) = \mathrm{S}(\delta\rho \,.\, \phi\mathrm{d}\rho),$$

in which $\mathrm{d}\rho$ and $\delta\rho$ may represent *any two vectors*; the *linear and vector function*, ϕ, which is *thus derived* from a *scalar function* $f\rho$ by *differentiation*, is therefore (as above asserted and exemplified) *always self-conjugate.**

(7.) The equation XXIII. may be thus briefly written,

$$\text{XXIV.} \ldots \mathrm{S}\mathrm{d}\rho\delta\nu = \mathrm{S}\delta\rho\mathrm{d}\nu;$$

and it will be found to be virtually equivalent to the following system of three known equations, in the calculus of partial differential coefficients,

$$\text{XXV.} \ldots \mathrm{D}_x\mathrm{D}_y = \mathrm{D}_y\mathrm{D}_x, \quad \mathrm{D}_y\mathrm{D}_z = \mathrm{D}_z\mathrm{D}_y, \quad \mathrm{D}_z\mathrm{D}_x = \mathrm{D}_x\mathrm{D}_z.\dagger$$

---

* [If n defined by the equation $\mathrm{d}f\rho = n\mathrm{S}\nu\mathrm{d}\rho$ (361, I.)
is not a constant scalar but a function of ρ, the function ϕ generally ceases to be self-conjugate. For example, comparing

$$\mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2F(\rho)\mathrm{S}\mu\mathrm{d}\rho,$$

since $\mathrm{d}\rho$ is arbitrary, $\nu = \mu F(\rho)$. Differentiating this again

$$\mathrm{d}\nu = \phi\mathrm{d}\rho = \mu\mathrm{d}F + \mathrm{d}\mu \,.\, F = \mu\mathrm{S}\lambda\mathrm{d}\rho + \theta\mathrm{d}\rho \,.\, F,$$

if $\mathrm{d}F = \mathrm{S}\lambda\mathrm{d}\rho$, and $\mathrm{d}\mu = \theta\mathrm{d}\rho$.

Here again, as $\mathrm{d}\rho$ is arbitrary, $\theta(\;) = (\phi(\;) - \mu\mathrm{S}\lambda(\;))F^{-1}$,

and the conjugate of θ is $\theta'(\;) = (\phi(\;) - \lambda\mathrm{S}\mu(\;))F^{-1}$.

Hence the spin-vector of θ is $\frac{1}{2}\mathrm{V}\lambda\mu F^{-1}$, or $\frac{1}{2}\mathrm{V}\lambda\nu F^{-2}$.

This vanishes only when F is some function of $f(\rho)$, or a constant as may be easily verified, and in this case θ is self-conjugate.]

† [In terms of the characteristic of operation ∇, defined in the Note to page 548, it is easy to see that

$$\delta\mathrm{d}f\rho = -\,\delta\mathrm{S}\mathrm{d}\rho\nabla.f = \mathrm{S}\delta\rho\nabla\mathrm{S}\mathrm{d}\rho\nabla.f$$
$$= \mathrm{d}\delta f\rho = -\,\mathrm{d}\mathrm{S}\delta\rho\nabla.f = \mathrm{S}\mathrm{d}\rho\nabla\mathrm{S}\delta\rho\nabla.f.$$

In the transformation of functions involving ∇, and operating on a single function $f(\rho)$, or

364. At the commencement of the present Section, we *reduced* (347) the problem of the *inversion* (346) of a *linear* (or *distributive*) *quaternion function of a quaternion*, to the corresponding problem for *vectors*; and, under this reduced or simplified *form*, have *resolved* it. Yet it may be interesting, and it will now be easy, to *resume* the *linear* and *quaternion equation*,

$$\text{I.} \ldots fq = r, \quad \text{with} \quad \text{II.} \ldots f(q + q') = fq + fq',$$

and to assign a *quaternion expression* for the *solution* of that equation, or for the *inverse quaternion function*,

$$\text{III.} \ldots q = f^{-1}r,$$

with the aid of notations already employed, and of results already established.

(1.) The *conjugate* of the linear and quaternion function fq being defined (comp. 347, IV.) by the equation,

$$\text{IV.} \ldots \mathrm{S}pfq = \mathrm{S}qf'p,$$

in which p and q are arbitrary quaternions, if we set out (comp. 347, XXXI.) with the *form*,

$$\text{V.} \ldots fq = tqs + t'qs' + \ldots = \Sigma tqs,$$

in which $s, s', \ldots$ and $t, t', \ldots$ are *arbitrary* but *constant quaternions*, and which is more than sufficiently general, we shall have (comp. 347, XXXII.) the *conjugate form*,

$$\text{VI.} \ldots f'p = spt + s'pt' + \ldots = \Sigma spt;$$

whence

$$\text{VII.} \ldots f1 = \Sigma ts, \quad \text{and} \quad \text{VIII.} \ldots f'1 = \Sigma st;$$

it is then possible, for each *given particular form* of the linear function fq, to assign *one scalar constant* e, and *two vector* constants, ε, ε', such that

$$\text{IX.} \ldots f1 = e + \varepsilon, \quad f'1 = e + \varepsilon';$$

$f(ix + jy + kz)$, $\nabla = i\mathrm{D}_x + j\mathrm{D}_y + k\mathrm{D}_z$ may be treated as an ordinary vector since D_x, D_y, and D_z obey symbolically the ordinary laws of scalar multiplication as expressed by XXV.

Comparing $\quad \delta d f\rho = 2\mathrm{S}\delta\rho\phi d\rho = \mathrm{S}d\rho\nabla\mathrm{S}\delta\rho\nabla . f,$

the vector function $\quad \phi(\ ) = \frac{1}{2}\nabla\mathrm{S}(\ )\nabla . f,$

since $d\rho$ and $\delta\rho$ are both arbitrary. Of course ∇ operates on f and not on the vector operated on by ϕ. This expression for ϕ shows again that it is self-conjugate. Again, as $\nabla f = -2\nu$, $\phi(\ ) = -\nabla\mathrm{S}(\ )\nu$, and in this ∇ operates on ν and not on the subject of ϕ.]

and then we shall have the general transformations (comp. 347, I.):

$$\text{X.} \ldots \mathrm{S}fq = \mathrm{S}\,.\,qf'1 = e\mathrm{S}q + \mathrm{S}\varepsilon'q\,;$$

$$\text{XI.} \ldots \mathrm{V}fq = \varepsilon\mathrm{S}q + \mathrm{V}.f\mathrm{V}q = \varepsilon\mathrm{S}q + \phi\mathrm{V}q\,;$$

and

$$\text{XII.} \ldots fq = (e + \varepsilon)\mathrm{S}q + \mathrm{S}\varepsilon'q + \phi\mathrm{V}q\,;$$

in which $\mathrm{S}\varepsilon'q = \mathrm{S}\,.\,\varepsilon'\mathrm{V}q$, and $\phi\mathrm{V}q$ or $\mathrm{V}f\mathrm{V}q$ is a *linear* and *vector function* of $\mathrm{V}q$, of the kind already considered in this Section; being also such that, with the form V. of fq, we have

$$\text{XIII.} \ldots \phi\rho = \Sigma\mathrm{V}t\rho s.^*$$

(2.) As regards the *number* of *independent* and *scalar constants* which enter, at least implicitly, into the composition of the quaternion function fq, it may in various ways be shown to be *sixteen*; and accordingly, in the expression XII., the *scalar* e is *one*; the *two vectors*, ε and ε', count *each* as *three*; and the *linear* and *vector function*, $\phi\mathrm{V}q$, counts as *nine* (comp. 347, (1.)).

(3.) Since we already know (347, &c.) how to *invert* a function of this last kind ϕ, we may in general write,

$$\text{XIV.} \ldots r = \mathrm{S}r + \mathrm{V}r = \mathrm{S}r + \phi\rho, \quad \text{where} \quad \text{XV.} \ldots \rho = \phi^{-1}\mathrm{V}r = m^{-1}\psi\mathrm{V}r\,;$$

the *scalar constant*, m, and the *auxiliary linear* and *vector function*, ψ, being deduced from the function ϕ by methods already explained. It is required then to express q, or $\mathrm{S}q$ and $\mathrm{V}q$, in terms of r, or of $\mathrm{S}r$ and ρ, so as to satisfy the linear equation,

$$\text{XVI.} \ldots (e + \varepsilon)\mathrm{S}q + \mathrm{S}\varepsilon'q + \phi\mathrm{V}q = \mathrm{S}r + \phi\rho\,;$$

the constants e, ε, ε', and the form of ϕ, being given.

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* [By a method analogous to that of the Note on page 507, if any three diplanar vectors β_1, β_2, and β_3 are chosen, any quaternion function fq may have its vector part resolved along these three vectors, so that $fq = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + x_4$, in which the coefficients x are scalar functions of q, and are moreover linear if fq is linear in q. So for a linear function,

$$fq = \beta_1\mathrm{S}p_1q + \beta_2\mathrm{S}p_2q + \beta_3\mathrm{S}p_3q + \mathrm{S}p_4q,$$

and in this expression p_1, p_2, p_3, and p_4 are four constant quaternions involving sixteen scalar constants and determining the function f. Denoting $\mathrm{S}p$ by a, and $\mathrm{V}p$ by α, on rearrangement,

$$fq = (\beta_1\mathrm{S}\alpha_1 + \beta_2\mathrm{S}\alpha_2 + \beta_3\mathrm{S}\alpha_3)\mathrm{V}q + (a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + a_4)\mathrm{S}q + \mathrm{S}\alpha_4\mathrm{V}q,$$

and this is manifestly of the type,

$$fq = (e + \epsilon)\mathrm{S}q + \mathrm{S}\epsilon'q + \phi\mathrm{V}q.]$$

(4.) Assuming for this purpose the expression,

$$\text{XVII.} \ldots q = q' + \rho,$$

in which q' is a new sought quaternion, we have the new equation,

$$\text{XVIII.} \ldots fq' = \text{S}r + \phi\rho - f\rho = \text{S}(r - \varepsilon'\rho);$$

whence

$$\text{XIX.} \ldots q' = \text{S}(r - \varepsilon'\rho) \,.\, f^{-1}1,$$

and

$$\text{XX.} \ldots q = \rho + \text{S}(r - \varepsilon'\rho) \,.\, f^{-1}1;$$

in which ρ is (by supposition) a *known vector*, and $\text{S}(r - \varepsilon'\rho)$ is a *known scalar*; so that it only remains to determine the *unknown* but *constant quaternion*, $f^{-1}1$, or to resolve the *particular equation*,

$$\text{XXI.} \ldots fq_0 = 1, \quad \text{in which} \quad \text{XXII.} \ldots q_0 = c + \gamma = f^{-1}1,$$

c being a *new* and *sought scalar constant*, and γ being a *new* and *sought vector constant.*

(5.) Taking scalar and vector parts, the quaternion equation XXI. breaks up into the two following (comp. X. and XI.):

$$\text{XXIII.} \ldots 1 = \text{S}f(c + \gamma) = ec + \text{S}\varepsilon'\gamma; \quad \text{XXIV.} \ldots 0 = \text{V}f(c + \gamma) = \varepsilon c + \phi\gamma;$$

which give the required values of c and γ, namely,

$$\text{XXV.} \ldots c = (e - \text{S}\varepsilon'\phi^{-1}\varepsilon)^{-1}, \quad \text{and} \quad \text{XXVI.} \ldots \gamma = -c\phi^{-1}\varepsilon;$$

whence

$$\text{XXVII.} \ldots f^{-1}1 = \frac{1 - \phi^{-1}\varepsilon}{e - \text{S}\varepsilon'\phi^{-1}\varepsilon};$$

and accordingly we have, by XII., the equation,

$$\text{XXVIII.} \ldots f(1 - \phi^{-1}\varepsilon) = e - \text{S}\varepsilon'\phi^{-1}\varepsilon = \text{V}^{-1}0.$$

(6.) The *problem* of *quaternion inversion* is therefore *reduced anew* to that of *vector inversion*, and *solved* thereby; but we can *now* advance some steps further, in the *elimination* of *inverse operations*, and in the *substitution* for them of *direct* ones. Thus, if we observe that $\phi^{-1} = m^{-1}\psi$, as before, and write for abridgment,

$$\text{XXIX.} \ldots n = me - \text{S}\varepsilon'\psi\varepsilon = f(m - \psi\varepsilon),$$

so that n is a *new* and *known scalar constant*, we shall have, by XV. XX. XXVII. XXIX.,

$$\text{XXX.}\ldots m\rho = \psi \mathrm{V}r; \quad \text{XXXI.}\ldots nf^{-1}1 = m - \psi\epsilon;$$

and

$$\text{XXXII.}\ldots mnq = n\psi\mathrm{V}r + (m\mathrm{S}r - \mathrm{S}\epsilon'\psi\mathrm{V}r).(m - \psi\epsilon),$$

an expression from which all *inverse* operations have disappeared, but which still admits of being simplified, through a division by m, as follows.

(7.) Substituting (by XXIX.), in the term $n\psi\mathrm{V}r$ of XXXII., the value $me - \mathrm{S}\epsilon'\psi\epsilon$ for n, and changing (by XXX.) $\psi\mathrm{V}r$ to $m\rho$, in the terms which are not obviously divisible by m, such a division gives,

$$\text{XXXIII.}\ldots nq = (m - \psi\epsilon)\mathrm{S}r + e\psi\mathrm{V}r - \mathrm{S}\epsilon'\psi\mathrm{V}r + \sigma,$$

where

$$\text{XXXIV.}\ldots \sigma = -\rho\mathrm{S}\epsilon'\psi\epsilon + \psi\epsilon\mathrm{S}\epsilon'\rho = \mathrm{V}.\epsilon'\mathrm{V}\rho\psi\epsilon.$$

But (by 348, VII., interchanging accents) we have the transformation,

$$\text{XXXV.}\ldots \mathrm{V}\rho\psi\epsilon = -\phi'\mathrm{V}\epsilon\phi\rho = -\phi'\mathrm{V}\epsilon\mathrm{V}r,$$

because $\phi\rho = \mathrm{V}r$, by XIV. or XV.; everything *inverse* therefore *again* disappears with this new elimination of the auxiliary vector ρ, and we have this final expression,

$$\text{XXXVI.}\ldots nq = nf^{-1}r = (me - \mathrm{S}\epsilon'\psi\epsilon).f^{-1}r$$
$$= (m - \psi\epsilon)\mathrm{S}r + e\psi\mathrm{V}r - \mathrm{S}\epsilon'\psi\mathrm{V}r - \mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\mathrm{V}r,$$

in which each symbol of operation governs all that follows it, except where a point indicates the contrary, and which it appears to be impossible further to reduce, as the *formula of solution* of the *linear equation* I., with the *form* XII. of the *quaternion function*, fq.*

---

* [The following solution is possibly more direct. Equating the scalar and vector parts of

the two equations

$$fq = (e + \epsilon)\mathrm{S}q + \mathrm{S}\epsilon'\mathrm{V}q + \phi\mathrm{V}q = r = \mathrm{S}r + \mathrm{V}r,$$

$$e\mathrm{S}q + \mathrm{S}\epsilon'\mathrm{V}q = \mathrm{S}r, \quad \text{and} \quad \epsilon\mathrm{S}q + \phi\mathrm{V}q = \mathrm{V}r$$

are found. Operating on the second equation by ϕ^{-1}, and replacing $\mathrm{V}q$ in the first, $\mathrm{S}q$ is seen to be given by

$$(e - \mathrm{S}\epsilon'\phi^{-1}\epsilon)\mathrm{S}q = \mathrm{S}r - \mathrm{S}\epsilon'\phi^{-1}\mathrm{V}r.$$

Now

$$q = \mathrm{S}q + \mathrm{V}q = (1 - \phi^{-1}\epsilon)\mathrm{S}q + \phi^{-1}\mathrm{V}r,$$

(8.) Such having been the *analysis* of the problem, the *synthesis*, by which an *à posteriori proof* of the correctness of the resulting formula is to be given, may be simplified by using the *scalar* value XXIX. of $f(m - \psi\epsilon)$; and it is sufficient to show (denoting $\mathrm{V}r$ by ω), that for every *vector* ω the following equation holds good, with the same form XII. of f:

$$\text{XXXVII.} \ldots f(e\psi\omega - \mathrm{S}\epsilon'\psi\omega) - f\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = (me - \mathrm{S}\epsilon'\psi\epsilon)\,.\,\omega.$$

(9.) Accordingly, that form of f gives, with the help of the principle employed in XXXV.,

$$\text{XXXVIII.} \ldots \begin{cases} ef\psi\omega = e(\mathrm{S}\epsilon'\psi\omega + m\omega), \quad -f\mathrm{S}\epsilon'\psi\omega = -\,(e+\epsilon)\mathrm{S}\epsilon'\psi\omega, \\ -f\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = -\,\phi\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = \mathrm{V}(\mathrm{V}\epsilon\omega\,.\,\psi'\epsilon') = \epsilon\mathrm{S}\epsilon'\psi\omega - \omega\mathrm{S}\epsilon'\psi\epsilon, \end{cases}$$

because $\mathrm{S}\omega\psi'\epsilon' = \mathrm{S}\epsilon'\psi\omega$, &c.; and thus the equation XXXVI. is proved, by actually operating with f.

(10.) As an *example*, if we take the particular form,

$$\text{XXXIX.} \ldots r = fq = pq + qp,$$

in which

$$\text{XL.} \ldots p = a + \mathrm{a} = \text{a given quaternion},$$

we have then,

$$\text{XLI.} \ldots f1 = f'1 = 2p, \quad e = 2a, \quad \epsilon = \epsilon' = 2\mathrm{a}, \quad \phi\rho = 2a\rho;$$

whence by the theory of linear and *vector* functions,

$$\text{XLII.} \ldots \phi'\rho = 2a\rho, \quad \psi\rho = 4a^2\rho, \quad m = 8a^3,$$

and therefore,

$$\text{XLIII.} \ldots \psi\epsilon = 8a^2\mathrm{a}, \quad m - \psi\epsilon = 8a^2(a - \mathrm{a}), \quad n = 16a^2(a^2 - \mathrm{a}^2);$$

so that, dividing by $8a$, the formula XXXVI. becomes,

$$\text{XLIV.} \ldots 2a(a^2 - \mathrm{a}^2)q = a(a - \mathrm{a})\mathrm{S}r + a^2\mathrm{V}r - a\mathrm{S}\,.\,\mathrm{a}\mathrm{V}r - \mathrm{a}\mathrm{V}\,.\,\mathrm{a}\mathrm{V}r,$$

or

$$\text{XLV.} \ldots 2a(a + \mathrm{a})q = a\mathrm{S}r + (a + \mathrm{a})\mathrm{V}r - \mathrm{S}\mathrm{a}r,$$

---

so on substituting the value of $\mathrm{S}q$ just found,

$$q = \frac{(1 - \phi^{-1}\epsilon)\,(\mathrm{S}r - \mathrm{S}\epsilon'\phi^{-1}\mathrm{V}r)}{e - \mathrm{S}\epsilon'\phi^{-1}\epsilon} + \phi^{-1}\mathrm{V}r.$$

It only remains to replace ϕ^{-1} by $m^{-1}\psi$ in order to recover XXXVI.]

or

$$\text{XLVI.} \ldots 2pq\mathrm{S}p = \mathrm{S}\,.\,r\mathrm{K}p + p\mathrm{V}r = r\mathrm{S}p + \mathrm{V}(\mathrm{V}p\,.\,\mathrm{V}r),$$

or

$$\text{XLVII.} \ldots 4pq\mathrm{S}p = 2r\mathrm{S}p + (pr - rp) = pr + r\mathrm{K}p\,;$$

or finally,

$$\text{XLVIII.} \ldots q = f^{-1}r = \frac{r + p^{-1}r\mathrm{K}p}{4\mathrm{S}p} = \frac{r + \mathrm{K}p\,.\,rp^{-1}}{4\mathrm{S}p}$$

Accordingly,

$$\text{XLIX.} \ldots (pr + r\mathrm{K}p) + (rp + \mathrm{K}p\,.\,r) = 2r(p + \mathrm{K}p) = 4r\mathrm{S}p.$$

(11.) In *so simple* an example as the last, we may with advantage avail ourselves of *special methods*; for instance (comp. 346), we may use that which was employed in 332, (6.), to *differentiate the square root of a quaternion*, and which conducted there more rapidly to a formula (332, XIX.) agreeing with the recent XLVIII.

(12.) We might also have observed, in the same case XXXIX., that

$$\text{L.} \ldots pr - rp = p^2q - qp^2 = 2\mathrm{V}(\mathrm{V}(p^2)\,.\mathrm{V}q) = 4\mathrm{S}p\,.\mathrm{V}(\mathrm{V}p\,.\mathrm{V}q) = 2\mathrm{S}p\,.\,(pq - qp)\,;$$

whence $pq - qp$, and therefore pq and qp, can be at once deduced, with the same resulting value for q, or for $f^{-1}r$, as before: and generally it is possible to *differentiate*, on a similar plan, the *n^{th} root of a quaternion.*

365. We shall conclude this Section on *Linear Functions*, of the kinds above considered, by proving the general existence of a *Symbolic and Biquadratic Equation*, of the form,

$$\text{I.} \ldots 0 = n - n'f + n''f^2 - n'''f^3 + f^4,$$

which is thus *satisfied by the Symbol* (f) *of Linear and Quaternion Operation on a Quaternion*, as the *Symbolic and Cubic Equation*,

$$\text{I}'. \ldots 0 = m - m'\phi + m''\phi^2 - \phi^3, \qquad \text{350, I.},$$

was satisfied by the *symbol* (ϕ) of *linear* and *vector operation* on a *vector*; the *four coefficients*, n, n', n'', n''', being *four scalar constants*, deduced from the function f in this extended or *quaternion theory*, as the *three* scalar coefficients m, m', m'' were constants deduced from ϕ, in the former or *vector theory*. And at the same time we shall see that there exists a *System* of *Three Auxiliary Functions*, *F, G, H*, of the *Linear* and *Quaternion* kind, analogous to the *two vector functions*, ψ and χ, which have been so useful in the foregoing theory of vectors, and like them connected with each other, and with the given quaternion function f, by several simple and useful relations.*

(1.) The formula of solution, 364, XXXVI., of the linear and quaternion equation $fq = r$, being denoted briefly as follows,

$$\text{II.} \ldots nq = nf^{-1}r = Fr,$$

so that (comp. 348, III′.) we may write, briefly and symbolically,

$$\text{III.} \ldots fF = Ff = n,$$

it may next be proposed to examine the changes which the scalar n and the function Fr undergo, when fr is changed to $fr + cr$, or f to $f + c$, where c is any scalar constant; that is, by 364, XII., when e is changed to $e + c$, and ϕ to $\phi + c$; ϕ', ψ, and m being at the same time changed, according to the laws of the earlier theory.

(2.) Writing, then,

$$\text{IV.} \ldots f_c = f + c, \quad e_c = e + c, \quad \phi_c = \phi + c, \quad \phi'_c = \phi' + c,$$

and

$$\text{V.} \ldots \psi_c = \psi + c\chi + c^2, \quad m_c = m + m'c + m''c^2 + c^3,$$

we may represent the new form of the equation 364, XXXVI. as follows:

$$\text{VI.} \ldots n_c f_c^{-1} r = F_c r, \quad \text{or} \quad \text{VII.} \ldots f_c F_c = n_c;$$

where

$$\text{VIII.} \ldots F_c r = (m_c - \psi_c \epsilon)\text{S}r + e_c \psi_c \text{V}r - \text{S}\epsilon' \psi_c \text{V}r - \text{V}\epsilon' \phi'_c \text{V}\epsilon \text{V}r,$$

* [That a linear quaternion function satisfies a symbolic quartic may be established as follows:

On inquiry whether it is possible to determine a scalar c and a quaternion q so that $fq + cq = 0$, the two equations

$$(e + c)\text{S}q + \text{S}\epsilon'\text{V}q = 0, \quad \text{and} \quad \epsilon\text{S}q + (\phi + c)\text{V}q = 0,$$

are found by equating to zero the scalar and vector parts. Hence from the second equation $\text{V}q = -(\phi + c)^{-1}\epsilon\text{S}q$, and, on substitution in the first, it appears that c must satisfy the relation $e + c - \text{S}\epsilon'(\phi + c)^{-1}\epsilon = 0$. It may be shown without difficulty, as in the text, that this leads to a quartic equation in c.

If c_n is any root of this quartic, and if $\alpha_n = -(\phi + c_n)^{-1}\epsilon$, the quaternion $q_n = 1 + \alpha_n$ will satisfy $(f + c_n)q_n = 0$. Corresponding to the four values of c_n are four quaternions, and in terms of these any arbitrary quaternion may in general be expressed.

Assuming

$$q = x_1 q_1 + x_2 q_2 + x_3 q_3 + x_4 q_4,$$

and deriving from this the equations

$$\text{V}q = \Sigma x_n \alpha_n, \quad \text{and} \quad \text{S}q = \Sigma x_n,$$

and again from these the equation

$$\text{V}q - \alpha_1 \text{S}q = x_2(\alpha_2 - \alpha_1) + x_3(\alpha_3 - \alpha_1) + x_4(\alpha_4 - \alpha_1),$$

and

$$\text{IX.} \ldots n_c = e_c m_c - S\varepsilon'\psi_c\varepsilon.$$

(3.) In this manner it is seen that we may write,

$$\text{X.} \ldots F_c = F + cG + c^2H + c^3,$$

and

$$\text{XI.} \ldots n_c = n + n'c + n''c^2 + n'''c^3 + c^4;$$

where F, G, H, are *three functional symbols*, such that

$$\text{XII.} \ldots \begin{cases} Fr = (m - \psi\varepsilon)Sr + e\psi Vr - S\varepsilon'\psi Vr - V\varepsilon'\phi' V\varepsilon Vr; \\ Gr = (m' - \chi\varepsilon)Sr + (e\chi + \psi)Vr - S\varepsilon'\chi Vr - V\varepsilon' V\varepsilon Vr; \\ Hr = (m'' - \varepsilon)Sr + (e + \chi)Vr - S\varepsilon' r; \end{cases}$$

and n, n', n'', n''' are *four scalar constants*, namely,

$$\text{XIII.} \ldots \begin{cases} n = em - S\varepsilon'\psi\varepsilon \quad (\text{as in 364, XXIX.}); \\ n' = m + em' - S\varepsilon'\chi\varepsilon; \\ n'' = m' + em'' - S\varepsilon'\varepsilon; \\ n''' = m'' + e. \end{cases}$$

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the scalar x_4 is given, on operating by $SV(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)$, by

$$x_4 S(\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1 + \alpha_4\alpha_1\alpha_2 - \alpha_1\alpha_2\alpha_3) = SVq(\alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2) - SqS\alpha_1\alpha_2\alpha_3,$$

and the values of the other scalars may be written down from symmetry (comp. p. 48). In general x_1, x_2, x_3, and x_4 are uniquely determinate provided the four vectors α_n do not terminate on a common plane. As c varies, the curve traced out by $\rho = -(\phi + c)^{-1}\epsilon$ is a twisted cubic and upon this curve the vectors α_n terminate, and consequently their four extremities do not lie on a plane.

To verify that $\rho = -(\phi + c)^{-1}\epsilon$ is a twisted cubic, the equation

$$S\lambda(\phi + c)^{-1}\epsilon = 1, \quad \text{or} \quad S\lambda\psi_c\epsilon = m_c,$$

is found determining the values of c for the points in which the curve cuts the arbitrary plane $S\lambda\rho + 1 = 0$. As this is a cubic equation in c, the curve cuts the plane in but three points.

In general then

$$q = x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4.$$

Operating on this by $f + c_1$, and

$$(f + c_1)q = x_2(c_1 - c_2)q_2 + x_3(c_1 - c_3)q_3 + x_4(c_1 - c_4)q_4,$$

from which q_1 has disappeared. Similarly operating by $f + c_2$ destroys the term in q_2, and finally

$$(f + c_4)(f + c_3)(f + c_2)(f + c_1)q = 0,$$

which is equivalent to I.]

(4.) Developing then the symbolical equation VII., with the help of X. and XI., and comparing powers of c, we obtain these new symbolical equations (comp. 350, XVI. XXI. XXIII.):

$$\text{XIV.} \ldots \begin{cases} H = n''' - f; \\ G = n'' - fH = n'' - n'''f + f^2; \\ F = n' - fG = n' - n''f + n'''f^2 - f^3; \end{cases}$$

and finally,

$$\text{XV.} \ldots n = Ff = n'f - n''f^2 + n'''f^3 - f^4,$$

which is only another way of writing the *symbolic and biquadratic equation* I.

(5.) *Other functional relations* exist, between these various symbols of operation, which we cannot here delay to develop: but we may remark that, as in the theory of linear and *vector* functions, these usually introduce a mixture of functions with their *conjugates* (comp. 347, XI., &c.).

(6.) This seems however to be a proper place for observing, that if we write, as temporary notations, for *any four quaternions*, p, q, r, s, the equations,

$$\text{XVI.} \ldots [pq] = pq - qp; \quad \text{XVII.} \ldots (pqr) = \mathrm{S}.p[qr];$$

$$\text{XVIII.} \ldots [pqr] = (pqr) + [rq]\mathrm{S}p + [pr]\mathrm{S}q + [qp]\mathrm{S}r;$$

and

$$\text{XIX.} \ldots (pqrs) = \mathrm{S}.p[qrs],$$

so that $[pq]$ is a vector, (pqr) and $(pqrs)$ are scalars, and $[pqr]$ is a quaternion, we shall have, in the first place, the relations:

$$\text{XX.} \ldots [pq] = -[qp], \quad [pp] = 0;$$

$$\text{XXI.} \ldots (pqr) = -(qpr) = (qrp) = \&\text{c.}, \quad (ppr) = 0;$$

$$\text{XXII.} \ldots [pqr] = -[qpr] = [qrp] = \&\text{c.}, \quad [ppr] = 0;$$

and

$$\text{XXIII.} \ldots (pqrs) = -(qprs) = (qrps) = -(qrsp) = \&\text{c.}, \quad (pprs) = 0.$$

(7.) In the next place, if t be *any fifth quaternion*, the *quaternion equation*,

$$\text{XXIV.} \ldots 0 = p(qrst) + q(rstp) + r(stpq) + s(tpqr) + t(pqrs),$$

which may also be thus written,

$$\text{XXV.} \ldots q(prst) = p(qrst) + r(pqst) + s(prqt) + t(prsq),^*$$

and which is analogous to the *vector equation*,

$$\text{XXVI.} \ldots 0 = \alpha S\beta\gamma\delta - \beta S\gamma\delta\alpha + \gamma S\delta\alpha\beta - \delta S\alpha\beta\gamma,$$

or to the continually† occurring transformation (comp. 294, XIV.),

$$\text{XXVII.} \ldots \delta S\alpha\beta\gamma = \alpha S\delta\beta\gamma + \beta S\alpha\delta\gamma + \gamma S\alpha\beta\delta,$$

is satisfied *generally*, because it is satisfied for the *four distinct suppositions*,

$$\text{XXVIII.} \ldots q = p, \quad q = r, \quad q = s, \quad q = t.$$

(8.) In the third place, we have this *other general quaternion equation*,

$$\text{XXIX.} \ldots q(prst) = [rst]Spq - [stp]Srq + [tpr]Ssq - [prs]Stq,$$

which is analogous to this *other*‡ *useful vector formula* (comp. 294, XV.),

$$\text{XXX.} \ldots \delta S\alpha\beta\gamma = V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta + V\alpha\beta S\gamma\delta;$$

because the equation XXIX. gives true results, when it is operated on by the *four distinct symbols* (comp. 312),

$$\text{XXXI.} \ldots S.p, \quad S.r, \quad S.s, \quad S.t.$$

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* [Or again as a determinant

$$\begin{vmatrix} p & q & r & s & t \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix} = 0,$$

if $p = w_1 + ix_1 + jy_1 + kz_1$, &c.]

† The equations XXVII. and XXX., which had been proved under slightly different forms in the sub-articles to 294, have been in fact freely employed as transformations in the course of the present Chapter, and are supposed to be *familiar* to the student. Compare the Note to page 485.

‡ Compare the Note immediately preceding.

(9.) *Assuming* then *any four quaternions*, p, r, s, t, which are *not* connected by the relation,

$$\text{XXXII.} \ldots (prst) = 0,$$

and *deducing* from them *four others*, p', r', s', t', by the equations,

$$\text{XXXIII.} \ldots \begin{cases} p'(prst) = f[rst], & r'(prst) = -f[stp], \\ s'(prst) = f[tpr], & t'(prst) = -f[prs], \end{cases}$$

in which f is still supposed to be a symbol of linear and quaternion operation on a quaternion, the formula XXIX. allows us to write generally, as an expression for the function fq, which may here be denoted by q' (because r is now otherwise used):

$$\text{XXXIV.} \ldots q' = fq = p'\text{S}pq + r'\text{S}rq + s'\text{S}sq + t'\text{S}tq\,;$$

and its *sixteen scalar constants* (comp. 364, (2.)) are now those which are involved in its *four quaternion constants*, p', r', s', t'.

(10.) Operating on this last equation with the four symbols,

$$\text{XXXV.} \ldots \text{S}\,.\,[r's't'], \quad \text{S}\,.\,[s't'p'], \quad \text{S}\,.\,[t'p'r'], \quad \text{S}\,.\,[p'r's'],$$

we obtain the four following results:

$$\text{XXXVI.} \ldots \begin{cases} (q'r's't') = (p'r's't')\text{S}pq\,; & (q's't'p') = (r's't'p')\text{S}rq\,; \\ (q't'p'r') = (s't'p'r')\text{S}sq\,; & (q'p'r's') = (t'p'r's')\text{S}tq\,; \end{cases}$$

and when the values thus found for the four scalars,

$$\text{XXXVII.} \ldots \text{S}pq, \quad \text{S}rq, \quad \text{S}sq, \quad \text{S}tq,$$

are substituted in the formula XXIX., we have the following *new formula of quaternion inversion*:

$$\text{XXXVIII.} \ldots (p'r's't')\,(prst)q = (p'r's't')\,(prst)f^{-1}q'$$
$$= [rst]\,(q'r's't') + [stp]\,(q's't'p') + [tpr]\,(q't'p'r') + [prs]\,(q'p'r's')\,;$$

which shows, in a new way, how to *resolve a linear equation in quaternions*, when put under what we may call (comp. 347, (1.)) the *Standard Quadrinomial Form*, XXXIV.

(11.) Accordingly, if we operate on the formula XXXVIII. with f, attending to the equations XXXIII., and dividing by $(prst)$, we get this new equation

$$\text{XXXIX.}\ldots (p'r's't')fq = p'(q'r's't') - r'(q's't'p') + s'(q't'p'r') - t'(q'p'r's');$$

whence

$$fq = q', \text{ by XXV.}$$

(12.) It has been remarked (9.), that p, r, s, t, in recent formulæ, may be *any four quaternions*, which do not satisfy the equation XXXII.; we may therefore assume,

$$\text{XL.}\ldots p = 1, \quad r = i, \quad s = j, \quad t = k,$$

with the laws of 182, &c., for the symbols i, j, k, because those laws give here,

$$\text{XLI.}\ldots (1ijk) = -2;$$

and then it will be found that the equations XXXIII. give simply,

$$\text{XLII.}\ldots p' = f1, \quad r' = -fi, \quad s' = -fj, \quad t' = -fk;$$

so that the *standard quadrinomial form* XXXIV. becomes, with this selection of $prst$,

$$\text{XLIII.}\ldots fq = f1 \,.\, \mathrm{S}q - fi \,.\, \mathrm{S}iq - fj \,.\, \mathrm{S}jq - fk \,.\, \mathrm{S}kq,$$

and admits of an immediate verification, because *any quaternion*, q, may be expressed (comp. 221) by the *quadrinomial*,

$$\text{XLIV.}\ldots q = \mathrm{S}q - i\mathrm{S}iq - j\mathrm{S}jq - k\mathrm{S}kq.$$

(13.) Conversely, if we *set out* with the expression,

$$\text{XLV.}\ldots q = w + ix + jy + kz, \qquad 221, \text{ III.},$$

which gives,

$$\text{XLVI.}\ldots fq = wf1 + xfi + yfj + zfk,$$

or briefly,

$$\text{XLVII.}\ldots e = aw + bx + cy + dz,$$

the letters *abcde* being *here* used to denote *five known quaternions*, while *wxyz* are *four sought scalars*, the *problem* of *quaternion inversion* comes to be that of the *separate determination* (comp. 312) of these *four* scalars, so as to satisfy the *one* equation XLVII.; and it is *resolved* (comp. XXV.) by the system of the four following formulæ:

$$\text{XLVIII.} \ldots \begin{cases} w(abcd) = (ebcd); & x(abcd) = (aecd); \\ y(abcd) = (abed); & z(abcd) = (abce); \end{cases}$$

the notations (6.) being retained.

(14.) Finally it may be shown, as follows, that the *biquadratic equation* I., for linear functions of *quaternions*, *includes** the *cubic* I'., or 350, I., for *vectors*. Suppose, for this purpose, that the linear and quaternion function, fq, reduces itself to the last term of the general expression 364, XII., or becomes,

$$\text{XLIX.} \ldots fq = \phi \mathrm{V}q, \quad \text{so that} \quad \text{L.} \ldots e = 0, \quad \varepsilon = \varepsilon' = 0, \quad f1 = f'1 = 0;$$

the coefficients n, n', n'', n''' take then, by XIII., the values,

$$\text{LI.} \ldots n = 0, \quad n' = m, \quad n'' = m', \quad n''' = m'';$$

and the biquadratic I. becomes,

$$\text{LII.} \ldots 0 = (-m + m'f - m''f^2 + f^3)f.$$

But fq is now a *vector*, by XLIX., and it *may* be *any* vector, ρ; also the *operation* f is now equivalent to that denoted by ϕ, when the *subject* of the

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* In like manner it may be said, that the *cubic* equation *includes* a *quadratic* one, when we confine ourselves to the consideration of *vectors* in *one plane*; for which case $m = 0$, and also $\psi\rho = 0$, if ρ be a line in the given plane: for we have then $\phi\chi = m' - \psi = m'$, or

$$\phi^2 - m''\phi + m' = 0,$$

with this understanding as to the operand. In fact, the *cubic* gives here (because $m = 0$),

$$(\phi^2 - m''\phi + m')\phi\rho = 0;$$

and therefore

$$(\phi^2 - m''\phi + m')\sigma = 0;$$

if σ be already the result of an operation with ϕ, on any vector ρ: that is if it be, as above supposed, a line in the given plane.

operation is a vector; we may therefore, in the case here considered, write this last equation LII. under the form,

$$\text{LIII.} \ldots 0 = (-m + m'\phi - m''\phi^2 + \phi^3)\rho,$$

which agrees with 351, I., and reproduces the *symbolical cubic*, when the symbol of the *operand* (ρ) is suppressed.*

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* [A few additional remarks may be made concerning the solutions of $Vq^{-1}fq = 0$, and of $Vq^{-1}f'q = 0$, and the relations connecting them.

It is easy to see, in various ways, that f and its conjugate f' satisfy the same symbolic biquadratic. If, for instance, q and q' are any arbitrary quaternions

$$Sq(f'^4 - n'''f'^3 + n''f'^2 - n'f' + n)q' = Sq'(f^4 - n'''f^3 + n''f^2 - n'f + n)q = 0$$

by I., and therefore as the quaternions are arbitrary,

$$(f'^4 - n'''f'^3 + n''f'^2 - n'f' + n)q' = 0.$$

Again, the same property follows from the equation

$$e + c = S\epsilon'(\phi + c)^{-1}\epsilon = S\epsilon(\phi' + c)^{-1}\epsilon'.$$

(See the Note to page 561.)

Now if, as in the Note just cited, q_1, q_2, q_3, and q_4 are the solutions of $Vq^{-1}fq = 0$, and q'_1, q'_2, q'_3, and q'_4 are those of $Vq^{-1}f'q = 0$,

$$c_1 Sq_1q'_2 = -Sfq_1q'_2 = -Sq_1f'q'_2 = c_2 Sq_1q'_2,$$

as the roots c are the same for f and for its conjugate f'. Hence if c_1 is not equal to c_2, it is necessary to have $Sq_1q'_2 = 0$; and, in general, $Sq_nq'_{n'} = 0$, where n is different from n'.

If then $\quad q_1 = 1 + \alpha_1, \quad$ and $\quad q'_1 = 1 + \alpha'_1$, &c., $\quad Sq_1q'_2 = 1 + S\alpha_1\alpha'_2 = 0.$

Interpreted geometrically this property shows that if vectors are drawn through the origin equal to α_1, α_2, α_3, α_4, and to α'_1, α'_2, α'_3, and α'_4; α'_2, α'_3, and α'_4 will terminate on the polar plane of α_1 with respect to the unit sphere $\rho^2 + 1 = 0$. In other words, the tetrahedron determined by the extremities of α_1, α_2, α_3, and α_4 is the polar reciprocal of that determined by α'_1, α'_2, α'_3, and α'_4. In the particular case in which f is self-conjugate, $1 + S\alpha_1\alpha_2 = 0$, and the tetrahedron is self-reciprocal with respect to the unit sphere; or, without reference to a sphere, the tetrahedron may be said to be orthocentric as the perpendiculars ($-\alpha_1^{-1}$, &c.) from the origin on the faces pass through the corresponding vertices.

Hence, any quaternion q may be expressed in the form (compare again the note to page 561)

$$q = q_1 \frac{Sqq'_1}{Sq_1q'_1} + q_2 \frac{Sqq'_2}{Sq_2q'_2} + q_3 \frac{Sqq'_3}{Sq_3q'_3} + q_4 \frac{Sqq'_4}{Sq_4q'_4},$$

and

$$fq = -c_1q_1 \frac{Sqq'_1}{Sq_1q'_1} - c_2q_2 \frac{Sqq'_2}{Sq_2q'_2} - c_3q_3 \frac{Sqq'_3}{Sq_3q'_3} - c_4q_4 \frac{Sqq'_4}{Sq_4q'_4},$$

may be regarded as a canonical form of a function f.

It is easy to see from the properties of the reciprocal tetrahedra that the vector

$$\alpha_1' = -\frac{V(\alpha_3\alpha_4 + \alpha_4\alpha_2 + \alpha_2\alpha_3)}{S\alpha_2\alpha_3\alpha_4}$$

being the negative of the reciprocal of the vector perpendicular on the plane through the extremities of α'_2, α'_3, and α'_4.]

INDEX TO VOLUME I.

(The Numbers refer to the Pages. Additional references are occasionally given to sections in which the subjects are specially treated.)

* The word *line* is frequently used for *vector* in the Elements.

END OF VOLUME I.